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Construction of a control and reconstruction of a source for linear and nonlinear heat equations

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Contents

Abstract

My thesis focuses on two main problems in studying the heat equation: Control problem and Inverse problem.

Our first concern is *the null controllability of a semilinear heat equation* which, if not controlled, can blow up in finite time. Roughly speaking, it consists in analyzing whether the solution of a semilinear heat equation, under the Dirichlet boundary condition, can be driven to zero by means of a control applied on a subdomain in which the equation evolves. Under an assumption on the smallness of the initial data, such control function is built up. The novelty of our method is computing the control function in a constructive way. Furthermore, another achievement of our method is providing a quantitative estimate for the smallness of the size of the initial data with respect to the control time that ensures the null controllability property.

Our second issue is *the local backward problem for a linear heat equation.* We study here the following question: Can we recover the source of a linear heat equation, under the Dirichlet boundary condition, from the observation on a subdomain at some time later? This inverse problem is well-known to be an ill-posed problem, i.e their solution (if exists) is unstable with respect to data perturbations. Here, we tackle this problem by two different regularization methods: The filtering method and The Tikhonov method. In both methods, the reconstruction formula of the approximate solution is explicitly given. Moreover, we also provide the error estimate between the exact solution and the regularized one.

In order to approach the two above results, *the observation estimate at one point of time* for a linear heat equation plays an significant role. This well-known estimate can already be found in many literatures. However, a full version of the proof for this estimate is presented here as the author desires to make a self-contained discussion.

Introduction

It is well-known that the heat equation, which describes the distribution of heat in a given region over time is a model for many diffusion phenomena. The interest on studying the heat equation relies not only in the fact that it is a model for a large class of physical phenomena but also one of the most significant partial differential equation of parabolic type.

My thesis focuses on the three following topics about the heat equation:

1. *The observation estimate at one point of time for a linear heat equation*: the estimate on the energy at some point of time on the whole domain in terms of the energy at the same time but on a subdomain;

2. *The null controllability for a semilinear cubic heat equation*: the property that there exists a control function which leads the solution of a cubic semilinear system from a small given data at initial time to be null at final time;

3. *The local backward heat problem*: the problem of reconstructing the solution at initial time from the observation on a subdomain at some time later.

Now, suppose that Ω is denoted an open, bounded domain in $\mathbb{R}^n (n \geq 1)$ with C^2 boundary $\partial\Omega$. We will give an abstract of our results as well as our methods for solving the three above problems.

1. **The observation estimate at one point of time for a linear heat equation.**

   This issue on the observation estimate at one point of time for a linear heat equation is studied in the first chapter of my thesis (see Subsection 1.2.3).

   i/ *Problem*

   We consider the following linear heat equation, under the Dirichlet boundary condition:

$$\begin{cases} \partial_t v - \Delta v = 0 & \text{in } \Omega \times (0, +\infty), \\ v = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ v(\cdot, 0) = v^0 \in L^2(\Omega). \end{cases} \tag{1}$$

   Our target is finding the answer for the question:

   What can we conclude about the energy at some point of time on the whole domain when we observe the energy at the same time on a subdomain?

   ii/ *Main result*

   The answer is presented in the following estimate, which named *the observation estimate at one point of time* (see Theorem 1.8):

Let ω be a nonempty, open subset of Ω and T be a positive number. Then there exist $\mathcal{K}_1 > 0$, $\mathcal{K}_2 > 0$ and $\mu \in (0,1)$ depending on Ω and ω such that:

$$\|v(\cdot,T)\|_{L^2(\Omega)} \leq \mathcal{K}_1 e^{\frac{\mathcal{K}_2}{T}} \|v(\cdot,T)\|^{\mu}_{L^2(\omega)} \|v^0\|^{1-\mu}_{L^2(\Omega)}. \tag{2}$$

Thus, if the energy at some point of time on a subdomain equals 0 then so does the energy at the same time on the whole domain.

The first application of this result is the observability estimate (see Theorem 1.7), which is the key tool for studying our first main concern - the null controllability.

The second application of this result is the impulse controllability (see Theorem 1.11), which plays an important role in tackling our second main issue - the local backward problem.

iii/ *Idea of method*

The idea of our method comes from the *logarithm convexity method*, which has been introduced by Agmon and Nirenberg [AgN]:
Let f be a positive smooth function defined on an interval D such that $\ln f$ is a convex function. Then for any $t_1, t_2 \in D$, any $k \in (0,1)$ so that $(1-k)t_1 + kt_2 \in D$, the following estimate holds

$$f\left((1-k)t_1 + kt_2\right) \leq f(t_1)^{1-k} f(t_2)^k. \tag{3}$$

By applying this method for the function $t \to \int_\Omega |v(x,t)|^2 dx$, we obtain the following well-known estimate:

$$\|v(\cdot,t)\|_{L^2(\Omega)} \leq \|v(\cdot,T)\|^{\frac{t}{T}}_{L^2(\Omega)} \|v^0\|^{1-\frac{t}{T}}_{L^2(\Omega)} \quad \forall 0 \leq t \leq T. \tag{4}$$

Now, with the observation restricted on a subdomain, we use a weight function $\xi = \xi(x,t) \in C^\infty(\Omega \times [0,T])$ in order to remove the energy on the domain $\Omega \setminus \omega$. Precisely, we consider the logarithm convexity of the function below:

$$\Psi(t) := \int_\Omega |v(x,t)|^2 e^{\xi(x,t)} dx. \tag{5}$$

Indeed, the computation of the second derivative of the function Ψ involves some boundary terms which can be dropped with a star-shaped assumption. To overcome this geometrical assumption, we follow the strategy below.

iv/ *Strategy*

The strategy for getting our main result is decomposed into three following steps:

- The first step is constructing the local observation estimate: for any $x_0 \in \Omega$, any $R > 0$ and any $\delta \in (0,1]$ such that $\Omega \cap B(x_0,(1+2\delta)R)$ is star-shaped with respect to x_0, then for any $0 < r < R$ satisfying $B(x_0,r) \Subset \Omega$, we obtain:

$$\|v(\cdot,T)\|_{L^2(\Omega\cap B(x_0,R))} \leq C e^{\frac{C}{T}} \|v(\cdot,T)\|^{\sigma}_{L^2(B(x_0,r))} \|v^0\|^{1-\sigma}_{L^2(\Omega)} \tag{6}$$

  for some $C > 0$ and $\sigma \in (0,1)$.

- The second step is replacing the ball $B(x_0,r)$ by the subdomain ω. The used technique is the propagation of smallness, i.e constructing a sequence of balls chained along the curve (see Figure 1).

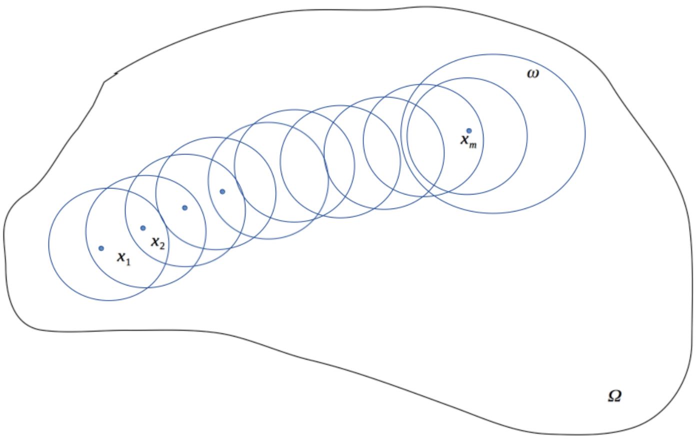


Figure 1 – Propagation of smallness

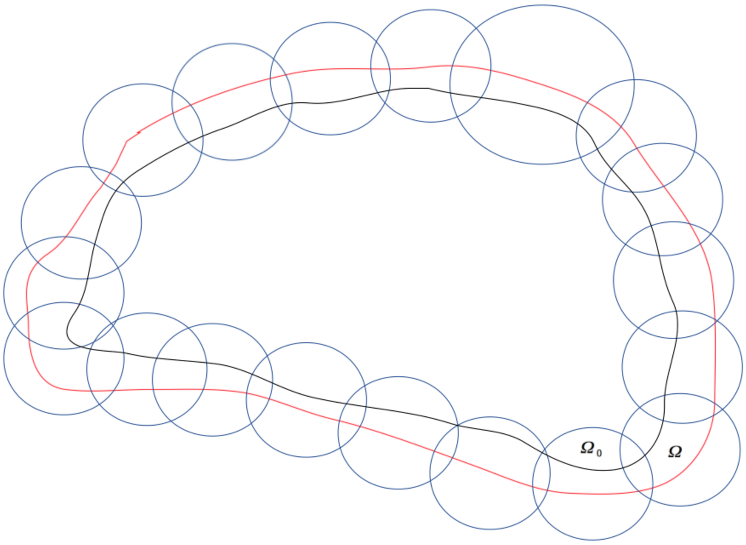


Figure 2 – Cover Ω

- The last step is covering Ω by dividing Ω into two parts: The interior Ω_0 which can be covered by balls being strictly inside Ω and the neighbourhood of $\partial\Omega$ (see Figure 2). Thanks to the fact Ω is bounded with C^2 boundary, there is a finite set of $(x_i, R_i, \delta_i) \in \Omega \times \mathbb{R}_+^* \times (0,1], i = 1, 2, ..., M$ such that $\Omega \cap B(x_i, (1+2\delta_i)R_i)$ is star-shaped with respect to x_i and

$$\partial\Omega \subset \bigcup_{i=1}^{M} (\Omega \cap B(x_i, R_i)) . \tag{7}$$

2. **The null controllability for a semilinear cubic heat equation.**

This topic on the null controllability for a semilinear cubic heat equation is the main concern in the second chapter of my thesis. This result is also presented in my first publication [Vo1].

i/ *Problem*

We consider the cubic semilinear heat equation complemented with initial and Dirichlet boundary condition in $\mathbb{R}^3 \times [0,T] \quad (T>0)$, which has the following form:

$$\begin{cases} \partial_t y - \Delta y + \gamma y^3 = \mathbb{1}_\omega f & \text{in} \quad \Omega \times (0,T) \ , \\ y = 0 & \text{on} \quad \partial\Omega \times (0,T) \ , \\ y(\cdot, 0) = y^0 \in L^2(\Omega) \ , \end{cases} \tag{8}$$

where $\gamma \in \{1, -1\}$, $\mathbb{1}_\omega$ denotes the characteristic function of ω and f denotes the control function acting on $\omega \times (0,T)$.

Our target is finding the answer for the following question (named *null controllable at time T* property):

Is there a control function $f \in L^2(\omega \times (0,T))$ which leads the solution of the above system from a given data y^0 at the initial time $t=0$ to be null at the final time $t=T$?

ii/ *Main result*

In this writing, we provide two answers for the null controllability of the cubic semilinear system, based on the fact that the blow up phenomenon appears or not:

— When the blow up phenomenon occurs ($\gamma = -1$), under an assumption on the smallness of the initial data in $H_0^1(\Omega)$, the answer is yes, i.e the system (8) is null controllable at time T (see Theorem 2.1).

— When the blow up phenomenon does not occur ($\gamma = 1$), under an assumption on the smallness of the initial data in $L^2(\Omega)$, the answer is yes, i.e the system (8) is null controllable at time T (see Corollary 2.1). This result is a direct corollary from the result for blow up case, thanks to the regularity property of the solution.

Furthermore, the construction of the control function is explicitly given and the smallness of the initial data is quantitatively estimated.

iii/ *Idea of method*

The idea of our method is based on an iterative algorithm of Liu, Takahashi and Tucsnak (see [LiTT]): Firstly, based on the null approximate controllability property of a homogeneous linear system (see Theorem 1.9), we construct the null controllability for a linear system with an outside force; Secondly, thanks to the idea of the Banach fixed-point theorem, we utilize an iteration argument by treating $-\gamma y^3$ as an outside force.

iv/ *Strategy*

For considering the null controllability of the linear system with an outside force (see Theorem 2.2 for case the initial data belongs to $L^2(\Omega)$ and Corollary 2.2 for case the initial data belongs to $H_0^1(\Omega)$), we follow the following strategy:

- The first step is dividing $[0,T]$ into small intervals of time $[T_k, T_{k+1}] \quad (k \geq 0)$ by taking

$$T_k = T - \frac{T}{a^k} \quad \text{for some} \quad a > 1. \tag{9}$$

- The second step is separating the controlled system with the outside force into two systems: One is with outside force, without control and another one is with control and without outside force such that the initial data of one system is the final data of the other (see more detail in page 52-54).

- The third step is using the null approximate controllability of the controlled system without outside force (see Subsection 1.3.1) to construct the null control function for the controlled system with outside force. The key tool is using the following argument:

  The fact that $\|\phi e^{\frac{M}{T-t}}\|_{C([0,T];L^2(\Omega))} < \infty$ for some $M > 0$ implies $\phi(\cdot, T) = 0$.

3. **The local backward heat problem.**

The local backward heat problem is solved in the third chapter of my thesis. This problem is also considered in my writing [Vo2]. Now, let us set up our problem.

i/ *Problem*

Let ω be a nonempty, open subset of Ω. We consider the following heat equation under the Dirichlet boundary condition:

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(\cdot, T) = \mathbb{f} \in L^2(\omega) \ . \end{cases} \tag{10}$$

Our target is finding the answer for the question (named *the local backward problem*):

Given $\delta > 0$ and $\mathbb{f}_\delta \in L^2(\omega)$ such that $\|\mathbb{f} - \mathbb{f}_\delta\|_{L^2(\omega)} \leq \delta$. Then can we construct an operator which maps $\mathbb{f}_\delta$ to some $\mathbb{g}_\delta$ such that $\|u(\cdot,0) - \mathbb{g}_\delta\|_{L^2(\Omega)} \leq \epsilon(\delta)$ where $\epsilon(\delta) \xrightarrow{\delta \to 0} 0$?

ii/ *Main result*

Under a priori condition on the initial data, the answer for above question is yes (see Theorem 3.2). Precisely, when $\delta < 1$, if $u(\cdot, 0) \in H_0^1(\Omega)$ then we can construct an approximation $\mathbb{g}_\delta$ from the given data $\mathbb{f}_\delta$ and δ such that

$$\|u(\cdot,0) - \mathbb{g}_\delta\|_{L^2(\Omega)} \leq C \left(\ln \frac{1}{\delta} \right)^{-\frac{1}{2}}, \tag{11}$$

where $C > 0$ depends on Ω, ω, T and $\|u(\cdot,0)\|_{H_0^1(\Omega)}$.

Furthermore, the reconstruction formula of the approximate solution $\mathbb{g}$ is explicitly given in (3.176). In addition, the error estimate between the exact solution $u(\cdot, 0)$ and the regularized solution $\mathbb{g}_\delta$ is also computed in (3.177).

iii/ *Idea of method*

In order to get our main result, a natural idea is firstly connecting the information on a subdomain with the solution on the whole domain, then secondly recovering the initial solution from what we have known about the solution on the whole domain.

The idea for the first step is from [GaOT], who determines the spatial dependence $f(x)$ of the source term in a heat equation $\partial_t u - \Delta u = f(x)\sigma(t)$, assuming $\sigma(t)$ is known, from a single internal measurements of the solution in $\omega \times (0,T)$. The reconstruction formula is associated to a family of null control acting on $(0, \tau)$ where $0 < \tau < T$. Here, our observation is available on $\omega \times \{T\}$. Hence, our reconstruction is involved in a family of impulse controls acting on $\omega \times \{T\}$ (see Lemma 3.1).

The idea for the second step comes from [Se2]. In this work, under the assumption that $u(\cdot, 0) \in L^2(\Omega)$, the author constructs the solution $u(\cdot, t)$ $(t \in (0, T))$ from the approximation data of $u(\cdot, T)|_\Omega$ by using a special filtering method. The idea of his method is using a filter in the eigenfunctions decomposition of the solution in order to eliminate the high frequency components. The error estimate by this method is of Hölder type $\delta^{\frac{t}{T}}$. This has no meaning for the case $t = 0$ but the convergence is optimal in sense of Tautenhahn (see Theorem 3.4). Hence, we will improve this method to recover the solution at time $t = 0$, whose convergence is also optimal (see Theorem 3.5).

iv/ *Strategy*

The strategy for constructing the regularized solution by the filtering method is presented in the following steps:

- In the first step, we will get the information of the solution at time $3T$ on the whole domain, based on the given data $\mathbb{f}_\delta$ at time T on a subdomain. The key tool to get this target is the null approximate impulse controllability (see Theorem 1.11) of the dual system. This property provides us a family of the impulse control functions acting on $\omega \times \{T\}$ which leads the solution from $e_i (i = 1, 2, ...)$ at initial time to be null approximately at time $2T$. Then, thanks to the property that $\sum\limits_{i \geq 1} e^{-2\lambda_i T} < \infty$, we construct the approximate data $\mathbb{f}_\eta$ of $u(\cdot, 3T)|_\Omega$. Here λ_i and e_i are respectively the eigenvalues and the corresponding eigenfunctions of the Laplacian, under the Dirichlet boundary condition.

- In the second step, we solve the backward problem (see Theorem 3.1): recovering the initial solution from the noisy data at time $3T$ on the whole domain. The approximate solution is constructed as below

$$\mathbb{g} := \sum_{i \geq 1} \min\{e^{3\lambda_i T}, \alpha\} \left(\int_\Omega \mathbb{f}_\eta(x) e_i(x) dx \right) e_i. \tag{12}$$

The important point in this step is choosing a suitable parameter α in order to get minimum error. In the error estimate, we have split the total error into an approximate error, which tends to 0 as $\alpha \to \infty$ and a data error, which explodes as $\alpha \to \infty$:

$$\text{total error} \leq \underbrace{\text{approximate error}}_{\lesssim (\ln \alpha)^{-\frac{1}{2}}} + \underbrace{\text{data error}}_{\lesssim \alpha}. \tag{13}$$

To get a good approximation, we have to balance these two error terms by a good choice of the parameter α.

Combining the results in two steps, we obtain our desired result.

Structure of the thesis

The content of the thesis is separated into three chapters:

Chapter 1: In this Chapter, we recall the important properties (well-posedness, spectral theory, controllability) as well as the necessary estimates (energy estimate, regularity estimate, backward estimate, stability estimate, observability estimate) for the linear heat equation, under the Dirichlet boundary condition. These results are the preliminaries for our concerns in next chapters.

Chapter 2: In this Chapter, we study the null controllability of a cubic semilinear heat equation. We present a constructive way to compute a control function which leads the solution of a cubic heat equation, under the Dirichlet condition, from a given data at time 0 (which is small enough) to null at a given final time T. Furthermore, the smallness of the initial data with respect to the final time T is also given in a quantitative estimate.

Chapter 3: In this Chapter, we discuss about the backward problem and local backward problem for the linear heat equation, under the Dirichlet boundary condition. Precisely, we approximately recover the initial data from the observation on the whole domain (backward problem) or on a subdomain (local backward problem). Two different regularization methods are used: the Filtering method and the Tikhonov method. Furthermore, we also study the optimality of our regularization method in sense of Tautenhahn, which concerns the best possible case error for identifying the approximate solution. In addition, by using a technique of changing variable, we also solve the backward and local backward problem for the time dependent thermal conductivity heat equation.

Chapter 1

Preliminaries

In this chapter, we recall the main results on the properties for a solution of the heat equation, which is the simplest example of a parabolic equation, under Dirichlet boundary condition. Furthermore, the topic of observability estimate, which has many important applications in control theory, is presented with detailed proof. One of these applications, null approximate controllability and null controllability, is reminded by a constructive way of a control function. They are primary results for studying null controllability for semilinear heat equation (see Chapter 2). Moreover, the null approximate impulse controllability which connects the backward problem and local backward one (see Chapter 3) is also studied. The main content of each section in this Chapter is shortly given as below:

Section 1.1: We focus on the well-posedness (Subsection 1.1.1) of the problem of finding the solution of heat equation, under Dirichlet boundary condition and the given initial data. Moreover, the explicit formula of the solution with respect to the initial data is given by the decomposition in Hilbert basis (Subsection 1.1.2). The classical estimates for solution of this problem are also mentioned: energy estimate (Subsection 1.1.3), regularity estimate (Subsection 1.1.4), backward estimate (Subsection 1.1.5) and stability estimate (Subsection 1.1.6).

Section 1.2: We study an interesting estimate which says: if $v|_{\omega\times(0,T)} = 0$ then $v \equiv 0$ where v is the solution of heat equation, under Dirichlet boundary condition and ω is any open subdomain (Subsection 1.2.1). In order to study how people solved this problem in the past, Subsection 1.2.2 is recommended . The main point to get this estimate is the observation estimate at one point of time (Subsection 1.2.3). In Subsection 1.2.3, we will firstly provide some preliminary lemmas (see 1.2.3.1) and then give the proof in two different geometry conditions: when Ω is convex (see 1.2.3.2) and when Ω is C^2, open and bounded (see 1.2.3.3). The proof of main results (Subsection 1.2.4) as well as the preliminary results (Subsection 1.2.5) will complete this section.

Section 1.3: We concern about an important issue in control theory: Controllability. Precisely, we construct a control function which leads the solution at a given point at initial time to a desired point at final time. When the control function acts on $\omega \times (0, T)$, if final data gets null approximately, we call null approximate controllability (see Subsection 1.3.1), if final data gets null exactly, we call null controllability (see Subsection 1.3.2). When the control function acts on $\omega \times \{T\}$ and final data approximates to zero, we name null approximate impulse controllability (Subsection 1.3.3).

1.1 Heat equation

1.1.1 Well-posedness

The term *well-posedness* stems from a definition given by Jacques Hadamard. He claims that a mathematical model for a physical problem has to be well-posed in the following sense.

Definition 1.1. *(see [Ev, p.7] or [Ki, p.9])*
A given problem for a partial differential equation is well-posed if

i/ The problem has a solution;

ii/ The solution is unique;

iii/ The solution depends continuously on the given data.

Now, we study on the well-posedness of a heat problem: Let Ω be an open bounded domain in $\mathbb{R}^n (n \geq 1)$ with a boundary $\partial\Omega$ of class C^2. We consider the following heat equation under the Dirichlet boundary condition:

(**HP**) *Given* $v^0 \in L^2(\Omega)$, *find a solution* $v : \overline{\Omega} \times [0, +\infty) \to \mathbb{R}$ *such that*

$$\begin{cases} \partial_t v - \Delta v = 0 & \text{in } \Omega \times (0, +\infty), \\ v = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ v(\cdot, 0) = v^0 & \text{in } \Omega. \end{cases} \tag{1.1}$$

It is well-known that (HP) is *well-posed* in sense of Hadamard, thanks to the following theorem:

Theorem 1.1. *(see [CaH, Pro.3.5.2, p.42])*
There exists a unique function $v(x, t)$ *satisfying (HP) and*

1. $v \in C([0, \infty); L^2(\Omega)) \cap C^1((0, \infty); L^2(\Omega))$,
2. $\Delta v \in C((0, \infty); L^2(\Omega))$,
3. $v \in C((0, \infty); H_0^1(\Omega))$.

In addition, we have:

$$\|v(\cdot, t)\|_{L^2(\Omega)} \leq \|v^0\|_{L^2(\Omega)} \quad \forall t > 0. \tag{1.2}$$

Furthermore, the explicit formula of the solution with respect to the initial data will be obtained by the following fundamental theory.

1.1.2 Spectral theory

The unique solution of problem (HP) can be given by a decomposition in a Hilbert basis, thanks to the following theorem:

Theorem 1.2. *(see [Br, Th.9.31, p.311])*
There exists a sequence of positive real eigenvalues of the operator $-\Delta$ *(with Dirichlet boundary condition), which denoted by* $\{\lambda_i\}_{i\geq 1}$ *where*

$$\begin{cases} 0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots, \\ \lambda_i \to \infty \text{ as } i \to \infty. \end{cases} \tag{1.3}$$

Moreover, there exists an orthonormal basis $\{e_i\}_{i\geq 1}$ *of* $L^2(\Omega)$, *where* $e_i \in H_0^1(\Omega) \cap C^\infty(\Omega)$ *is an eigenfunction corresponding to* λ_i *such that*

$$-\Delta e_i = \lambda_i e_i \quad \text{in } \Omega.$$

In order to study more about the property of eigenvalues $\{\lambda_i\}$, we recommend the readers to [He] or [BuH]. Here, we use this theorem to solve (HP) by a decomposition in a Hilbert basis of $L^2(\Omega)$. Precisely, we seek a solution v of (HP) in the form of a series

$$v(x,t) = \sum_{i\geq 1} a_i(t) e_i(x). \tag{1.4}$$

The fact that v is a solution of (HP) requires the functions $a_i(t)$ satisfy

$$\begin{cases} a_i'(t) + \lambda_i a_i(t) = 0, \\ \sum\limits_{i\geq 1} a_i(0) e_i(x) = v^0. \end{cases}$$

Thus, we get $a_i(t) = a_i(0) e^{-\lambda_i t}$ and $a_i(0) = \int_\Omega v^0(x) e_i(x) dx$. In conclusion, the unique solution of (HP) is

$$v(\cdot,t) = \sum_{i\geq 1} \left(\int_\Omega v^0(x) e_i(x) dx \right) e^{-\lambda_i t} e_i. \tag{1.5}$$

1.1.3 Energy estimate

Here, we recall a basic estimate for the heat equation, which is based on the non-increasing property of the "energy" function $E(t) := \frac{1}{2}\int_\Omega |v(x,t)|^2 dx$.

Theorem 1.3. *(see [CaH, Pro.3.5.5, p.43])*
Let v be the solution of (HP). Then the following estimate holds

$$\|v(\cdot,t)\|_{L^2(\Omega)} \leq e^{-\lambda_1 t} \|v^0\|_{L^2(\Omega)} \quad \forall t > 0.$$

1.1.4 Regularity estimate

Let us move to another basic estimate for solution of (HP), which is called the regularity estimate on smoothing effect of the heat equation.

Theorem 1.4. *(see [CaH, Pro.3.5.2, p.42])*
Let v be the solution of (HP). Then the following estimate holds

$$\|\nabla v(\cdot,t)\|_{L^2(\Omega)} \leq \frac{1}{\sqrt{2t}} \|v^0\|_{L^2(\Omega)} \quad \forall t > 0.$$

1.1.5 Backward estimate

Our target in this subsection is looking for an estimate of the form

$$\|v^0\|_{L^2(\Omega)} \leq constant \|v(\cdot,T)\|_{L^2(\Omega)} \tag{1.6}$$

where v solves (HP) and T denotes a positive number. This estimate (1.6) is called "backward estimate" for heat equation, which gives the uniqueness of solution for the backward heat problem.

Theorem 1.5. *(see [BaT])*
Let v be the solution of (HP) and T be a positive number. If $v^0 \in H_0^1(\Omega)$ and $\|v^0\|_{L^2(\Omega)} \neq 0$ then

$$\|v^0\|_{L^2(\Omega)} \leq e^{\frac{\|v^0\|^2_{H_0^1(\Omega)}}{\|v^0\|^2_{L^2(\Omega)}} T} \|v(\cdot,T)\|_{L^2(\Omega)}. \tag{1.7}$$

1.1.6 Stability estimate

The backward estimate shows how $\|v^0\|_{L^2(\Omega)}$ depends on $\|v(\cdot,T)\|_{L^2(\Omega)}$. Next, the stability estimate in this subsection will tell us how $\|v(\cdot,t)\|_{L^2(\Omega)}(t \in (0,T))$ depends on $\|v(\cdot,T)\|_{L^2(\Omega)}$.

Theorem 1.6. *(see [Ve, p.432] or [Pa, p.11])*
Let v be the solution of (HP) and T be a positive number. Then the following estimate holds for any $t \in (0,T)$:

$$\|v(\cdot,t)\|_{L^2(\Omega)} \leq \|v^0\|_{L^2(\Omega)}^{1-\frac{t}{T}} \|v(\cdot,T)\|_{L^2(\Omega)}^{\frac{t}{T}}. \tag{1.8}$$

The result in Theorem 1.6 is obtained by using the logarithm convexity method for the function $f(t) := \int_\Omega |v(x,t)|^2 dx$.

1.2 Observability estimate

In this section, we concern about another well-known estimate named observability estimate which is not only mathematically interesting but also has important applications in the control theory of the heat equation, such as: Bang-bang control (see [PhW2]); Time optimal control (see [PhW2], [PhWZ] or [ApEWZ]); Null controllability (see [Zu3] or [ApEWZ]); etc. Now, let us state the main result of this section.

1.2.1 Main result

Theorem 1.7. *(see [DuZZ, Th.A, p.2])*
Let ω be a nonempty, open subset of Ω, v be the solution of (HP) and T be a positive number. Then there exist positive constants $\mathcal{C}_1$ and $\mathcal{C}_2$ depending on Ω and ω such that the following assertion holds:

$$\|v(\cdot,T)\|_{L^2(\Omega)} \leq \mathcal{C}_1 e^{\frac{\mathcal{C}_2}{T}} \|v\|_{L^2(\omega\times(0,T))}. \tag{1.9}$$

Remark 1.1.
1. *The estimate (1.9) is so called an observability estimate which asserts that the energy of solution concentrated in ω yields an upper bound of the energy everywhere in Ω. Here, because of the strong irreversibility of the heat equation, the constant in (1.9) grows exponentially as $T \to 0$.*
2. *The constants $\mathcal{C}_1$ and $\mathcal{C}_2$ in (1.9) depend on the geometric properties of Ω and ω. Under the special geometric condition on the domain, that is: Ω is convex or star-shaped with respect to $x_0 \in \Omega$ such that $\{x : |x-x_0| < r\} \subset \omega$ for some $r \in (0,1)$, how the constant $\mathcal{C}_1$ and $\mathcal{C}_2$ depend on ω are explicitly computed. Precisely, in [Ph, Th.1.1, p.2], the authors gives that:*

$$\|v(\cdot,T)\|_{L^2(\Omega)} \leq C_\epsilon e^{\frac{C_\epsilon}{r^\epsilon}\frac{1}{T}} \|v\|_{L^2(\omega\times(0,T))}. \tag{1.10}$$

for some positive constant C_ϵ depending on ϵ and $\max\{|x-x_0| : x \in \overline{\Omega}$. Recently, the authors in [LauL] improve the result of [Ph] by providing

$$\|v(\cdot,T)\|_{L^2(\Omega)} \leq Ce^{(C|\ln r|^2+C)\frac{1}{T}} \|v\|_{L^2(\omega\times(0,T))}. \tag{1.11}$$

for some positive geometric constant C (see [LauL, TH.1.3,p.4]).

1.2.2 State of art

There are extensive literatures on the subject of finding the observability estimate (1.9). Let us now discuss some of well-known methods.

1. **Global Carleman inequalities:** The idea to use Global Carleman inequalities for establishing the observability estimate is firstly given by Emanuvilov (see [Em]) in 1995. Then, thanks to the advantage that getting explicit bounds on the observability constants and application for general parabolic equations, this method is widely used by many researchers such as: Fursikov ([FuI] or [Fu]), Fernández-Cara, Zuazua ([FeZ2], [DuZZ]), etc.The complete proof can also be found in the book of Tucsnak and Weiss (see [TuW, Th.9.5.1, p.313]).
2. **Spectral estimate:** An interesting characterization of the observability estimate in terms of the spectrum of Laplacian, which also named Lebeau-Robbiano spectral inequality, has been derived by Lebeau and Robbiano in [LeR]. This method can also be found in [Zh], [Mi3], [ApEWZ], [FeZ1] or [MiZ].
3. **Logarithm convexity method:** Recently, Phung and all (see [PhW1], [PhW2], [PhWX], [PhWZ] or [BaP]) provide a new method, which is independent with the two above methods. Their strategy is based on the logarithm convexity method to find an observation estimate at one point of time, and then apply the telescoping series method to get the desired estimate. This method can also work for the parabolic equations with space-time coefficients (see [BaP]). In what following, we will use this method to give the proof of Theorem 1.7.

1.2.3 Observation estimate at one point of time

In this subsection, we will study an estimate which is the key point for the proof of Theorem 1.7.

Theorem 1.8. *(see [PhWZ, Le.5] or [BaP, Th.4.1])*
Let ω be a nonempty, open subset of Ω, v be the solution of (HP) and T be a positive number. Then there exist positive constants $\mathcal{K}_1$, $\mathcal{K}_2$ and $\mu \in (0,1)$ depending on Ω, ω such that the following estimate holds:

$$\|v(\cdot,T)\|_{L^2(\Omega)} \leq \mathcal{K}_1 e^{\frac{\mathcal{K}_2}{T}} \|v(\cdot,T)\|^{\mu}_{L^2(\omega)} \|v^0\|^{1-\mu}_{L^2(\Omega)}. \tag{1.12}$$

Remark 1.2. *1. Theorem 1.8 has an interesting meaning that is: If $v \equiv 0$ on $\omega \times \{T\}$ then $v \equiv 0$ on $\Omega \times \{T\}$.*

2. When Ω is convex, the constants $\mathcal{K}_1$, $\mathcal{K}_2$ and μ are explicitly computed (see Subsection 1.2.3.2). The interested readers can compare with [PhW1, Pro.2.1], [PhW2, Pro.2.2] or [BaP, Th.4.2].

3. The estimate (1.12) is equivalent to the following Lebeau-Robbiano spectral inequality (see [PhWX, Th.2.1]): There exist positive constants $\mathcal{K}_3$, $\mathcal{K}_4$, depending on Ω and ω so that for each $\lambda > 0$ and each sequence of real numbers $\{a_j\}_{j\geq 1} \subset \mathbb{R}$, we get

$$\sum_{\lambda_j < \lambda} |a_j|^2 \leq \mathcal{K}_3 e^{\mathcal{K}_4 \sqrt{\lambda}} \int_\omega |\sum_{\lambda_j<\lambda} a_j e_j|^2. \tag{1.13}$$

It is also called the observability estimate for the spectrum of Laplacian. The key ingredient for this equivalence is the eigenfunctions decomposition of the solution of (HP) given by (1.5).

4. The estimate (1.12) can be improved by the following estimate (see also [PhWX, Th.2.1]): There exist positive constants $\mathcal{K}_5$, $\mathcal{K}_6$, depending on Ω and ω so that for any $\beta \in (0,1)$, one has

$$\|v(\cdot,T)\|_{L^2(\Omega)} \leq \mathcal{K}_5 e^{\frac{\mathcal{K}_6}{T\beta}} \|v(\cdot,T)\|^{\beta}_{L^2(\omega)} \|v^0\|^{1-\beta}_{L^2(\Omega)}. \tag{1.14}$$

One application of this estimate is minimal norm impulse control (see [PhWX, Th.3.4]).

1.2.3.1 Preliminary lemmas

The key tool for the proof of Theorem 1.8 is the *logarithm convexity method* for the following function

$$\int_\Omega |v(x,t)|^2 e^{\xi(x,t)} dx, \tag{1.15}$$

where ξ is some weight function. In [PhW1], [PhW2], [PhWZ], [PhWX] or [BaP], they use the following weight function

$$\xi(x,t) := \frac{-|x-x_0|^2}{4(T-t+\rho)} - \frac{n}{2}\ln(T-t+\rho) \tag{1.16}$$

for $x_0 \in \Omega$ and $\rho > 0$. With such choice of the weight function, the constants in final estimate (1.12) depend on the dimension n of the domain. In [Ph], the author improves their result by removing the dependence on the dimension n with another choice of weight function, which is

$$\xi(x,t) := \frac{-|x-x_0|^2}{4(T-t+\rho)}. \tag{1.17}$$

In [Ph], the author combines the logarithmic convexity with the Carleman commutator in order to get an abstract result for any weight function satisfying some assumptions. Here, for simplicity, we will use the same technique in the previous works but with the new weight function (1.17). First of all, let us count on some properties of the weight function defined in (1.17):

(P1) $\partial_t\xi + |\nabla\xi|^2 = 0$,

(P2) $\nabla\xi = \frac{-(x-x_0)}{2(T-t+\rho)}$,

(P3) $\Delta\xi = \frac{-n}{2(T-t+\rho)}$,

(P4) $\nabla^2\xi = \frac{-1}{2(T-t+\rho)}I_n$ where I_n is the identity matrix of size n.

Now, we consider the first derivative of the function defined in (1.15) by Lemma below.

Lemma 1.1. *Let ϑ be an open set in $\mathbb{R}^n$ $(n \geq 1)$, $x_0 \in \vartheta$, $w \in H^1(0,T;H_0^1(\vartheta))$, $\rho > 0$, ξ be defined in (1.17) and $\Psi : [0,T] \to \mathbb{R}$ such that*

$$\Psi(t) := \int_\vartheta |w(x,t)|^2 e^{\xi(x,t)}dx. \tag{1.18}$$

Then the following assertion holds for any time $t > 0$

$$\begin{aligned}\Psi'(t) &= 2\int_\vartheta w(x,t)(\partial_t w - \Delta w)(x,t)e^{\xi(x,t)}dx \\ &\quad -2\int_\vartheta |\nabla w(x,t)|^2 e^{\xi(x,t)}dx - \frac{n}{2(T-t+\rho)}\Psi(t).\end{aligned} \tag{1.19}$$

Next, we define another function $N : [0,T] \to \mathbb{R}$ satisfying

$$N(t) := \frac{2\int_\vartheta |\nabla w(x,t)|^2 e^{\xi(x,t)}dx}{\int_\vartheta |w(x,t)|^2 e^{\xi(x,t)}dx} + \frac{n}{2(T-t+\rho)}. \tag{1.20}$$

Then the estimate (1.19) can be written as

$$\Psi'(t) + N(t)\Psi(t) = 2\int_\vartheta w(x,t)(\partial_t w - \Delta w)(x,t)e^{\xi(x,t)}dx. \tag{1.21}$$

The next lemma will provide us the estimate of $N'(t)$.

Lemma 1.2. *Suppose all the assumptions of Lemma 1.1 are satisfied and ϑ is convex or star-shaped with respect to x_0. Let N be defined as in (1.20). Then we get the following estimate*

$$N'(t) \leq \frac{N(t)}{T-t+\rho} + \frac{\int_\vartheta |\partial_t w(x,t) - \Delta w(x,t)|^2 e^{\xi(x,t)}dx}{\int_\vartheta |w(x,t)|^2 e^{\xi(x,t)}dx}. \tag{1.22}$$

In the proof of Theorem 1.8 when Ω is C^2, open and bounded (convex or non-convex), we need two more technical lemmas. The first one is used to estimate the rest term, which appears in the localization.

Lemma 1.3. *(see [Ph, Le.2.2, p.12])*
Let $x_0 \in \Omega$, $R > 0$, $\delta \in (0, 1]$ and v be the solution of (HP). Then there exists $0 < \hbar \le \min\{1, \frac{T}{2}\}$ such that the following estimate holds for any $T - \hbar \le t \le T$:

$$\frac{\int_\Omega |v^0(x)|^2 \, dx}{\int_{\Omega \cap B(x_0,(1+\delta)R)} |v(x,t)|^2 \, dx} \le e^{\frac{(1+\delta)\delta R^2}{2\hbar}}.$$

Here

$$\frac{1}{\hbar} = \frac{2}{(\delta R)^2} \ln \left(e^{\frac{R^2}{2}(1+\frac{2}{T})} \frac{2\int_\Omega |v^0(x)|^2 dx}{\int_{\Omega \cap B(x_0,R)} |v(x,T)|^2 dx} \right).$$

The second lemma is used for covering the neighbourhood of the boundary.

Lemma 1.4. *For any $z \in \partial\Omega$, there exist $x_0 \in \Omega$ and $R > 0$ such that $z \in B(x_0, R)$ and $(x - x_0)\nu \ge 0$ for any $x \in \partial\Omega \cap B(x_0, R)$ where ν is the unit outward normal vector to x.*

Now we can start the proof of Theorem 1.8 by the simple case when Ω is convex.

1.2.3.2 Case when Ω is convex

We divide the proof into several steps: Step 1 constructs ordinary differential equations (ODEs) which are applications of Lemma 1.1 and Lemma 1.2; Solving these ODEs, Step 2 provides us Hölder estimate; In Step 3, we take off the weight function and make appear the local term; Lastly, by choosing suitable parameters, Step 4 and Step 5 will give us the final result.

Step 1: Construct ODEs.
Taking $x_0 \in \omega$ and $\rho > 0$, we define the weight function $\xi : \Omega \times (0, T) \to \mathbb{R}$ such that

$$\xi(x,t) := \frac{-|x - x_0|^2}{4(T - t + \rho)}. \tag{1.23}$$

Let v be the solution of (HP). We consider the following functions

$$\Psi(t) := \int_\Omega |v(x,t)|^2 e^{\xi(x,t)} dx \tag{1.24}$$

and

$$N(t) := \frac{2\int_\Omega |\nabla v(x,t)|^2 e^{\xi(x,t)} dx}{\int_\Omega |v(x,t)|^2 e^{\xi(x,t)} dx} + \frac{n}{2(T - t + \rho)}. \tag{1.25}$$

Applying Lemma 1.1 and Lemma 1.2 with $\vartheta := \Omega$, $w := v$, we get from (1.21) that

$$\Psi'(t) + N(t)\Psi(t) = 0 \tag{1.26}$$

and from (1.22) that

$$N'(t) \le \frac{N(t)}{T - t + \rho}. \tag{1.27}$$

Step 2: Solve ODEs.
Let $0 \le t_1 < t_2 < t_3 \le T$, we will solve ODEs (1.26) and (1.27) from Step 1 on (t_1, t_2) and (t_2, t_3) respectively. From (1.27), we get

$$(N(t)(T - t + \rho))' \le 0. \tag{1.28}$$

For $t_1 < t < t_2$

Integrating (1.28) over (t, t_2) gives us

$$N(t) \geq N(t_2)\frac{T - t_2 + \rho}{T - t + \rho}. \tag{1.29}$$

Combining (1.26) and (1.29), we obtain

$$\Psi'(t) + N(t_2)\frac{T - t_2 + \rho}{T - t + \rho}\Psi(t) \leq 0.$$

It is equivalent to

$$\left(\Psi(t)e^{N(t_2)(T-t_2+\rho)\int_0^t \frac{ds}{T-s+\rho}}\right)' \leq 0. \tag{1.30}$$

Integrating (1.30) over (t_1, t_2), one has

$$\Psi(t_1) \geq \Psi(t_2)e^{N(t_2)(T-t_2+\rho)\ln\frac{T-t_1+\rho}{T-t_2+\rho}}. \tag{1.31}$$

It deduces from (1.31) that

$$e^{N(t_2)(T-t_2+\rho)} \leq \left(\frac{\Psi(t_1)}{\Psi(t_2)}\right)^{\frac{1}{\ln\frac{T-t_1+\rho}{T-t_2+\rho}}}. \tag{1.32}$$

For $t_2 < t < t_3$

Integrating (1.28) over (t_2, t) gives us

$$N(t) \leq N(t_2)\frac{T - t_2 + \rho}{T - t + \rho}. \tag{1.33}$$

It deduces from (1.26) and (1.33) that

$$\Psi'(t) + N(t_2)\frac{T - t_2 + \rho}{T - t + \rho}\Psi(t) \geq 0. \tag{1.34}$$

It is equivalent to

$$\left(\Psi(t)e^{N(t_2)(T-t_2+\rho)\int_0^t \frac{ds}{T-s+\rho}}\right)' \geq 0. \tag{1.35}$$

Integrating (1.35) over (t_2, t_3) gives us

$$\Psi(t_2) \leq \Psi(t_3)e^{N(t_2)(T-t_2+\rho)\ln\frac{T-t_2+\rho}{T-t_3+\rho}}. \tag{1.36}$$

Now, combining (1.32) and (1.36), one gets

$$\Psi(t_2) \leq \Psi(t_3)\left(\frac{\Psi(t_1)}{\Psi(t_2)}\right)^M \tag{1.37}$$

where

$$M := \frac{\ln\frac{T-t_2+\rho}{T-t_3+\rho}}{\ln\frac{T-t_1+\rho}{T-t_2+\rho}}. \tag{1.38}$$

It implies from (1.37) that

$$\Psi(t_2)^{1+M} \leq \Psi(t_3)\Psi(t_1)^M. \tag{1.39}$$

Step 3: Choose suitable t_1, t_2, t_3, take off the weight function and make appear ω. Remind that

$$\Psi(t) = \int_\Omega |v(x,t)|^2 e^{-\frac{|x-x_0|^2}{4(T-t+\rho)}}dx. \tag{1.40}$$

Now, for any $\ell > 1$ such that $\ell\rho \leq \frac{T}{2}$, choosing $t_1 = T - 2\ell\rho,\ t_2 = T - \ell\rho,\ t_3 = T$, we get from (1.39) that

$$\begin{aligned} &\left(\int_\Omega |v(x, T-\ell\rho)|^2 e^{-\frac{|x-x_0|^2}{4(1+\ell)\rho}} dx\right)^{1+M} \\ \leq\ &\left(\int_\Omega |v(x,T)|^2 e^{-\frac{|x-x_0|^2}{4\rho}} dx\right)\left(\int_\Omega |v(x,T-2\ell\rho)|^2 e^{-\frac{|x-x_0|^2}{4(1+2\ell)\rho}} dx\right)^M . \end{aligned} \tag{1.41}$$

On one hand, we consider the term on the left-hand side of (1.41):
Put $R := \max_{x\in\overline{\Omega}} |x - x_0|$ then

$$\left(\int_\Omega |v(x, T-\ell\rho)|^2 e^{-\frac{|x-x_0|^2}{4(1+\ell)\rho}} dx\right)^{1+M} \geq e^{-\frac{R^2(1+M)}{4(1+\ell)\rho}} \left(\int_\Omega |v(x, T-\ell\rho)|^2 dx\right)^{1+M} . \tag{1.42}$$

Applying the energy estimate, that is

$$\int_\Omega |v(x, T-\ell\rho)|^2 dx \geq \int_\Omega |v(x,T)|^2 dx, \tag{1.43}$$

one obtains

$$\left(\int_\Omega |v(x, T-\ell\rho)|^2 e^{-\frac{|x-x_0|^2}{4(1+\ell)\rho}} dx\right)^{1+M} \geq e^{-\frac{R^2(1+M)}{4(1+\ell)\rho}} \left(\int_\Omega |v(x,T)|^2 dx\right)^{1+M} . \tag{1.44}$$

On the other hand, we consider the terms on the right-hand side of (1.41):

- For the first term, in order to make appear ω, take $0 < r < R$ such that $B(x_0, r) \subset \omega$, we have

$$\begin{aligned} &\int_\Omega |v(x,T)|^2 e^{\frac{-|x-x_0|^2}{4\rho}} dx \\ =\ &\int_{B(x_0,r)} |v(x,T)|^2 e^{\frac{-|x-x_0|^2}{4\rho}} dx + \int_{\Omega\setminus B(x_0,r)} |v(x,T)|^2 e^{\frac{-|x-x_0|^2}{4\rho}} dx \\ \leq\ &\int_{B(x_0,r)} |v(x,T)|^2 dx + e^{-\frac{r^2}{4\rho}} \int_{\Omega\setminus B(x_0,r)} |v(x,T)|^2 dx \\ \leq\ &\int_\omega |v(x,T)|^2 dx + e^{-\frac{r^2}{4\rho}} \int_\Omega |v(x,T)|^2 dx \\ \leq\ &\int_\omega |v(x,T)|^2 dx + e^{-\frac{r^2}{4\rho}} \int_\Omega |v^0(x)|^2 dx. \end{aligned} \tag{1.45}$$

  The first inequality comes from the fact that

$$e^{\frac{-|x-x_0|^2}{4\rho}} \leq 1 \quad \forall x \in B(x_0, r) \tag{1.46}$$

  and

$$e^{\frac{-|x-x_0|^2}{4\rho}} \leq e^{-\frac{r^2}{4\rho}} \quad \forall x \in \Omega \setminus B(x_0, r). \tag{1.47}$$

  The second inequality is based on the fact that $B(x_0, r) \subset \omega$ and $\Omega \setminus B(x_0, r) \subset \Omega$. The last inequality is obtained thanks to the energy estimate.

- For the second term, using the fact that $e^{-\frac{|x-x_0|^2}{4(1+2\ell)\rho}} \leq 1$ and the energy estimate, which is

$$\int_\Omega |v(x, T-2\ell\rho)|^2 dx \leq \int_\Omega |v^0(x)|^2 dx, \tag{1.48}$$

  we get

$$\left(\int_\Omega |v(x, T-2\ell\rho)|^2 e^{-\frac{|x-x_0|^2}{4(1+\ell)\rho}} dx\right)^M \leq \left(\int_\Omega |v^0(x)|^2 dx\right)^M . \tag{1.49}$$

Now, combining (1.41), (1.44), (1.45) and (1.49) gives us

$$\begin{aligned}\left(\int_\Omega |v(x,T)|^2dx\right)^{1+M} \quad &\leq \quad e^{\frac{R^2(1+M)}{4(1+\ell)\rho}}\left(\int_\omega |v(x,T)|^2dx\right)\left(\int_\Omega |v^0(x)|^2dx\right)^{M}\\ &\quad +e^{\frac{R^2(1+M)}{4(1+\ell)\rho}-\frac{r^2}{4\rho}}\left(\int_\Omega |v^0(x)|^2dx\right)^{1+M}. \end{aligned} \tag{1.50}$$

Step 4: Choose suitable ℓ.

Now, recall that

$$M := \frac{\ln\frac{T-t_2+\rho}{T-t_3+\rho}}{\ln\frac{T-t_1+\rho}{T-t_2+\rho}}. \tag{1.51}$$

With above choice of t_1, t_2, t_3, one gets

$$M = \frac{\ln(1+\ell)}{\ln\frac{1+2\ell}{1+\ell}}. \tag{1.52}$$

Thanks to the fact that $\ell > 1$, we obtain $\frac{1+2\ell}{1+\ell} = 2 - \frac{1}{1+\ell} > \frac{3}{2}$. Hence, one has

$$1 < M < \frac{\ln(1+\ell)}{\ln\frac{3}{2}} := M_\ell. \tag{1.53}$$

Notice that the estimate (1.50) still holds with M replaced by M_ℓ. Indeed, the estimate (1.50) can be written as

$$\left(\frac{\int_\Omega |v(x,T)|^2dx}{\int_\Omega |v^0(x)|^2dx}\right)^{1+M} \leq e^{\frac{R^2(1+M)}{4(1+\ell)\rho}}\frac{\int_\omega |v(x,T)|^2dx}{\int_\Omega |v^0(x)|^2dx} + e^{\frac{R^2(1+M)}{4(1+\ell)\rho}-\frac{r^2}{4\rho}}. \tag{1.54}$$

Thanks to the fact that $\int_\Omega |v(x,T)|^2dx \leq \int_\Omega |v^0(x)|^2dx$, one gets

$$\left(\frac{\int_\Omega |v(x,T)|^2dx}{\int_\Omega |v^0(x)|^2dx}\right)^{1+M} \geq \left(\frac{\int_\Omega |v(x,T)|^2dx}{\int_\Omega |v^0(x)|^2dx}\right)^{1+M_\ell}. \tag{1.55}$$

Hence, the following estimate holds

$$\begin{aligned}\left(\int_\Omega |v(x,T)|^2dx\right)^{1+M_\ell} \quad &\leq \quad e^{\frac{R^2(1+M_\ell)}{4(1+\ell)\rho}}\int_\omega |v(x,T)|^2dx\left(\int_\Omega |v^0(x)|^2dx\right)^{M_\ell}\\ &\quad +e^{\frac{R^2(1+M_\ell)}{4(1+\ell)\rho}-\frac{r^2}{4\rho}}\left(\int_\Omega |v^0(x)|^2dx\right)^{1+M_\ell}. \end{aligned} \tag{1.56}$$

Now, in order to make the estimate (1.56) has the form $A \leq e^{C(\rho)}B + e^{-C(\rho)}D$ for some positive constant $C(\rho)$ depending on ρ, we choose $\ell > 1$ such that

$$\frac{R^2(1+M_\ell)}{4(1+\ell)\rho} \leq \frac{r^2}{8\rho}. \tag{1.57}$$

Remind that

$$1 < M_\ell = \frac{\ln(1+\ell)}{\ln\frac{3}{2}}. \tag{1.58}$$

Hence, we get

$$\frac{R^2(1+M_\ell)}{4(1+\ell)\rho} < \frac{2R^2M_\ell}{4(1+\ell)\rho} = \frac{R^2\ln(1+\ell)}{2(1+\ell)\rho\ln\frac{3}{2}}. \tag{1.59}$$

Moreover, using the fact that $\ln(1+\ell) \leq \frac{(1+\ell)^\varepsilon}{\varepsilon}\quad \forall 0<\varepsilon<1$, one yields

$$\frac{R^2(1+M_\ell)}{4(1+\ell)\rho} < \frac{R^2}{2\varepsilon(1+\ell)^{1-\varepsilon}\rho\ln\frac{3}{2}} < \frac{R^2}{2\varepsilon\ell^{1-\varepsilon}\rho\ln\frac{3}{2}}. \tag{1.60}$$

Combining (1.57) and (1.60), we can choose ℓ such that

$$\frac{R^2}{2\varepsilon\ell^{1-\varepsilon}\rho\ln\frac{3}{2}} = \frac{r^2}{8\rho}. \tag{1.61}$$

Precisely, we can take ℓ as below

$$\ell = \left(\frac{4R^2}{\varepsilon r^2\ln\frac{3}{2}}\right)^{\frac{1}{1-\varepsilon}}. \tag{1.62}$$

Such choice of ℓ implies from (1.56) that

$$\begin{aligned}\left(\int_\Omega |v(x,T)|^2dx\right)^{1+M_\ell} \quad&\leq\quad e^{\frac{r^2}{8\rho}}\left(\int_\omega |v(x,T)|^2dx\right)\left(\int_\Omega |v^0(x)|^2dx\right)^{M_\ell}\\&\quad+e^{-\frac{r^2}{8\rho}}\left(\int_\Omega |v^0(x)|^2dx\right)^{1+M_\ell}.\end{aligned} \tag{1.63}$$

Step 5: Choose suitable ρ.
The estimate (1.63) holds for $\rho \leq \frac{T}{2\ell}$. Now, for $\rho > \frac{T}{2\ell}$ which implies $\frac{r^2}{8\rho} < \frac{r^2\ell}{4T}$, we can get the following estimate be true for any $\rho > 0$

$$\begin{aligned}\left(\int_\Omega |v(x,T)|^2dx\right)^{1+M_\ell} \quad&\leq\quad e^{\frac{r^2}{8\rho}}\left(\int_\omega |v(x,T)|^2dx\right)\left(\int_\Omega |v^0(x)|^2dx\right)^{M_\ell}\\&\quad+e^{-\frac{r^2}{8\rho}+\frac{r^2\ell}{4T}}\left(\int_\Omega |v^0(x)|^2dx\right)^{1+M_\ell}.\end{aligned} \tag{1.64}$$

Now, in order to minimize the right-hand side of (1.64) with respect to ρ, we choose ρ such that

$$e^{-\frac{r^2}{8\rho}+\frac{r^2\ell}{4T}}\left(\int_\Omega |v^0(x)|^2dx\right)^{1+M_\ell} = \frac{1}{2}\left(\int_\Omega |v(x,T)|^2dx\right)^{1+M_\ell}, \tag{1.65}$$

that is

$$e^{\frac{r^2}{8\rho}} = 2e^{\frac{r^2\ell}{4T}}\left(\frac{\int_\Omega |v^0(x)|^2dx}{\int_\Omega |v(x,T)|^2dx}\right)^{1+M_\ell}. \tag{1.66}$$

With such choice of ρ, it deduces from (1.64) that

$$\left(\int_\Omega |v(x,T)|^2dx\right)^{2(1+M_\ell)} \leq 4e^{\frac{r^2\ell}{4T}}\left(\int_\omega |v(x,T)|^2dx\right)\left(\int_\Omega |v^0(x)|^2dx\right)^{1+2M_\ell}. \tag{1.67}$$

It is equivalent to

$$\int_\Omega |v(x,T)|^2dx \leq \left(4e^{\frac{r^2\ell}{4T}}\int_\omega |v(x,T)|^2dx\right)^{\frac{1}{2(1+M_\ell)}}\left(\int_\Omega |v^0(x)|^2dx\right)^{\frac{1+2M_\ell}{2(1+M_\ell)}}. \tag{1.68}$$

This completes the proof of Theorem 1.8 with $\mathcal{K}_1 = 4^{\frac{1}{4(1+M_\ell)}}$, $\mathcal{K}_2 = \frac{r^2\ell}{16(1+M_\ell)}$ and $\mu = \frac{1}{2(1+M_\ell)}$.

1.2.3.3 Case when Ω is C^2, open and bounded

When Ω is C^2, open and bounded, we will use the covering argument to cover Ω by finite number sets $\Omega\cap B(x_i,R_i)(i=1,2,...,N)$ where $N\in\mathbb{N}^*$ and $\Omega\cap B(x_i,(1+2\delta_i)R_i)$ is star-shaped with respect to $x_i\in\Omega$ for some $\delta_i\in(0,1]$ (see Step 8). Furthermore, in order to reach ω, we will use the propagation of smallness (see Step 7). However, the difficulty when we work on the local star-shaped $\Omega\cap B(x_i,(1+2\delta_i)R_i)$ is that the Dirichlet boundary condition: $v=0$ on $\partial(\Omega\cap B(x_i,(1+2\delta_i)R_i)$ does not hold any more. Hence, we need to use a cut off function χ which is null on $\partial B(x_i,(1+2\delta_i)R_i)$. The appearance of this cut off function makes appear another term $\partial_t(\chi v)-\Delta(\chi v)$. In order to estimate this term, we will use the technical Lemma 1.3 (see Step 3).

For the rest of the proof, we use the same technique in proof when Ω is convex (see Subsection 1.2.3.2).

Step 1: Construct ODEs.
Let $x_0 \in \Omega$, $R > 0$, $\delta \in (0, 1]$ such that $\Omega \cap B(x_0, (1+2\delta)R)$ is star-shaped with respect to x_0. Let us define the cut off function: Define $\chi \in C_0^2(B(x_0, (1+2\delta)R))$ satisfying $\chi = 1$ in $B(x_0, \left(1+\frac{3}{2}\delta\right) R)$ and $0 < \chi(x) \leq 1 \quad \forall x \in B(x_0, (1+2\delta)R)$. Then $\chi v \in H^1((0, T); H_0^1(\Omega \cap B(x_0, (1+2\delta)R)))$. Let $\rho > 0$, we define two following functions:

$$\Psi(t) := \int_{\Omega \cap B(x_0,(1+2\delta)R)} |\chi(x)v(x,t)|^2 e^{\xi(x,t)} dx. \tag{1.69}$$

and

$$N(t) := \frac{2\int_{\Omega \cap B(x_0,(1+2\delta)R)} |\nabla(\chi(x)v(x,t))|^2 e^{\xi(x,t)} dx}{\int_{\Omega \cap B(x_0,(1+2\delta)R)} |\chi(x)v(x,t)|^2 e^{\xi(x,t)} dx} + \frac{n}{2(T-t+\rho)}. \tag{1.70}$$

Apply Lemma 1.1 and Lemma 1.2 with $\vartheta := \Omega \cap B(x_0, (1+2\delta)R)$ and $w := \chi v$, we get

$$\Psi'(t) + N(t)\Psi(t) = 2\int_{\Omega \cap B(x_0,(1+2\delta)R)} \chi(x)v(x,t)(\partial_t(\chi(x)v(x,t)) - \Delta(\chi(x)v(x,t)))e^{\xi(x,t)} dx \tag{1.71}$$

and

$$N'(t) \leq \frac{N(t)}{T-t+\rho} + \frac{\int_{\Omega \cap B(x_0,(1+2\delta)R)} |\partial_t(\chi(x)v(x,t)) - \Delta(\chi(x)v(x,t))|^2 e^{\xi(x,t)} dx}{\int_{\Omega \cap B(x_0,(1+2\delta)R)} |\chi(x)v(x,t)|^2 e^{\xi(x,t)} dx}. \tag{1.72}$$

Now, using the Cauchy-Schwarz inequality and the inequality that $2ab \leq a^2 + b^2 \quad \forall a, b > 0$ for the right-hand side of (1.71), we get

$$\begin{aligned} |\Psi'(t) + N(t)\Psi(t)| \quad \leq \quad & \int_{\Omega \cap B(x_0,(1+2\delta)R)} |\chi(x)v(x,t)|^2 e^{\xi(x,t)} dx \\ & + \int_{\Omega \cap B(x_0,(1+2\delta)R)} |\partial_t(\chi(x)v(x,t)) - \Delta(\chi(x)v(x,t))|^2 e^{\xi(x,t)} dx. \end{aligned} \tag{1.73}$$

Put

$$G(t) := \frac{\int_{\Omega \cap B(x_0,(1+2\delta)R)} |\partial_t(\chi(x)v(x,t)) - \Delta(\chi(x)v(x,t))|^2 e^{\xi(x,t)} dx}{\int_{\Omega \cap B(x_0,(1+2\delta)R)} |\chi(x)v(x,t)|^2 e^{\xi(x,t)} dx}. \tag{1.74}$$

Then, one gets from (1.73) and (1.74) that

$$|\Psi'(t) + N(t)\Psi(t)| \leq (1 + G(t))\Psi(t). \tag{1.75}$$

Moreover, we also get from (1.72) and (1.74) that

$$N'(t) \leq \frac{N(t)}{T-t+\rho} + G(t). \tag{1.76}$$

Step 2: Solve ODEs.
Let $0 \leq t_1 < t_2 < t_3 \leq T$, we will solve ODEs (1.75) and (1.76) from Step 1 on (t_1, t_2) and (t_2, t_3) respectively. From (1.76), we get

$$(N(t)(T-t+\rho))' \leq G(t)(T-t+\rho). \tag{1.77}$$

<u>For $t_1 < t < t_2$</u>

Integrating (1.77) over (t,t_2) gives us

$$\begin{aligned} N(t_2)(T-t_2+\rho)-N(t)(T-t+\rho) &\leq \int_t^{t_2} G(s)(T-s+\rho)ds \\ &\leq (T-t+\rho)\int_t^{t_2} G(s)ds. \end{aligned} \tag{1.78}$$

It implies from (1.78) that

$$N(t) \geq N(t_2)\frac{T-t_2+\rho}{T-t+\rho} - \int_t^{t_2} G(s)ds. \tag{1.79}$$

On the other hand, it implies from (1.75) that

$$\Psi'(t)+N(t)\Psi(t) \leq (1+G(t))\,\Psi(t). \tag{1.80}$$

Combining (1.79) and (1.80), we obtain

$$\Psi'(t)+\left(N(t_2)\frac{T-t_2+\rho}{T-t+\rho}-\int_{t_1}^{t_2} G(s)ds-1-G(t)\right)\Psi(t)\leq 0. \tag{1.81}$$

It follows from (1.81) that

$$\left(\Psi(t)e^{N(t_2)(T-t_2+\rho)\int_0^t \frac{ds}{T-s+\rho}-\left(\int_{t_1}^{t_2} G(s)ds\right)t-t-\int_0^t G(s)ds}\right)'\leq 0. \tag{1.82}$$

Integrating (1.82) over (t_1,t_2), one has

$$\Psi(t_1)\geq \Psi(t_2)e^{N(t_2)(T-t_2+\rho)\ln\frac{T-t_1+\rho}{T-t_2+\rho}-\left(\int_{t_1}^{t_2} G(s)ds\right)(t_2-t_1)-(t_2-t_1)-\int_{t_1}^{t_2} G(s)ds}. \tag{1.83}$$

It deduces from (1.83) that

$$e^{N(t_2)(T-t_2+\rho)}\leq \left(\frac{\Psi(t_1)}{\Psi(t_2)}e^{\left(\int_{t_1}^{t_2} G(s)ds\right)(t_2-t_1)+(t_2-t_1)+\int_{t_1}^{t_2} G(s)ds}\right)^{\frac{1}{\ln\frac{T-t_1+\rho}{T-t_2+\rho}}}. \tag{1.84}$$

<u>For $t_2<t<t_3$</u>

Integrating (1.77) over (t_2,t) gives us

$$\begin{aligned} N(t) &\leq N(t_2)\frac{T-t_2+\rho}{T-t+\rho}+\frac{1}{T-t+\rho}\int_{t_2}^{t} G(s)(T-s+\rho)ds \\ &\leq N(t_2)\frac{T-t_2+\rho}{T-t+\rho}+\frac{T-t_2+\rho}{T-t_3+\rho}\int_{t_2}^{t_3} G(s)ds. \end{aligned} \tag{1.85}$$

On the other hand, it follows from (1.75) that

$$\Psi'(t)+N(t)\Psi(t)\geq -(1+G(t))\,\Psi(t). \tag{1.86}$$

Combining (1.85) and (1.86), yields

$$\Psi'(t)+\left(N(t_2)\frac{T-t_2+\rho}{T-t+\rho}+\frac{T-t_2+\rho}{T-t_3+\rho}\int_{t_2}^{t_3} G(s)ds+1+G(t)\right)\Psi(t)\geq 0. \tag{1.87}$$

It follows from (1.87) that

$$\left(\Psi(t)e^{N(t_2)(T-t_2+\rho)\int_0^t \frac{ds}{T-s+\rho}+\left(\frac{T-t_2+\rho}{T-t_3+\rho}\int_{t_2}^{t_3} G(s)ds+1\right)t+\int_0^t G(s)ds}\right)'\geq 0. \tag{1.88}$$

Integrating (1.88) over (t_2,t_3) gives us

$$\Psi(t_2)\leq \Psi(t_3)e^{N(t_2)(T-t_2+\rho)\ln\frac{T-t_2+\rho}{T-t_3+\rho}+\left(\frac{T-t_2+\rho}{T-t_3+\rho}\int_{t_2}^{t_3} G(s)ds+1\right)(t_3-t_2)+\int_{t_2}^{t_3} G(s)ds}. \tag{1.89}$$

Now, combining (1.84) and (1.89), one gets

$$\Psi(t_2) \leq \Psi(t_3)\left(\frac{\Psi(t_1)}{\Psi(t_2)}e^{\left(\int_{t_1}^{t_2}G(s)ds+1\right)(t_2-t_1)+\int_{t_1}^{t_2}G(s)ds}\right)^M \times e^{\left(\frac{T-t_2+\rho}{T-t_3+\rho}\int_{t_2}^{t_3}G(s)ds+1\right)(t_3-t_2)+\int_{t_2}^{t_3}G(s)ds} \tag{1.90}$$

where

$$M := \frac{\ln\frac{T-t_2+\rho}{T-t_3+\rho}}{\ln\frac{T-t_1+\rho}{T-t_2+\rho}}. \tag{1.91}$$

It implies from (1.90) that

$$\Psi(t_2)^{1+M} \leq \Psi(t_3)\Psi(t_1)^M e^{\left[\left(1+\frac{T-t_2+\rho}{T-t_3+\rho}\int_{t_1}^{t_3}G(s)ds\right)(t_3-t_1)+\int_{t_1}^{t_3}G(s)ds\right](1+M)}. \tag{1.92}$$

Step 3: Estimate $\int_{t_1}^{t_3}G(s)ds$ for some $0 \leq t_1 < t_3 \leq T$.
Recall that

$$G(t) := \frac{\int_{\Omega\cap B(x_0,(1+2\delta)R)}|\partial_t(\chi(x)v(x,t)) - \Delta(\chi(x)v(x,t))|^2 e^{\xi(x,t)}dx}{\int_{\Omega\cap B(x_0,(1+2\delta)R)}|\chi(x)v(x,t)|^2 e^{\xi(x,t)}dx}. \tag{1.93}$$

Firstly, we estimate $\int_{\Omega\cap B(x_0,(1+2\delta)R)}|\partial_t(\chi(x)v(x,t)) - \Delta(\chi(x)v(x,t))|^2 e^{\xi(x,t)}dx$.
We have

$$\begin{aligned}\partial_t(\chi v) - \Delta(\chi v) &= \chi\partial_t v - \nabla(\nabla v\chi + v\nabla\chi) \\ &= \chi\partial_t v - 2\nabla\chi\nabla v - \Delta\chi v - \chi\Delta v.\end{aligned} \tag{1.94}$$

Thanks to the fact that $\partial_t v - \Delta v = 0$, one gets

$$\partial_t(\chi v) - \Delta(\chi v) = -\Delta\chi v - 2\nabla\chi\nabla v. \tag{1.95}$$

Moreover, the fact that $\chi = 1$ on $B(x_0, \left(1+\frac{3}{2}\delta\right)R)$ implies that $\Delta\chi = 0$ and $\nabla\chi = 0_{\mathbb{R}^n}$ on $B(x_0, \left(1+\frac{3}{2}\delta\right)R)$. Here, $0_{\mathbb{R}^n}$ denotes the null vector in $\mathbb{R}^n$. Thus, we obtain the following equality

$$\begin{aligned}&\int_{\Omega\cap B(x_0,(1+2\delta)R)}|\partial_t(\chi(x)v(x,t)) - \Delta(\chi(x)v(x,t))|^2 e^{\xi(x,t)}dx \\ =& \int_{\Omega\cap B(x_0,(1+2\delta)R)}|\Delta\chi(x)v(x,t) + 2\nabla\chi(x)\nabla v(x,t)|^2 e^{\xi(x,t)}dx \\ =& \int_{\Omega\cap B(x_0,(1+2\delta)R)\setminus B(x_0,\left(1+\frac{3}{2}\delta\right)R)}|\Delta\chi(x)v(x,t) + 2\nabla\chi(x)\nabla v(x,t)|^2 e^{\xi(x,t)}dx.\end{aligned} \tag{1.96}$$

Now, taking off the weight function $\xi = \frac{-|x-x_0|^2}{4(T-t+\rho)}$ from the right-hand side of (1.96) and using the fact that $\Omega\cap B(x_0,(1+2\delta)R)\setminus B(x_0,\left(1+\frac{3}{2}\delta\right)R) \subset \Omega$ gives us

$$\begin{aligned}&\int_{\Omega\cap B(x_0,(1+2\delta)R)\setminus B(x_0,\left(1+\frac{3}{2}\delta\right)R)}|\Delta\chi(x)v(x,t) + 2\nabla\chi(x)\nabla v(x,t)|^2 e^{\xi(x,t)}dx \\ \leq& \ Ce^{-\frac{\left(1+\frac{3}{2}\delta\right)^2R^2}{4(T-t+\rho)}}\left(\int_\Omega|v(x,t)|^2dx + \int_\Omega|\nabla v(x,t)|^2dx\right)\end{aligned} \tag{1.97}$$

where $C := 4\max\{\|\Delta\chi\|_\infty^2, 4\|\nabla\chi\|_\infty^2\}$. By using the energy estimate

$$\int_\Omega|v(x,t)|^2dx \leq \int_\Omega|v^0(x)|^2dx$$

and the regularity estimate

$$\int_{\Omega} |\nabla v(x,t)|^2 dx \leq \frac{1}{2t} \int_{\Omega} |v^0(x)|^2 dx,$$

we obtain

$$\int_{\Omega \cap B(x_0,(1+2\delta)R)} |\partial_t(\chi(x)v(x,t)) - \Delta(\chi(x)v(x,t))|^2 e^{\xi(x,t)} dx \leq Ce^{-\frac{\left(1+\frac{3}{2}\delta\right)^2 R^2}{4(T-t+\rho)}} \left(1+\frac{1}{2t}\right) \int_{\Omega} |v^0(x)|^2 dx. \tag{1.98}$$

Secondly, we estimate $\int_{\Omega \cap B(x_0,(1+2\delta)R)} |\chi(x)v(x,t)|^2 e^{\xi(x,t)} dx$.
The fact that $\chi = 1$ on $B(x_0, \left(1+\frac{3}{2}\delta\right) R)$ gives us

$$\begin{aligned} \int_{\Omega \cap B(x_0,(1+2\delta)R)} |\chi(x)v(x,t)|^2 e^{\xi(x,t)} dx &\geq \int_{\Omega \cap B(x_0,\left(1+\frac{3}{2}\delta\right)R)} |\chi(x)v(x,t)|^2 e^{\xi(x,t)} dx \\ &= \int_{\Omega \cap B(x_0,\left(1+\frac{3}{2}\delta\right)R)} |v(x,t)|^2 e^{\xi(x,t)} dx \\ &\geq e^{-\frac{(1+\delta)^2 R^2}{4(T-t+\rho)}} \int_{\Omega \cap B(x_0,(1+\delta)R)} |v(x,t)|^2 dx. \end{aligned} \tag{1.99}$$

Lastly, combining (1.98) and (1.99), one yields

$$G(t) \leq Ce^{-\frac{\left(2+\frac{5}{2}\delta\right)\frac{\delta}{2}R^2}{4(T-t+\rho)}} \frac{\int_{\Omega} |v^0(x)|^2 dx}{\int_{\Omega \cap B(x_0,(1+\delta)R)} |v(x,t)|^2 dx} \left(1+\frac{1}{2t}\right). \tag{1.100}$$

Now, apply Lemma 1.3, one gets: there exists $\hbar \leq \frac{T}{2}$ such that

$$\frac{\int_{\Omega} |v^0(x)|^2 dx}{\int_{\Omega \cap B(x_0,(1+\delta)R)} |v(x,t)|^2 dx} \leq e^{\frac{(1+\delta)\delta R^2}{2\hbar}} \quad \forall T-\hbar \leq t \leq T. \tag{1.101}$$

Thus, it follows from (1.100) and (1.101) that

$$G(t) \leq Ce^{-\frac{\left(2+\frac{5}{2}\delta\right)\frac{\delta}{2}R^2}{4(T-t+\rho)} + \frac{(1+\delta)\delta R^2}{2\hbar}} \left(1+\frac{1}{2t}\right). \tag{1.102}$$

Suppose $t_1 < t < t_3$ satisfy

$$-\frac{\left(2+\frac{5}{2}\delta\right)\frac{\delta}{2}R^2}{4(T-t+\rho)} + \frac{(1+\delta)\delta R^2}{2\hbar} \leq 0. \tag{1.103}$$

It is equivalent to

$$T-t+\rho \leq \frac{2+\frac{5}{2}\delta}{1+\delta}\hbar. \tag{1.104}$$

Thus, with $\mathcal{E} := \frac{2+\frac{5}{2}\delta}{1+\delta} = \frac{5}{2} - \frac{\frac{1}{2}}{1+\delta} < \frac{5}{2}$, if $t \geq T+\rho-\mathcal{E}\hbar$ then we get

$$G(t) \leq C\left(1+\frac{1}{2t}\right). \tag{1.105}$$

Hence, for any $t_1, t_3 > 0$ satisfying $\max\{T+\rho-\mathcal{E}\hbar, T-\hbar\} \leq t_1 < t_3$, we conclude that

$$\int_{t_1}^{t_3} G(s) ds \leq C\left[t_3 - t_1 + \frac{1}{2}\ln\frac{t_3}{t_1}\right]. \tag{1.106}$$

Step 4: Choose suitable t_1, t_2, t_3, take off the weight function and make appear ω. Now, for any $\ell > 1$ such that $\ell\rho \leq \min\{\frac{1}{2}, \frac{T}{4}, \frac{\mathcal{E}\hbar}{5}\}$, we choose

$$t_1 = T - 2\ell\rho \ \ ; t_2 = T - \ell\rho \ \ ; t_3 = T. \tag{1.107}$$

In order to estimate $\int_{t_1}^{t_3} G(s)ds$ with t_1, t_3 chosen as above, we check $t_1 \geq \max\{T + \rho - \mathcal{E}\hbar, T - \hbar\}$. Indeed, we have

$$t_1 = T - 2\ell\rho \geq T - \frac{2}{5}\mathcal{E}\hbar \geq T - \hbar. \tag{1.108}$$

On the other hand, the fact that $\ell > 1$ gives us

$$t_1 - T - \rho = -(1 + 2\ell)\rho \geq -3\ell\rho \geq -\frac{3}{5}\mathcal{E}\hbar \geq -\mathcal{E}\hbar. \tag{1.109}$$

It implies that $t_1 \geq T + \rho - \mathcal{E}\hbar$. Thus, we get from (1.106) that

$$\int_{t_1}^{t_3} G(s)ds \leq C\left[2\ell\rho + \frac{1}{2}\ln\frac{T}{T - 2\ell\rho}\right] \leq C\left(1 + \frac{1}{2}\ln 2\right) := Const. \tag{1.110}$$

Let us recall the Hölder estimate (1.92) from Step 3

$$\Psi(t_2)^{1+M} \quad \leq \quad \Psi(t_3)\Psi(t_1)^M e^{\left[\left(1+\frac{T-t_2+\rho}{T-t_3+\rho}\int_{t_1}^{t_3} G(s)ds\right)(t_3-t_1)+\int_{t_1}^{t_3} G(s)ds\right](1+M)}. \tag{1.111}$$

Thanks to (1.110), the term in (1.111) is estimated as below

$$\begin{aligned} & e^{\left[\left(1+\frac{T-t_2+\rho}{T-t_3+\rho}\int_{t_1}^{t_3} G(s)ds\right)(t_3-t_1)+\int_{t_1}^{t_3} G(s)ds\right](1+M)} \\ \leq \ & e^{[(1+(1+\ell)Const)2\ell\rho+Const](1+M)} \\ \leq \ & e^{C\ell(1+M)} \end{aligned} \tag{1.112}$$

for some constant $C > 0$. Remind that

$$\Psi(t) = \int_{\Omega\cap B(x_0,(1+2\delta)R)} |\chi(x)v(x,t)|^2 e^{-\frac{|x-x_0|^2}{4(T-t+\rho)}} dx. \tag{1.113}$$

As a consequence, it deduces from (1.111) that

$$\begin{aligned} & \left(\int_{\Omega\cap B(x_0,(1+2\delta)R)} |\chi(x)v(x,T-\ell\rho)|^2 e^{-\frac{|x-x_0|^2}{4(1+\ell)\rho}} dx\right)^{1+M} \\ \leq \ & e^{C\ell(1+M)} \left(\int_{\Omega\cap B(x_0,(1+2\delta)R)} |\chi(x)v(x,T)|^2 e^{\frac{-|x-x_0|^2}{4\rho}} dx\right) \\ & \times \left(\int_{\Omega\cap B(x_0,(1+2\delta)R)} |\chi(x)v(x,T-2\ell\rho)|^2 e^{-\frac{|x-x_0|^2}{4(1+2\ell)\rho}} dx\right)^M . \end{aligned} \tag{1.114}$$

On one hand, we consider the term on the left-hand side of (1.114): We have

$$\begin{aligned} & \int_{\Omega\cap B(x_0,(1+2\delta)R)} |\chi(x)v(x,T-\ell\rho)|^2 e^{-\frac{|x-x_0|^2}{4(1+\ell)\rho}} dx \\ \geq \ & e^{-\frac{(1+2\delta)^2R^2}{4(1+\ell)\rho}} \int_{\Omega\cap B(x_0,(1+2\delta)R)} |\chi(x)v(x,T-\ell\rho)|^2 dx. \end{aligned} \tag{1.115}$$

Thanks to the fact that $\chi = 1$ on $B(x_0, \left(1 + \frac{3}{2}\delta\right) R)$, one gets

$$\begin{aligned} \int_{\Omega\cap B(x_0,(1+2\delta)R)} |\chi(x)v(x,T-\ell\rho)|^2 dx \ & \geq \int_{\Omega\cap B(x_0,\left(1+\frac{3}{2}\delta\right)R)} |\chi(x)v(x,T-\ell\rho)|^2 dx \\ & = \int_{\Omega\cap B(x_0,\left(1+\frac{3}{2}\delta\right)R)} |v(x,T-\ell\rho)|^2 dx \\ & \geq \int_{\Omega\cap B(x_0,(1+\delta)R)} |v(x,T-\ell\rho)|^2 dx. \end{aligned} \tag{1.116}$$

Thus, it follows from (1.115) and (1.116) that

$$\begin{aligned}&\left(\int_{\Omega\cap B(x_0,(1+2\delta)R)}|\chi(x)v(x,T-\ell\rho)|^2e^{-\frac{|x-x_0|^2}{4(1+\ell)\rho}}dx\right)^{1+M}\\ \geq\ & e^{-\frac{(1+2\delta)^2R^2(1+M)}{4(1+\ell)\rho}}\left(\int_{\Omega\cap B(x_0,(1+\delta)R)}|v(x,T-\ell\rho)|^2dx\right)^{1+M}.\end{aligned}\tag{1.117}$$

On the other hand, we consider the terms on the right-hand side of (1.114):

- For the first term, thanks to the fact that $0<\chi(x)\leq 1\quad\forall x\in B(x_0,(1+2\delta)R)$, one has

$$\int_{\Omega\cap B(x_0,(1+2\delta)R)}|\chi(x)v(x,T)|^2e^{\frac{-|x-x_0|^2}{4\rho}}dx\leq\int_{\Omega\cap B(x_0,(1+2\delta)R)}|v(x,T)|^2e^{\frac{-|x-x_0|^2}{4\rho}}dx.\tag{1.118}$$

  Now, take $0<r<R$ such that $B(x_0,r)\subset\Omega$, we have

$$\begin{aligned}&\int_{\Omega\cap B(x_0,(1+2\delta)R)}|v(x,T)|^2e^{\frac{-|x-x_0|^2}{4\rho}}dx\\ \leq\ & \int_{B(x_0,r)}|v(x,T)|^2e^{\frac{-|x-x_0|^2}{4\rho}}dx+\int_{(\Omega\cap B(x_0,(1+2\delta)R))\setminus B(x_0,r)}|v(x,T)|^2e^{\frac{-|x-x_0|^2}{4\rho}}dx\\ \leq\ & \int_{B(x_0,r)}|v(x,T)|^2dx+e^{-\frac{r^2}{4\rho}}\int_\Omega|v(x,T)|^2dx\\ \leq\ & \int_{B(x_0,r)}|v(x,T)|^2dx+e^{-\frac{r^2}{4\rho}}\int_\Omega|v^0(x)|^2dx.\end{aligned}\tag{1.119}$$

- For the second term, thanks to the fact that $0<\chi(x)\leq 1\quad\forall x\in B(x_0,(1+2\delta)R)$ and $e^{-\frac{|x-x_0|^2}{4(1+2\ell)\rho}}\leq 1$, we get

$$\begin{aligned}&\left(\int_{\Omega\cap B(x_0,(1+2\delta)R)}|\chi(x)v(x,T-2\ell\rho)|^2e^{-\frac{|x-x_0|^2}{4(1+\ell)\rho}}dx\right)^{M}\\ \leq\ & \left(\int_{\Omega\cap B(x_0,(1+2\delta)R)}|v(x,T-2\ell\rho)|^2dx\right)^{M}\\ \leq\ & \left(\int_\Omega|v(x,T-2\ell\rho)|^2dx\right)^{M}\\ \leq\ & \left(\int_\Omega|v^0(x)|^2dx\right)^{M}.\end{aligned}\tag{1.120}$$

Thus, combining (1.114), (1.117), (1.119) and (1.120) gives us

$$\begin{aligned}&\left(\int_{\Omega\cap B(x_0,(1+2\delta)R)}|v(x,T-\ell\rho)|^2dx\right)^{1+M}\\ \leq\ & e^{C\ell(1+M)}e^{\frac{(1+2\delta)^2R^2(1+M)}{4(1+\ell)\rho}}\left(\int_{B(x_0,r)}|v(x,T)|^2dx\right)\left(\int_\Omega|v^0(x)|^2dx\right)^{M}\\ &+e^{C\ell(1+M)}e^{\frac{(1+2\delta)^2R^2(1+M)}{4(1+\ell)\rho}-\frac{r^2}{4\rho}}\left(\int_\Omega|v^0(x)|^2dx\right)^{1+M}.\end{aligned}\tag{1.121}$$

Step 5: Choose suitable ℓ.

Recall that for $\ell>1$, we have $1<M<M_\ell:=\frac{\ln(1+\ell)}{\ln\frac{3}{2}}$ (see Step 4 of Subsection 1.2.3.2 for case

Ω convex). We claim that the estimate (1.121) still holds when M is replaced by M_ℓ. Indeed, the estimate (1.121) can be written as

$$\begin{aligned} &\left(\frac{\int_{\Omega\cap B(x_0,(1+2\delta)R)}|v(x,T-\ell\rho)|^2dx}{\int_\Omega |v^0(x)|^2dx}\right)^{1+M} \\ \leq\quad & e^{C\ell(1+M)}e^{\frac{(1+2\delta)^2R^2(1+M)}{4(1+\ell)\rho}}\frac{\int_{B(x_0,r)}|v(x,T)|^2dx}{\int_\Omega |v^0(x)|^2dx}+e^{C\ell(1+M)}e^{\frac{(1+2\delta)^2R^2(1+M)}{4(1+\ell)\rho}-\frac{r^2}{4\rho}}. \end{aligned} \tag{1.122}$$

We have

$$\int_{\Omega\cap B(x_0,(1+2\delta)R)}|v(x,T-\ell\rho)|^2dx \leq \int_\Omega |v(x,T-\ell\rho)|^2dx \leq \int_\Omega |v^0(x)|^2dx. \tag{1.123}$$

Hence, with $M_\ell > M$, one gets

$$\left(\frac{\int_{\Omega\cap B(x_0,(1+2\delta)R)}|v(x,T-\ell\rho)|^2dx}{\int_\Omega |v^0(x)|^2dx}\right)^{1+M_\ell} \leq \left(\frac{\int_{\Omega\cap B(x_0,(1+2\delta)R)}|v(x,T-\ell\rho)|^2dx}{\int_\Omega |v^0(x)|^2dx}\right)^{1+M}. \tag{1.124}$$

It means the following estimate holds

$$\begin{aligned} &\left(\int_{\Omega\cap B(x_0,(1+2\delta)R)}|v(x,T-\ell\rho)|^2dx\right)^{1+M_\ell} \\ \leq\quad & e^{C\ell(1+M_\ell)}e^{\frac{(1+2\delta)^2R^2(1+M_\ell)}{4(1+\ell)\rho}}\left(\int_{B(x_0,r)}|v(x,T)|^2dx\right)\left(\int_\Omega |v^0(x)|^2dx\right)^{M_\ell} \\ & +e^{C\ell(1+M_\ell)}e^{\frac{(1+2\delta)^2R^2(1+M_\ell)}{4(1+\ell)\rho}-\frac{r^2}{4\rho}}\left(\int_\Omega |v^0(x)|^2dx\right)^{1+M_\ell}. \end{aligned} \tag{1.125}$$

Our target is choosing $\ell > 1$ such that

$$\frac{(1+2\delta)^2R^2(1+M_\ell)}{4(1+\ell)\rho} \leq \frac{r^2}{8\rho}. \tag{1.126}$$

With the same argument in Step 4 of Subsection 1.2.3.2 for case Ω convex, we can choose ℓ as below

$$\ell = \left(\frac{4(1+2\delta)^2R^2}{\varepsilon r^2\ln\frac{3}{2}}\right)^{\frac{1}{1-\varepsilon}} \quad \forall\varepsilon\in(0,1). \tag{1.127}$$

Such choice of ℓ implies from (1.125) that

$$\begin{aligned} &\left(\int_{\Omega\cap B(x_0,(1+2\delta)R)}|v(x,T-\ell\rho)|^2dx\right)^{1+M_\ell} \\ \leq\quad & e^{C\ell(1+M_\ell)}e^{\frac{r^2}{8\rho}}\left(\int_{B(x_0,r)}|v(x,T)|^2dx\right)\left(\int_\Omega |v^0(x)|^2dx\right)^{M_\ell} \\ & +e^{C\ell(1+M_\ell)}e^{-\frac{r^2}{8\rho}}\left(\int_\Omega |v^0(x)|^2dx\right)^{1+M_\ell}. \end{aligned} \tag{1.128}$$

On the other hand, in order to estimate the term on the left-hand side of (1.128), we apply again Lemma 1.3:

$$\begin{aligned} \int_{\Omega\cap B(x_0,(1+2\delta)R)}|v(x,T-\ell\rho)|^2dx &\geq \int_{\Omega\cap B(x_0,(1+\delta)R)}|v(x,T-\ell\rho)|^2dx \\ &\geq e^{-\frac{(1+\delta)\delta R^2}{2h}}\int_\Omega |v^0(x)|^2dx. \end{aligned} \tag{1.129}$$

Applying the energy estimate, which says: $\int_\Omega |v^0(x)|^2 dx \geq \int_\Omega |v(x,T)|^2 dx$, yields

$$\int_{\Omega\cap B(x_0,(1+2\delta)R)} |v(x,T-\ell\rho)|^2 dx \geq e^{-\frac{(1+\delta)\delta R^2}{2\hbar}} \int_\Omega |v(x,T)|^2 dx. \tag{1.130}$$

Combining (1.128) and (1.130), one obtains

$$\begin{aligned}\left(\int_\Omega |v(x,T)|^2 dx\right)^{1+M_\ell} &\leq Ce^{\frac{C}{\hbar}} e^{\frac{r^2}{8\rho}} \left(\int_{B(x_0,r)} |v(x,T)|^2 dx\right)\left(\int_\Omega |v^0(x)|^2 dx\right)^{M_\ell} \\ &\quad + Ce^{\frac{C}{\hbar}} e^{-\frac{r^2}{8\rho}} \left(\int_\Omega |v^0(x)|^2 dx\right)^{1+M_\ell}. \end{aligned} \tag{1.131}$$

Here, the constant C depends on R, ℓ and δ, not depends on $\hbar$ and ρ.

Step 6: Choose suitable ρ.

The estimate (1.131) holds for $\rho \leq \frac{1}{\ell}\min\{\frac{1}{2}, \frac{T}{4}, \frac{\mathcal{E}\hbar}{5}\}$. Now, for $\rho > \frac{1}{\ell}\min\{\frac{1}{2}, \frac{T}{4}, \frac{\mathcal{E}\hbar}{5}\}$ which implies $\frac{r^2}{8\rho} < \frac{r^2\ell}{4\left(\frac{1}{2}+\frac{T}{4}+\frac{\mathcal{E}\hbar}{5}\right)}$, we can get the following estimate be true for any $\rho > 0$.

$$\begin{aligned}\left(\int_\Omega |v(x,T)|^2 dx\right)^{1+M_\ell} &\leq Ce^{\frac{C}{\hbar}} e^{\frac{r^2}{8\rho}} \left(\int_{B(x_0,r)} |v(x,T)|^2 dx\right)\left(\int_\Omega |v^0(x)|^2 dx\right)^{M_\ell} \\ &\quad + Ce^{\frac{C}{\hbar}} e^{-\frac{r^2}{8\rho}+\frac{r^2\ell}{4\left(\frac{1}{2}+\frac{T}{4}+\frac{\mathcal{E}\hbar}{5}\right)}} \left(\int_\Omega |v^0(x)|^2 dx\right)^{1+M_\ell}. \end{aligned} \tag{1.132}$$

Now, we choose ρ such that

$$Ce^{\frac{C}{\hbar}} e^{-\frac{r^2}{8\rho}+\frac{r^2\ell}{4\left(\frac{1}{2}+\frac{T}{4}+\frac{\mathcal{E}\hbar}{5}\right)}} \left(\int_\Omega |v^0(x)|^2 dx\right)^{1+M_\ell} = \frac{1}{2}\left(\int_\Omega |v(x,T)|^2 dx\right)^{1+M_\ell},$$

that is

$$e^{\frac{r^2}{8\rho}} = 2Ce^{\frac{C}{\hbar}} e^{\frac{r^2\ell}{4\left(\frac{1}{2}+\frac{T}{4}+\frac{\mathcal{E}\hbar}{5}\right)}} \left(\frac{\int_\Omega |v^0(x)|^2 dx}{\int_\Omega |v(x,T)|^2 dx}\right)^{1+M_\ell}.$$

With such choice of ρ, it deduces from (1.132) that

$$\left(\int_\Omega |v(x,T)|^2 dx\right)^{2(1+M_\ell)} \leq Ce^{C\left(\frac{1}{T}+\frac{1}{\hbar}\right)} \left(\int_{B(x_0,r)} |v(x,T)|^2 dx\right)\left(\int_\Omega |v^0(x)|^2 dx\right)^{1+2M_\ell}, \tag{1.133}$$

for some constant $C = C(r,\ell,R,\delta) > 0$. Let us recall Lemma 1.3 that

$$\frac{1}{\hbar} = \frac{2}{(\delta R)^2}\ln\left(e^{\frac{R^2}{2}\left(1+\frac{2}{T}\right)} \frac{2\int_\Omega |v^0(x)|^2 dx}{\int_{\Omega\cap B(x_0,R)} |v(x,T)|^2 dx}\right). \tag{1.134}$$

Hence, one has

$$e^{\frac{1}{\hbar}} = \left(e^{\frac{R^2}{2}\left(1+\frac{2}{T}\right)} \frac{2\int_\Omega |v^0(x)|^2 dx}{\int_{\Omega\cap B(x_0,R)} |v(x,T)|^2 dx}\right)^{\frac{2}{(\delta R)^2}}. \tag{1.135}$$

Thus, it implies from (1.133) and (1.135) that

$$\begin{aligned}\left(\int_\Omega |v(x,T)|^2 dx\right)^{2(1+M_\ell)} &\leq Ce^{\frac{C}{T}} \left(\frac{\int_\Omega |v^0(x)|^2 dx}{\int_{\Omega\cap B(x_0,R)} |v(x,T)|^2 dx}\right)^{\frac{2C}{(\delta R)^2}} \\ &\quad \times \left(\int_{B(x_0,r)} |v(x,T)|^2 dx\right)\left(\int_\Omega |v^0(x)|^2 dx\right)^{1+2M_\ell}, \end{aligned} \tag{1.136}$$

for another positive constant C not depending on T. On the other hand, using the fact that

$$\int_\Omega |v(x,T)|^2 dx \geq \int_{\Omega\cap B(x_0,R)} |v(x,T)|^2 dx, \tag{1.137}$$

one has

$$\begin{aligned} &\left(\int_{\Omega\cap B(x_0,R)} |v(x,T)|^2 dx\right)^{2(1+M_\ell)+\frac{2C}{(\delta R)^2}} \\ \leq\; &\left(Ce^{\frac{C}{T}} \int_{B(x_0,r)} |v(x,T)|^2 dx\right) \left(\int_\Omega |v^0(x)|^2 dx\right)^{1+2M_\ell+\frac{2C}{(\delta R)^2}}. \end{aligned} \tag{1.138}$$

Therefore, we obtain the following local observation estimate: For any $x_0 \in \Omega$, any $R > 0$, any $0 < \delta \leq 1$ satisfying $\Omega \cap B(x_0,(1+2\delta)R)$ is star-shaped with respect to x_0, any $0 < r < R$ satisfying $B(x_0,r) \Subset \Omega$, there exist $C > 0$ and $\sigma \in (0,1)$ only depending on R, r and δ such that

$$\int_{\Omega\cap B(x_0,R)} |v(x,T)|^2 dx \leq \left(Ce^{\frac{C}{T}} \int_{B(x_0,r)} |v(x,T)|^2 dx\right)^\sigma \left(\int_\Omega |v^0(x)|^2 dx\right)^{1-\sigma}. \tag{1.139}$$

Step 7 Make appear ω by propagation of smallness.
Suppose $x_0 \in \Omega$, $R > 0$ and $0 < \delta \leq 1$ satisfy $\Omega \cap B(x_0,(1+2\delta)R)$ is star-shaped with respect to x_0. Let $0 < \kappa < \frac{R}{2}$ and $x_j \in \Omega (j = 1,2,...,m)(m \in \mathbb{N}; m \geq 1)$, we can construct a sequence of balls $\{B(x_j,\kappa)\}_{j\in\overline{1,m}}$ such that the following inclusions hold

1. $B(x_m,\kappa) \subset \omega$;
2. $B(x_{j-1},\kappa) \subset B(x_j,2\kappa)\ \forall j = 1,2,..,m$;
3. $B(x_j,2\kappa) \Subset \Omega\ \forall j = 1,2,...m+1$.

According to the locally observation estimate in Step 6, one has: There exist $C > 0$ and $\sigma \in (0,1)$ such that

$$\begin{aligned} &\int_{\Omega\cap B(x_0,R)} |v(x,T)|^2 dx \\ \leq\; &\left(Ce^{\frac{C}{T}} \int_{B(x_0,\kappa)} |v(x,T)|^2 dx\right)^\sigma \left(\int_\Omega |v^0(x)|^2 dx\right)^{1-\sigma}. \end{aligned} \tag{1.140}$$

Thanks to the fact $B(x_0,\kappa) \subset B(x_1,2\kappa)$, we get

$$\int_{B(x_0,\kappa)} |v(x,T)|^2 dx \leq \int_{B(x_1,2\kappa)} |v(x,T)|^2 dx. \tag{1.141}$$

Now, due to the fact $B(x_1,2\kappa) \Subset \Omega$, one has: there exists $\delta \in (0,1]$ small enough such that $B(x_1,2(1+2\delta)\kappa) \Subset \Omega$. Now, applying the local result from Step 6, one obtains: There exists $C_1 > 0$ and $\sigma_1 \in (0,1)$ such that

$$\begin{aligned} &\int_{B(x_1,2\kappa)} |v(x,T)|^2 dx \\ \leq\; &\left(C_1 e^{\frac{C_1}{T}} \int_{B(x_1,\kappa)} |v(x,T)|^2 dx\right)^{\sigma_1} \left(\int_\Omega |v^0(x)|^2 dx\right)^{1-\sigma_1}. \end{aligned} \tag{1.142}$$

Repeat the same technique, one yields

$$\begin{aligned} &\int_{\Omega\cap B(x_0,R)} |v(x,T)|^2 dx \\ \leq\; &K_1 e^{\frac{K_2}{T}} \left(\int_{B(x_m,\kappa)} |v(x,T)|^2 dx\right)^{k_1} \left(\int_\Omega |v^0(x)|^2 dx\right)^{k_2}, \end{aligned} \tag{1.143}$$

where

$$K_1 = C^{\sigma} C_1^{\sigma\sigma_1} C_2^{\sigma\sigma_1\sigma_2} ... C_m^{\sigma\sigma_1\sigma_2...\sigma_m}, \tag{1.144}$$

$$K_2 = C\sigma + C_1\sigma\sigma_1 + C_2\sigma_2\sigma_1\sigma + ... + C_m\sigma_m...\sigma_2\sigma_1\sigma, \tag{1.145}$$

$$k_1 = \sigma\sigma_1\sigma_2...\sigma_m, \tag{1.146}$$

$$\begin{aligned} k_2 &= 1-\sigma+(1-\sigma_1)\sigma+(1-\sigma_2)\sigma_1\sigma+...+(1-\sigma_m)\sigma_{m-1}...\sigma_1\sigma \\ &= 1-\sigma\sigma_1\sigma_2...\sigma_m. \end{aligned} \tag{1.147}$$

Thanks to the fact that $B(x_m,\kappa)\subset\omega$, one gets

$$\begin{aligned} &\int_{\Omega\cap B(x_0,R)} |v(x,T)|^2 dx \\ \leq\ & K_1 e^{\frac{K_2}{T}} \left(\int_\omega |v(x,T)|^2 dx\right)^{k_1} \left(\int_\Omega |v^0(x)|^2 dx\right)^{1-k_1}. \end{aligned} \tag{1.148}$$

Now, we can conclude that: For any non-empty subset ω of Ω, for any $x_0\in\Omega$, any $R>0$ and any $\delta\in(0,1]$ such that $\Omega\cap B(x_0,(1+2\delta)R)$ is star-shaped with respect to x_0, there exist $K_1>0$, $K_2>0$ and $k\in(0,1)$ satisfying

$$\int_{\Omega\cap B(x_0,R)} |v(x,T)|^2 dx \leq K_1 e^{\frac{K_2}{T}} \left(\int_\omega |v(x,T)|^2 dx\right)^{k} \left(\int_\Omega |v^0(x)|^2 dx\right)^{1-k}. \tag{1.149}$$

Step 8: Recover Ω.

Thanks to the fact that Ω is bounded, we can cover the interior of Ω by finite number of balls which are inside Ω. For covering the neighbourhood Θ of the boundary, we use Lemma 1.4 to cover Θ by finite sets $\Omega\cap B(x_i,R_i)$ $(i=1,2,...,M)$ where $M\in\mathbb{N}^*$ and $\Omega\cap B(x_i,(1+2\delta_i)R_i)$ is star-shaped with respect to x_i for some small $\delta_i>0$ $(i=1,2,...,M)$. Thus, there exists $N\in\mathbb{N}^*$ satisfying for $i=1,2,...,N$, there exist $x_i\in\Omega$, $R_i>0$ and $\delta_i\in(0,1]$ such that $\Omega\cap B(x_i,(1+2\delta_i)R_i)$ is star-shaped with respect to x_i and

$$\Omega\subset\bigcup_{i=1}^{N}(\Omega\cap B(x_i,R_i)) \tag{1.150}$$

Hence, applying the local result from Step 7, we get

$$\begin{aligned} \int_\Omega |v(x,T)|^2 dx &\leq \sum_{i=1}^{N}\int_{\Omega\cap B(x_i,R_i)} |v(x,T)|^2 dx \\ &\leq \sum_{i=1}^{N} K_{1,i} e^{\frac{K_{2,i}}{T}} \left(\int_\omega |v(x,T)|^2 dx\right)^{k_i} \left(\int_\Omega |v^0(x)|^2 dx\right)^{1-k_i} \\ &\leq \mathcal{K}_1 e^{\frac{\mathcal{K}_2}{T}} \left(\int_\omega |v(x,T)|^2 dx\right)^{\mu} \left(\int_\Omega |v^0(x)|^2 dx\right)^{1-\mu}. \end{aligned} \tag{1.151}$$

Here, $\mathcal{K}_1 = N\max_{i\in[1,N]} K_{1,i}$, $\mathcal{K}_2=\max_{i\in[1,N]} K_{2,i}$, $\mu=\min_{i\in[1,N]} k_i$.
This completes the proof of Theorem 1.8.

1.2.4 Proof of Theorem 1.7

Now, we move to the proof of Theorem 1.7. First of all, we state the following lemma, which is a direct corollary from Theorem 1.8.

Lemma 1.5. *Let v be the solution of (HP) and T be a positive number. Then there exist positive constants $\mathcal{M}_1, \mathcal{M}_2$ and θ depending on Ω and ω such that the following estimate holds for any $\varepsilon > 0$:*

$$\|v(\cdot,T)\|^2_{L^2(\Omega)} \leq \left(\frac{\mathcal{M}_1 e^{\frac{\mathcal{M}_2}{T}}}{\varepsilon^\theta}\right)^2 \|v(\cdot,T)\|^2_{L^2(\omega)} + \varepsilon^2 \|v^0\|^2_{L^2(\Omega)}. \tag{1.152}$$

The proof of Lemma 1.5 will be found in Subsection 1.2.5.5.
Now, we start the proof of Theorem 1.7. The idea of this proof is based on the telescoping series method (see [PhW2]), i.e using the following fact

$$\sum_{m\geq 1}(c_m - c_{m+1}) = c_1 \text{ if } \lim_{m\to\infty} c_m = 0.$$

First of all, we construct a sequence of time as below: Let $l_m := \frac{T}{a^m}$ ($a > 1$ will be chosen later). Then $\{l_m\}_{m\geq 1}$ is a decreasing sequence and $l_m \xrightarrow{m\to\infty} 0$.

Step 1: Prove that: There exist $\mathcal{M}_1 > 0$, $\mathcal{M}_2 > 0$ and $\theta > 0$ such that for any $\varepsilon > 0$, the following estimate

$$\|v(\cdot,l_m)\|^2_{L^2(\Omega)} - \varepsilon^2\|v(\cdot,l_{m+2})\|^2_{L^2(\Omega)} \leq \left(\frac{\mathcal{M}_1 e^{\frac{\mathcal{M}_2}{t-l_{m+2}}}}{\varepsilon^\theta}\right)^2 \|v(\cdot,t)\|^2_{L^2(\omega)}$$

holds for $0 < l_{m+2} < l_{m+1} < t < l_m$.
Indeed, Lemma 1.5 gives us: There exist $\mathcal{M}_1 > 0$, $\mathcal{M}_2 > 0$ and $\theta > 0$ depending on Ω, ω, such that for any $\varepsilon > 0$ and any $t > l_{m+2}$ the following estimate holds

$$\|v(\cdot,t)\|^2_{L^2(\Omega)} \leq \left(\frac{\mathcal{M}_1 e^{\frac{\mathcal{M}_2}{t-l_{m+2}}}}{\varepsilon^\theta}\right)^2 \|v(\cdot,t)\|^2_{L^2(\omega)} + \varepsilon^2\|v(\cdot,l_{m+2})\|^2_{L^2(\Omega)}. \tag{1.153}$$

On the other hand, with $t < l_m$, the energy estimate says

$$\|v(\cdot,l_m)\|^2_{L^2(\Omega)} \leq \|v(\cdot,t)\|^2_{L^2(\Omega)}. \tag{1.154}$$

Combining (1.153) and (1.154), one yields

$$\|v(\cdot,l_m)\|^2_{L^2(\Omega)} - \varepsilon^2\|v(\cdot,l_{m+2})\|^2_{L^2(\Omega)} \leq \left(\frac{\mathcal{M}_1 e^{\frac{\mathcal{M}_2}{t-l_{m+2}}}}{\varepsilon^\theta}\right)^2 \|v(\cdot,t)\|^2_{L^2(\omega)}. \tag{1.155}$$

Step 2: Make appear integration with respect to t.
Integrating (1.155) over (l_{m+1}, l_m) gives us

$$(l_m - l_{m+1})\left(\|v(\cdot,l_m)\|^2_{L^2(\Omega)} - \varepsilon^2\|v(\cdot,l_{m+2})\|^2_{L^2(\Omega)}\right) \leq \left(\frac{\mathcal{M}_1}{\varepsilon^\theta}\right)^2 \int_{l_{m+1}}^{l_m} e^{\frac{2\mathcal{M}_2}{t-l_{m+2}}}\|v(\cdot,t)\|^2_{L^2(\omega)}dt. \tag{1.156}$$

Using the fact that $t \geq l_{m+1}$, we get

$$\|v(\cdot,l_m)\|^2_{L^2(\Omega)} - \varepsilon^2\|v(\cdot,l_{m+2})\|^2_{L^2(\Omega)} \leq \left(\frac{\mathcal{M}_1}{\varepsilon^\theta}\right)^2 \frac{e^{\frac{2\mathcal{M}_2}{l_{m+1}-l_{m+2}}}}{l_m - l_{m+1}} \int_{l_{m+1}}^{l_m} \|v(\cdot,t)\|^2_{L^2(\omega)}dt. \tag{1.157}$$

Furthermore, we also have

$$l_m - l_{m+1} = \frac{T(a-1)}{a^{m+1}}. \tag{1.158}$$

Combining (1.157) and (1.158), we obtain

$$\|v(\cdot,l_m)\|^2_{L^2(\Omega)} - \varepsilon^2\|v(\cdot,l_{m+2})\|^2_{L^2(\Omega)} \quad \leq \quad \left(\frac{\mathcal{M}_1}{\varepsilon^\theta}\right)^2 \frac{a^{m+1}}{T(a-1)} e^{\frac{2\mathcal{M}_2 a^{m+2}}{T(a-1)}} \int_{l_{m+1}}^{l_m} \|v(\cdot,t)\|^2_{L^2(\omega)} dt. \tag{1.159}$$

Thanks to the fact that $a > 1$ and $x \leq e^x \quad \forall x > 0$, one obtains

$$\frac{a^{m+1}}{T(a-1)} \leq \frac{a^{m+2}}{T(a-1)} \leq e^{\frac{a^{m+2}}{T(a-1)}}. \tag{1.160}$$

It deduces from (1.159) and (1.160) that

$$\|v(\cdot,l_m)\|^2_{L^2(\Omega)} - \varepsilon^2\|v(\cdot,l_{m+2})\|^2_{L^2(\Omega)} \leq \left(\frac{\mathcal{M}_1}{\varepsilon^\theta}\right)^2 e^{\frac{(2\mathcal{M}_2+1)a^{m+2}}{T(a-1)}} \int_{l_{m+1}}^{l_m} \|v(\cdot,t)\|^2_{L^2(\omega)} dt. \tag{1.161}$$

Step 3: Make appear terms in form $c_k - c_{k+1}$.
We can rewrite (1.161) with $m = 2k$ as below

$$\begin{aligned} & \varepsilon^{2\theta} e^{-\frac{(2\mathcal{M}_2+1)a^{2k+2}}{T(a-1)}} \|v(\cdot,l_{2k})\|^2_{L^2(\Omega)} - \varepsilon^{2(1+\theta)} e^{-\frac{(2\mathcal{M}_2+1)a^{2k+2}}{T(a-1)}} \|v(\cdot,l_{2k+2})\|^2_{L^2(\Omega)} \\ \leq \quad & \mathcal{M}_1^2 \int_{l_{2k+1}}^{l_{2k}} \|v(\cdot,t)\|^2_{L^2(\omega)} dt. \end{aligned} \tag{1.162}$$

We choose $\varepsilon := e^{-\frac{(2\mathcal{M}_2+1)a^{2k+2}}{T(a-1)}}$ in order to get

$$\begin{aligned} & e^{-(1+2\theta)\frac{(2\mathcal{M}_2+1)a^{2k+2}}{T(a-1)}} \|v(\cdot,l_{2k})\|^2_{L^2(\Omega)} - e^{-(3+2\theta)\frac{(2\mathcal{M}_2+1)a^{2k+2}}{T(a-1)}} \|v(\cdot,l_{2k+2})\|^2_{L^2(\Omega)} \\ \leq \quad & \mathcal{M}_1^2 \int_{l_{2k+1}}^{l_{2k}} \|v(\cdot,t)\|^2_{L^2(\omega)} dt. \end{aligned} \tag{1.163}$$

Now, our target is making appear the term $c_k - c_{k+1}$ on the left-hand side of (1.163). Hence, we choose $a := \sqrt{\frac{3+2\theta}{1+2\theta}} > 1$ for obtaining

$$\begin{aligned} & e^{-\frac{(3+2\theta)(2\mathcal{M}_2+1)a^{2k}}{T(a-1)}} \|v(\cdot,l_{2k})\|^2_{L^2(\Omega)} - e^{-\frac{(3+2\theta)(2\mathcal{M}_2+1)a^{2k+2}}{T(a-1)}} \|v(\cdot,l_{2k+2})\|^2_{L^2(\Omega)} \\ \leq \quad & \mathcal{M}_1 \int_{l_{2k+1}}^{l_{2k}} \|v(\cdot,t)\|^2_{L^2(\omega)} dt. \end{aligned} \tag{1.164}$$

Step 4: Use the telescoping series method .
Put

$$c_k := e^{-\frac{(3+2\theta)(2\mathcal{M}_2+1)a^{2k}}{T(a-1)}} \|v(\cdot,l_{2k})\|^2_{L^2(\Omega)}, \tag{1.165}$$

Then, thanks to the fact that $a > 1$, we have $\lim_{k\to\infty} c_k = 0$. Now, taking infinite sum both side of (1.164), one has

$$\sum_{k\geq 1}(c_k - c_{k+1}) \leq \mathcal{M}_1 \sum_{k\geq 1} \int_{l_{2k+1}}^{l_{2k}} \|v(\cdot,t)\|^2_{L^2(\omega)} dt. \tag{1.166}$$

Using the fact $\sum_{k\geq 1}(c_k - c_{k+1}) = c_1$ if $\lim_{k\to\infty} c_k = 0$, one gets

$$c_1 = e^{-\frac{(3+2\theta)(2\mathcal{M}_2+1)a^2}{T(a-1)}} \|v(\cdot,l_2)\|^2_{L^2(\Omega)} \leq \mathcal{M}_1 \int_0^{l_2} \|v(\cdot,t)\|^2_{L^2(\omega)} dt. \tag{1.167}$$

For the left-hand side term of (1.167), we use the energy estimate, which is

$$\|v(\cdot,l_2)\|^2_{L^2(\Omega)} \geq \|v(\cdot,T)\|^2_{L^2(\Omega)}. \quad (1.168)$$

For the right-hand side term of (1.167), we use the fact that

$$\int_0^{l_2} \|v(\cdot,t)\|^2_{L^2(\omega)}dt \leq \int_0^T \|v(\cdot,t)\|^2_{L^2(\omega)}dt. \quad (1.169)$$

Combining (1.167), (1.168) and (1.169), we obtain

$$\|v(\cdot,T)\|^2_{L^2(\Omega)} \leq \mathcal{M}_1 e^{\frac{(3+2\theta)(2\mathcal{M}_2+1)a^2}{T(a-1)}} \int_0^T \|v(\cdot,t)\|^2_{L^2(\omega)}dt. \quad (1.170)$$

This completes the proof of Theorem 1.7.

1.2.5 Proof of preliminary lemmas

1.2.5.1 Proof of Lemma 1.1

Remind that

$$\Psi(t) = \int_\vartheta |w(x,t)|^2 e^{\xi(x,t)}dx. \quad (1.171)$$

We have

$$\Psi'(t) = 2\int_\vartheta w(x,t)\partial_t w(x,t)e^{\xi(x,t)}dx + \int_\vartheta |w(x,t)|^2\partial_t\xi(x,t)e^{\xi(x,t)}dx.$$

In order to make appear the term $\partial_t w - \Delta w$, one has

$$\begin{aligned}\Psi'(t) \quad = \quad & 2\int_\vartheta w(x,t)(\partial_t w - \Delta w)(x,t)e^{\xi(x,t)}dx + 2\int_\vartheta w(x,t)\Delta w(x,t)e^{\xi(x,t)}dx \\ & + \int_\vartheta |w(x,t)|^2\partial_t\xi(x,t)e^{\xi(x,t)}dx. \quad (1.172)\end{aligned}$$

Let us compute the second term on the right-hand side of (1.172) by using integration by parts

$$\begin{aligned} & 2\int_\vartheta w(x,t)\Delta w(x,t)e^{\xi(x,t)}dx \\ = \quad & -2\int_\vartheta |\nabla w(x,t)|^2 e^{\xi(x,t)}dx - 2\int_\vartheta w(x,t)\nabla w(x,t)\nabla\xi(x,t)e^{\xi(x,t)}dx \\ = \quad & -2\int_\vartheta |\nabla w(x,t)|^2 e^{\xi(x,t)}dx - \int_\vartheta \nabla(|w(x,t)|^2)\nabla\xi(x,t)e^{\xi(x,t)}dx. \end{aligned} \quad (1.173)$$

We use the fact that $2w\nabla w = \nabla(|w|^2)$ to get the second equality. Integrating by parts the second term in (1.173) gives

$$\begin{aligned} & -\int_\vartheta \nabla(|w(x,t)|^2)\nabla\xi(x,t)e^{\xi(x,t)}dx \\ = \quad & \int_\vartheta |w(x,t)|^2\Delta\xi(x,t)e^{\xi(x,t)}dx + \int_\vartheta |w(x,t)|^2|\nabla\xi(x,t)|^2 e^{\xi(x,t)}dx. \end{aligned} \quad (1.174)$$

Combining (1.172), (1.173 and (1.174), we obtain

$$\begin{aligned}\Psi'(t) \quad = \quad & 2\int_\vartheta w(x,t)(\partial_t w - \Delta w)(x,t)e^{\xi(x,t)}dx \\ & -2\int_\vartheta |\nabla w(x,t)|^2 e^{\xi(x,t)}dx + \int_\vartheta |w(x,t)|^2\Delta\xi(x,t)e^{\xi(x,t)}dx \\ & + \int_\vartheta |w(x,t)|^2|\nabla\xi(x,t)|^2 e^{\xi(x,t)}dx + \int_\vartheta |w(x,t)|^2\partial_t\xi(x,t)e^{\xi(x,t)}dx. \quad (1.175)\end{aligned}$$

Thanks to the property (P1) of the weight function, which is $\partial_t \xi + |\nabla \xi|^2 = 0$, one gets

$$\begin{aligned}\Psi'(t) &= 2\int_\vartheta w(x,t)(\partial_t w - \Delta w)(x,t)e^{\xi(x,t)}dx \\ &\quad -2\int_\vartheta |\nabla w(x,t)|^2 e^{\xi(x,t)}dx + \int_\vartheta |w(x,t)|^2 \Delta\xi(x,t) e^{\xi(x,t)}dx.\end{aligned}$$

Using the property (P3) of the weight function, which is $\Delta \xi = -\frac{n}{2(T-t+\rho)}$, we get

$$\begin{aligned}\Psi'(t) &= 2\int_\vartheta w(x,t)(\partial_t w - \Delta w)(x,t)e^{\xi(x,t)}dx \\ &\quad -2\int_\vartheta |\nabla w(x,t)|^2 e^{\xi(x,t)}dx - \frac{n}{2(T-t+\rho)}\Psi(t).\end{aligned} \tag{1.176}$$

This completes the proof of Lemma 1.1.

1.2.5.2 Proof of Lemma 1.2

Step 1: Compute $\frac{d}{dt}\left(\int_\vartheta |\nabla w(x,t)|^2 e^{\xi(x,t)}dx\right)$.
We have

$$\begin{aligned}&\frac{d}{dt}\left(\int_\vartheta |\nabla w(x,t)|^2 e^{\xi(x,t)}dx\right) \\ &= 2\int_\vartheta \nabla w(x,t)\partial_t(\nabla w(x,t))e^{\xi(x,t)}dx + \int_\vartheta |\nabla w(x,t)|^2 \partial_t \xi(x,t) e^{\xi(x,t)}dx.\end{aligned} \tag{1.177}$$

Step 1.1: Compute $A := 2\int_\vartheta \nabla w(x,t)\partial_t(\nabla w(x,t))e^{\xi(x,t)}dx$.
By using integration by parts, we get

$$\begin{aligned}A &= 2\int_\vartheta \nabla w(x,t)\nabla(\partial_t w(x,t))e^{\xi(x,t)}dx \\ &= -2\int_\vartheta \Delta w(x,t)\partial_t w(x,t)e^{\xi(x,t)}dx - 2\int_\vartheta \nabla w(x,t)\partial_t w(x,t)\nabla\xi(x,t)e^{\xi(x,t)}dx.\end{aligned} \tag{1.178}$$

In order to make appear $\partial_t w - \Delta w$, one gets

$$\begin{aligned}A &= -2\int_\vartheta |\partial_t w(x,t)|^2 e^{\xi(x,t)}dx + 2\int_\vartheta (\partial_t w - \Delta w)(x,t)\partial_t w(x,t)e^{\xi(x,t)}dx \\ &\quad -2\int_\vartheta \partial_t w(x,t)\nabla w(x,t)\nabla\xi(x,t)e^{\xi(x,t)}dx.\end{aligned} \tag{1.179}$$

Step 1.2: Compute $B := \int_\vartheta |\nabla w(x,t)|^2 \partial_t\xi(x,t)e^{\xi(x,t)}dx$.
Thanks to property (P1) of the weight function, which is $\partial_t\xi = -|\nabla\xi|^2$, we get

$$B = -\int_\vartheta |\nabla w(x,t)|^2 |\nabla\xi(x,t)|^2 e^{\xi(x,t)}dx. \tag{1.180}$$

Notice that $\nabla(e^\xi) = \nabla\xi e^\xi$, hence B can be written as

$$B = -\int_\vartheta |\nabla w(x,t)|^2 \nabla\xi(x,t)\nabla(e^{\xi(x,t)})dx. \tag{1.181}$$

Now, by integrating by parts, one has

$$\begin{aligned}B &= \int_\vartheta \nabla(|\nabla w(x,t)|^2)\nabla\xi(x,t)e^{\xi(x,t)}dx + \int_\vartheta |\nabla w(x,t)|^2\Delta\xi(x,t)e^{\xi(x,t)}dx \\ &\quad -\int_{\partial\vartheta} |\nabla w(x,t)|^2 \partial_\nu\xi(x,t)e^{\xi(x,t)}dx.\end{aligned} \tag{1.182}$$

Next step is computing $B_1 := \int_\vartheta \nabla(|\nabla w(x,t)|^2)\nabla\xi(x,t)e^{\xi(x,t)}dx$ by using standard summation notations

$$\begin{aligned} B_1 &= \int_\vartheta \partial_i(|\partial_j w(x,t)|^2)\partial_i\xi(x,t)e^{\xi(x,t)}dx \\ &= 2\int_\vartheta \partial_j w(x,t)\partial^2_{ij}w(x,t)\partial_i\xi(x,t)e^{\xi(x,t)}dx \\ &= -2\int_\vartheta \partial^2_{jj}w(x,t)\partial_i w(x,t)\partial_i\xi(x,t)e^{\xi(x,t)}dx \\ &\quad -2\int_\vartheta \partial_j w(x,t)\partial^2_{ij}\xi(x,t)\partial_i w(x,t)e^{\xi(x,t)}dx \\ &\quad -2\int_\vartheta \partial_j w(x,t)\partial_i w(x,t)\partial_i\xi(x,t)\partial_j\xi(x,t)e^{\xi(x,t)}dx \\ &\quad +2\int_{\partial\vartheta} \partial_j w(x,t)\partial_i w(x,t)\partial_i\xi(x,t)\nu_j e^{\xi(x,t)}dx. \end{aligned} \tag{1.183}$$

Thus, we can write

$$\begin{aligned} B_1 &= -2\int_\vartheta \Delta w(x,t)\nabla w(x,t)\nabla\xi(x,t)e^{\xi(x,t)}dx - 2\int_\vartheta \nabla w(x,t)\nabla^2\xi(x,t)\nabla w(x,t)e^{\xi(x,t)}dx \\ &\quad -2\int_\vartheta |\nabla w(x,t)\nabla\xi(x,t)|^2 e^{\xi(x,t)}dx + 2\int_{\partial\vartheta}|\nabla w(x,t)|^2\partial_\nu\xi(x,t)e^{\xi(x,t)}dx. \end{aligned} \tag{1.184}$$

It follows from (1.182) and (1.184) that

$$\begin{aligned} B &= -2\int_\vartheta \Delta w(x,t)\nabla w(x,t)\nabla\xi(x,t)e^{\xi(x,t)}dx - 2\int_\vartheta \nabla w(x,t)\nabla^2\xi(x,t)\nabla w(x,t)e^{\xi(x,t)}dx \\ &\quad -2\int_\vartheta |\nabla w(x,t)\nabla\xi(x,t)|^2 e^{\xi(x,t)}dx + \int_{\partial\vartheta}|\nabla w(x,t)|^2\partial_\nu\xi(x,t)e^{\xi(x,t)}dx \\ &\quad + \int_\vartheta |\nabla w(x,t)|^2\Delta\xi(x,t)e^{\xi(x,t)}dx. \end{aligned} \tag{1.185}$$

Step 1.3: Compute $A + B$.
We get from results (1.179) in Step 1.1 and (1.185) in Step 1.2 that

$$\begin{aligned} &\frac{d}{dt}\left(\int_\vartheta |\nabla w(x,t)|^2 e^{\xi(x,t)}dx\right) = A + B \\ &= -2\int_\vartheta |\partial_t w(x,t)|^2 e^{\xi(x,t)}dx + 2\int_\vartheta (\partial_t w - \Delta w)(x,t)\partial_t w(x,t)e^{\xi(x,t)}dx \\ &\quad -2\int_\vartheta \partial_t w(x,t)\nabla w(x,t)\nabla\xi(x,t)e^{\xi(x,t)}dx - 2\int_\vartheta \Delta w(x,t)\nabla w(x,t)\nabla\xi(x,t)e^{\xi(x,t)}dx \\ &\quad -2\int_\vartheta \nabla w(x,t)\nabla^2\xi(x,t)\nabla w(x,t)e^{\xi(x,t)}dx - 2\int_\vartheta |\nabla w(x,t)\nabla\xi(x,t)|^2 e^{\xi(x,t)}dx \\ &\quad + \int_{\partial\vartheta}|\nabla w(x,t)|^2\partial_\nu\xi(x,t)e^{\xi(x,t)}dx + \int_\vartheta |\nabla w(x,t)|^2\Delta\xi(x,t)e^{\xi(x,t)}dx. \end{aligned} \tag{1.186}$$

In order to make appear the term $\partial_t w - \Delta w$, we replace the fourth term in (1.186) by

$$\begin{aligned} &-2\int_\vartheta \Delta w(x,t)\nabla w(x,t)\nabla\xi(x,t)e^{\xi(x,t)}dx \\ &= 2\int_\vartheta (\partial_t w(x,t) - \Delta w(x,t))\nabla w(x,t)\nabla\xi(x,t)e^{\xi(x,t)}dx \\ &\quad -2\int_\vartheta \partial_t w(x,t)\nabla w(x,t)\nabla\xi(x,t)e^{\xi(x,t)}dx. \end{aligned} \tag{1.187}$$

It deduces from (1.186) and (1.187) that

$$\begin{aligned}
&\frac{d}{dt}\left(\int_\vartheta |\nabla w(x,t)|^2 e^{\xi(x,t)}dx\right)\\
=\;& -2\int_\vartheta |\partial_t w(x,t)|^2 e^{\xi(x,t)}dx + 2\int_\vartheta (\partial_t w - \Delta w)(x,t)\partial_t w(x,t)e^{\xi(x,t)}dx\\
&-4\int_\vartheta \partial_t w(x,t)\nabla w(x,t)\nabla\xi(x,t)e^{\xi(x,t)}dx + 2\int_\vartheta(\partial_t w(x,t)-\Delta w(x,t))\nabla w(x,t)\nabla\xi(x,t)e^{\xi(x,t)}dx\\
&-2\int_\vartheta |\nabla w(x,t)\nabla\xi(x,t)|^2 e^{\xi(x,t)}dx - 2\int_\vartheta \nabla w(x,t)\nabla^2\xi(x,t)\nabla w(x,t)e^{\xi(x,t)}dx\\
&+\int_{\partial\vartheta}|\nabla w(x,t)|^2\partial_\nu\xi(x,t)e^{\xi(x,t)}dx + \int_\vartheta |\nabla w(x,t)|^2\Delta\xi(x,t)e^{\xi(x,t)}dx. \qquad (1.188)
\end{aligned}$$

The sum of five first terms in (1.188) can be written as

$$\begin{aligned}
&-2\int_\vartheta \left(\partial_t w(x,t) + \nabla w(x,t)\nabla\xi(x,t) - \frac{1}{2}(\partial_t w(x,t) - \Delta w(x,t))\right)^2 e^{\xi(x,t)}dx\\
&+\frac{1}{2}\int_\vartheta |\partial_t w(x,t) - \Delta w(x,t)|^2 e^{\xi(x,t)}dx. \qquad (1.189)
\end{aligned}$$

Thus, (1.188) is equivalent to

$$\begin{aligned}
&\frac{d}{dt}\left(\int_\vartheta |\nabla w(x,t)|^2 e^{\xi(x,t)}dx\right)\\
=\;& -2\int_\vartheta\left(\partial_t w(x,t) + \nabla w(x,t)\nabla\xi(x,t) - \frac{1}{2}(\partial_t w - \Delta w)(x,t)\right)^2 e^{\xi(x,t)}dx\\
&+\frac{1}{2}\int_\vartheta |\partial_t w(x,t) - \Delta w(x,t)|^2 e^{\xi(x,t)}dx - 2\int_\vartheta \nabla w(x,t)\nabla^2\xi(x,t)\nabla w(x,t)e^{\xi(x,t)}dx\\
&+\int_{\partial\vartheta}|\nabla w(x,t)|^2\partial_\nu\xi(x,t)e^{\xi(x,t)}dx + \int_\vartheta|\nabla w(x,t)|^2\Delta\xi(x,t)e^{\xi(x,t)}dx. \qquad (1.190)
\end{aligned}$$

Step 2: Estimate $D := \left(\frac{d}{dt}\int_\vartheta |\nabla w(x,t)|^2 e^{\xi(x,t)}dx\right)\left(\int_\vartheta |w(x,t)|^2 e^{\xi(x,t)}dx\right)$.
We will use properties of the weight function ξ to compute the three last terms on the right-hand side of (1.190).
Firstly, using the property (P4), which is $\nabla^2\xi = \frac{-1}{2(T-t+\rho)}I_n$, one obtains

$$-2\int_\vartheta \nabla w(x,t)\nabla^2\xi(x,t)\nabla w(x,t)e^{\xi(x,t)}dx = \frac{1}{T-t+\rho}\int_\vartheta |\nabla w(x,t)|^2 e^{\xi(x,t)}dx. \qquad (1.191)$$

Secondly, thanks to the property (P2), which is $\nabla\xi = -\frac{x-x_0}{2(T-t+\rho)}$ and the assumption that ϑ is star-shaped with respect to x_0, we get a good sign for the boundary term

$$\int_{\partial\vartheta}|\nabla w(x,t)|^2\partial_\nu\xi(x,t)e^{\xi(x,t)}dx = -\frac{1}{2(T-t+\rho)}\int_{\partial\vartheta}|\nabla w(x,t)|^2\nu(x-x_0)e^{\xi(x,t)}dx \le 0. \quad (1.192)$$

Thirdly, using the property (P3), which is $\Delta\xi = \frac{-n}{2(T-t+\rho)}$, we get

$$\int_\vartheta |\nabla w(x,t)|^2\Delta\xi(x,t)e^{\xi(x,t)}dx = \frac{-n}{2(T-t+\rho)}\int_\vartheta |w(x,t)|^2 e^{\xi(x,t)}dx. \qquad (1.193)$$

Combining (1.190), (1.191), (1.192) and (1.193), one obtains

$$\begin{aligned}
D \;\le\;& -2\int_\vartheta\left(\partial_t w(x,t) + \nabla w(x,t)\nabla\xi(x,t) - \frac{1}{2}(\partial_t w - \Delta w)(x,t)\right)^2 e^{\xi(x,t)}dx\int_\vartheta |w(x,t)|^2 e^{\xi(x,t)}dx\\
&+\frac{1}{2}\int_\vartheta |(\partial_t w - \Delta w)(x,t)|^2 e^{\xi(x,t)}dx\int_\vartheta |w(x,t)|^2 e^{\xi(x,t)}dx + \frac{1}{T-t+\rho}\int_\vartheta |\nabla w(x,t)|^2 e^{\xi(x,t)}dx\\
&-\frac{n}{2(T-t+\rho)}\int_\vartheta |w(x,t)|^2 e^{\xi(x,t)}dx\int_\vartheta |\nabla w(x,t)|^2 e^{\xi(x,t)}dx. \qquad (1.194)
\end{aligned}$$

Step 3: Estimate $E := -\left(\frac{d}{dt}\int_\vartheta |w(x,t)|^2 e^{\xi(x,t)}dx\right)\left(\int_\vartheta |\nabla w(x,t)|^2 e^{\xi(x,t)}dx\right)$.
Our target is making appear the following term

$$\int_\vartheta \left(\partial_t w(x,t) + \nabla w(x,t)\nabla\xi(x,t) - \frac{1}{2}(\partial_t w - \Delta w)(x,t)\right) w(x,t)e^{\xi(x,t)}dx. \tag{1.195}$$

Step 3.1: Compute $\int_\vartheta |\nabla w(x,t)|^2 e^{\xi(x,t)}dx$.
By integrating by parts, we have

$$\begin{aligned}
&\int_\vartheta |\nabla w(x,t)|^2 e^{\xi(x,t)}dx \\
= &-\int_\vartheta \Delta w(x,t)w(x,t)e^{\xi(x,t)}dx - \int_\vartheta \nabla w(x,t)w(x,t)\nabla\xi(x,t)e^{\xi(x,t)}dx \\
= &-\int_\vartheta \left(\partial_t w(x,t) + \nabla w(x,t)\nabla\xi(x,t) - \frac{1}{2}(\partial_t w - \Delta w)(x,t)\right) w(x,t)e^{\xi(x,t)}dx \\
&+\frac{1}{2}\int_\vartheta (\partial_t w - \Delta w)(x,t)w(x,t)e^{\xi(x,t)}dx.
\end{aligned} \tag{1.196}$$

Step 3.2: Compute $\frac{d}{dt}\left(\int_\vartheta |w(x,t)|^2 e^{\xi(x,t)}dx\right)$.
Lemma 1.1 gives us (see (1.176))

$$\begin{aligned}
&\frac{d}{dt}\int_\vartheta |w(x,t)|^2 e^{\xi(x,t)}dx \\
= \ &2\int_\vartheta w(x,t)(\partial_t w - \Delta w)(x,t)e^{\xi(x,t)}dx \\
&-2\int_\vartheta |\nabla w(x,t)|^2 e^{\xi(x,t)}dx + \int_\vartheta |w(x,t)|^2 \Delta\xi(x,t)e^{\xi(x,t)}dx.
\end{aligned} \tag{1.197}$$

Using the result (1.196) in Step 3.1, one obtains

$$\begin{aligned}
&\frac{d}{dt}\int_\vartheta |w(x,t)|^2 e^{\xi(x,t)}dx \\
= \ &2\int_\vartheta \left(\partial_t w(x,t) + \nabla w(x,t)\nabla\xi(x,t) - \frac{1}{2}(\partial_t w - \Delta w)(x,t)\right) w(x,t)e^{\xi(x,t)}dx \\
&+\int_\vartheta (\partial_t w - \Delta w)(x,t)w(x,t)e^{\xi(x,t)}dx + \int_\vartheta |w(x,t)|^2\Delta\xi(x,t)e^{\xi(x,t)}dx.
\end{aligned} \tag{1.198}$$

Step 3.3: Compute E.
Combining (1.196), (1.198) and using the fact that $2(a+\frac{1}{2}b)(a-\frac{1}{2}b) = 2a^2 - \frac{1}{2}b^2$, we have

$$\begin{aligned}
E \ = \ &2\left(\int_\vartheta \left(\partial_t w(x,t) + \nabla w(x,t)\nabla\xi(x,t) - \frac{1}{2}(\partial_t w - \Delta w)(x,t)\right) w(x,t)e^{\xi(x,t)}dx\right)^2 \\
&-\frac{1}{2}\left(\int_\vartheta (\partial_t w - \Delta w)(x,t)w(x,t)e^{\xi(x,t)}dx\right)^2 \\
&-\left(\int_\vartheta |w(x,t)|^2\Delta\xi(x,t)e^{\xi(x,t)}dx\right)\left(\int_\vartheta |\nabla w(x,t)|^2 e^{\xi(x,t)}dx\right).
\end{aligned} \tag{1.199}$$

Step3.4: Estimate E.
For the first term of E, using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
&2\left(\int_\vartheta \left(\partial_t w(x,t) + \nabla w(x,t)\nabla\xi(x,t) - \frac{1}{2}(\partial_t w - \Delta w)(x,t)\right) w(x,t)e^{\xi(x,t)}dx\right)^2 \\
\leq \ &2\int_\vartheta \left(\partial_t w(x,t) + \nabla w(x,t)\nabla\xi(x,t) - \frac{1}{2}(\partial_t w - \Delta w)(x,t)\right)^2 e^{\xi(x,t)}dx \int_\vartheta |w(x,t)|^2 e^{\xi(x,t)}dx.
\end{aligned} \tag{1.200}$$

For the second term of E, it is nonpositive.
For the third term of E, using the property (P3), which is $\Delta\xi = \frac{-n}{2(T-t+\rho)}$, we get

$$\begin{aligned} & -\int_{\vartheta} |w(x,t)|^2 \Delta\xi(x,t) e^{\xi(x,t)} dx \int_{\vartheta} |\nabla w(x,t)|^2 e^{\xi(x,t)} dx \\ = & \frac{n}{2(T-t+\rho)} \int_{\vartheta} |w(x,t)|^2 e^{\xi(x,t)} dx \int_{\vartheta} |\nabla w(x,t)|^2 e^{\xi(x,t)} dx. \end{aligned} \tag{1.201}$$

Thus, E is estimated as below

$$\begin{aligned} E \leq & 2\int_{\vartheta} \left(\partial_t w(x,t) + \nabla w(x,t)\nabla\xi(x,t) - \frac{1}{2}(\partial_t w - \Delta w)(x,t)\right)^2 e^{\xi(x,t)} dx \int_{\vartheta} |w(x,t)|^2 e^{\xi(x,t)} dx \\ & + \frac{n}{2(T-t+\rho)} \int_{\vartheta} |w(x,t)|^2 e^{\xi(x,t)} dx \int_{\vartheta} |\nabla w(x,t)|^2 e^{\xi(x,t)} dx. \end{aligned} \tag{1.202}$$

Step 4: Compute $N'(t)$.
Combining (1.194) in Step 2 and (1.202) in Step 3, one gets

$$\begin{aligned} D+E \leq & \frac{1}{2}\int_{\vartheta} |(\partial_t w - \Delta w)(x,t)|^2 e^{\xi(x,t)} dx \int_{\vartheta} |w(x,t)|^2 e^{\xi(x,t)} dx \\ & + \frac{1}{T-t+\rho} \int_{\vartheta} |\nabla w(x,t)|^2 e^{\xi(x,t)} dx \int_{\vartheta} |w(x,t)|^2 e^{\xi(x,t)} dx. \end{aligned} \tag{1.203}$$

Now, we can compute $N'(t)$. We have

$$\begin{aligned} N'(t) = & \frac{2(D+E)}{\left(\int_{\vartheta} |w(x,t)|^2 e^{\xi(x,t)} dx\right)^2} + \frac{n}{2(T-t+\rho)^2} \\ \leq & \frac{1}{T-t+\rho}\left(\frac{2\int_{\vartheta} |\nabla w(x,t)|^2 e^{\xi(x,t)} dx}{\int_{\vartheta} |w(x,t)|^2 e^{\xi(x,t)} dx} + \frac{n}{2(T-t+\rho)}\right) \\ & + \frac{\int_{\vartheta} |(\partial_t w - \Delta w)(x,t)|^2 e^{\xi(x,t)} dx}{\int_{\vartheta} |w(x,t)|^2 e^{\xi(x,t)} dx} \\ = & \frac{N(t)}{T-t+\rho} + \frac{\int_{\vartheta} |(\partial_t w - \Delta w)(x,t)|^2 e^{\xi(x,t)} dx}{\int_{\vartheta} |w(x,t)|^2 e^{\xi(x,t)} dx}. \end{aligned} \tag{1.204}$$

This completes the proof of Lemma 1.2.

1.2.5.3 Proof of Lemma 1.3

Step 1: Prove $\frac{d}{dt}\left(\int_{\Omega} |v(x,t)|^2 e^{\frac{-|x-x_0|^2}{2(T-t+\rho)}} dx\right) \leq 0 \quad \forall\rho > 0 \quad \forall x_0 \in \Omega$.
We claim that

$$\frac{d}{dt}\left(\int_{\Omega} |v(x,t)|^2 e^{\tilde{\xi}(x,t)}\right) dx \leq 0, \tag{1.205}$$

with $\tilde{\xi} \in C^{\infty}(\Omega \times [0,T])$ satisfying

$$\partial_t\tilde{\xi} + \frac{1}{2}|\nabla\tilde{\xi}|^2 \leq 0. \tag{1.206}$$

The weight function which is defined as below

$$\tilde{\xi}(x,t) := \frac{-|x-x_0|^2}{2(T-t+\rho)} \quad \forall\rho > 0 \tag{1.207}$$

satisfies $\partial_t\tilde{\xi} + \frac{1}{2}|\nabla\tilde{\xi}|^2 = 0$. Hence we get

$$\frac{d}{dt}\left(\int_{\Omega} |v(x,t)|^2 e^{\frac{-|x-x_0|^2}{2(T-t+\rho)}} dx\right) \leq 0. \tag{1.208}$$

Now, we will prove our claim. We have

$$\begin{aligned}
&\frac{d}{dt}\left(\int_\Omega |v(x,t)|^2 e^{\tilde{\xi}(x,t)}\right) dx \\
= \;& 2\int_\Omega v(x,t)\partial_t v(x,t) e^{\tilde{\xi}(x,t)} dx + \int_\Omega |v(x,t)|^2 \partial_t \xi(x,t) e^{\tilde{\xi}(x,t)} dx \\
= \;& 2\int_\Omega v(x,t)\Delta v(x,t) e^{\tilde{\xi}(x,t)} dx + \int_\Omega |v(x,t)|^2 \partial_t \xi(x,t) e^{\tilde{\xi}(x,t)} dx \\
= \;& -2\int_\Omega |\nabla v(x,t)|^2 e^{\tilde{\xi}(x,t)} dx - 2\int_\Omega v(x,t)\nabla v(x,t)\nabla\xi(x,t) e^{\tilde{\xi}(x,t)} dx \\
& + \int_\Omega |v(x,t)|^2 \partial_t \xi(x,t) e^{\tilde{\xi}(x,t)} dx.
\end{aligned} \tag{1.209}$$

In the second equality of (1.209), we use the fact that $\partial_t v = \Delta v$. In the third equality of (1.209), we apply integration by parts for the first term. Now, thanks to assumption (1.206), yields

$$\begin{aligned}
&\frac{d}{dt}\left(\int_\Omega |v(x,t)|^2 e^{\tilde{\xi}(x,t)}\right) dx \\
\leq \;& -2\int_\Omega |\nabla v(x,t)|^2 e^{\tilde{\xi}(x,t)} dx - 2\int_\Omega v(x,t)\nabla v(x,t)\nabla\xi(x,t) e^{\tilde{\xi}(x,t)} dx \\
& -\frac{1}{2}\int_\Omega |v(x,t)|^2 |\nabla\xi(x,t)|^2 e^{\tilde{\xi}(x,t)} dx \\
= \;& -\frac{1}{2}\int_\Omega \left(4|\nabla v(x,t)|^2 + 4v(x,t)\nabla v(x,t)\nabla\xi(x,t) + |v(x,t)|^2|\nabla\xi(x,t)|^2\right) e^{\tilde{\xi}(x,t)} dx \\
= \;& -\frac{1}{2}\int_\Omega \left(v(x,t)\nabla\xi(x,t) + 2\nabla v(x,t)\right)^2 e^{\tilde{\xi}(x,t)} dx \\
\leq \;& 0.
\end{aligned} \tag{1.210}$$

This completes the proof of Step 1.

Step 2: Make appear $B(x_0,(1+\delta)R)$.
Integrating (**??**) over (t,T) with $0<t<T$, one obtains

$$\int_\Omega |v(x,T)|^2 e^{\frac{-|x-x_0|^2}{2\rho}} dx \leq \int_\Omega |v(x,t)|^2 e^{\frac{-|x-x_0|^2}{2(T-t+\rho)}} dx. \tag{1.211}$$

Now, in order to make appear $\Omega\cap B(x_0,(1+\delta)R)$, we use the following fact

$$\begin{aligned}
&\int_\Omega |v(x,t)|^2 e^{\frac{-|x-x_0|^2}{2(T-t+\rho)}} dx \\
= \;& \int_{\Omega\cap B(x_0,(1+\delta)R)} |v(x,t)|^2 e^{\frac{-|x-x_0|^2}{2(T-t+\rho)}} dx + \int_{\Omega\setminus B(x_0,(1+\delta)R)} |v(x,t)|^2 e^{\frac{-|x-x_0|^2}{2(T-t+\rho)}} dx \\
\leq \;& \int_{\Omega\cap B(x_0,(1+\delta)R)} |v(x,t)|^2 dx + e^{\frac{-(1+\delta)^2R^2}{2(T-t+\rho)}} \int_\Omega |v(x,t)|^2 dx.
\end{aligned} \tag{1.212}$$

Thanks to the energy estimate, which is $\int_\Omega |v(x,t)|^2 dx \leq \int_\Omega |v^0(x)|^2 dx$, we get from (1.211) and (1.212) that

$$\int_\Omega |v(x,T)|^2 e^{\frac{-|x-x_0|^2}{2\rho}} dx \leq \int_{\Omega\cap B(x_0,(1+\delta)R)} |v(x,t)|^2 dx + e^{\frac{-(1+\delta)^2R^2}{2(T-t+\rho)}} \int_\Omega |v^0(x)|^2 dx. \tag{1.213}$$

Step 3: Take off the weight function.
On the other hand, we also have

$$\int_\Omega |v(x,T)|^2 e^{\frac{-|x-x_0|^2}{2\rho}} dx \geq \int_{\Omega\cap B(x_0,R)} |v(x,T)|^2 e^{\frac{-|x-x_0|^2}{2\rho}} dx \geq e^{\frac{-R^2}{2\rho}} \int_{\Omega\cap B(x_0,R)} |v(x,T)|^2 dx. \tag{1.214}$$

Combining (1.213) and (1.214), one yields

$$\int_{\Omega\cap B(x_0,R)} |v(x,T)|^2 dx \le e^{\frac{R^2}{2\rho}} \int_{\Omega\cap B(x_0,(1+\delta)R)} |v(x,t)|^2 dx + e^{\frac{-(1+\delta)^2 R^2}{2(T-t+\rho)}+\frac{R^2}{2\rho}} \int_\Omega |v^0(x)|^2 dx. \quad (1.215)$$

Step 4: Make appear $A \le e^{C(\rho)}B + e^{-C(\rho)}D$ for some positive constant $C(\rho)$ depending on ρ. Under the assumption that $T-\delta\rho \le t \le T$, we get

$$-\frac{(1+\delta)^2R^2}{2(T-t+\rho)} + \frac{R^2}{2\rho} \le -\frac{\delta R^2}{2\rho}. \quad (1.216)$$

Hence, it follows from (1.215) that

$$\int_{\Omega\cap B(x_0,R)} |v(x,T)|^2 dx \le e^{\frac{R^2}{2\rho}} \int_{\Omega\cap B(x_0,(1+\delta)R)} |v(x,t)|^2 dx + e^{-\frac{\delta R^2}{2\rho}} \int_\Omega |v^0(x)|^2 dx. \quad (1.217)$$

Step 5: Choose suitable ρ.
Now, we choose $\rho \le \min\{1, \frac{T}{2}\}$, i.e $\frac{1}{\rho} \ge 1 + \frac{2}{T}$ such that

$$e^{-\frac{\delta R^2}{2\rho}} \int_\Omega |v^0(x)|^2 dx = \frac{1}{2} e^{-\frac{R^2}{2}(1+\frac{2}{T})} \int_{\Omega\cap B(x_0,R)} |v(x,T)|^2 dx \quad (1.218)$$

or

$$\frac{1}{\rho} = \frac{2}{\delta R^2} \ln\left(e^{\frac{R^2}{2}(1+\frac{2}{T})} \frac{2\int_\Omega |v^0(x)|^2 dx}{\int_{\Omega\cap B(x_0,R)} |v(x,T)|^2 dx} \right). \quad (1.219)$$

With such choice of ρ, it follows from (1.217) that

$$\left(1 - \frac{1}{2} e^{-\frac{R^2}{2}(1+\frac{2}{T})}\right) \int_{\Omega\cap B(x_0,R)} |v(x,T)|^2 dx \le e^{\frac{R^2}{2\rho}} \int_{\Omega\cap B(x_0,(1+\delta)R)} |v(x,t)|^2 dx. \quad (1.220)$$

On the other hand, it deduces from (1.218) that

$$\int_{\Omega\cap B(x_0,R)} |v(x,T)|^2 dx = 2e^{-\frac{\delta R^2}{2\rho}+\frac{R^2}{2}(1+\frac{2}{T})} \int_\Omega |v^0(x)|^2 dx. \quad (1.221)$$

Combining (1.220) and (1.221), one yields

$$2\left(1 - \frac{1}{2} e^{-\frac{R^2}{2}(1+\frac{2}{T})}\right) e^{-\frac{\delta R^2}{2\rho}+\frac{R^2}{2}(1+\frac{2}{T})} \int_\Omega |v^0(x)|^2 dx \le e^{\frac{R^2}{2\rho}} \int_{\Omega\cap B(x_0,(1+\delta)R)} |v(x,t)|^2 dx. \quad (1.222)$$

Thanks to the fact that

$$2\left(1 - \frac{1}{2} e^{-\frac{R^2}{2}(1+\frac{2}{T})}\right) \ge 1 \quad (1.223)$$

and

$$e^{\frac{R^2}{2}(1+\frac{2}{T})} \ge 1, \quad (1.224)$$

we get

$$\int_\Omega |v^0(x)|^2 dx \le e^{\frac{(1+\delta)R^2}{2\rho}} \int_{\Omega\cap B(x_0,(1+\delta)R)} |v(x,t)|^2 dx. \quad (1.225)$$

Put $\hbar = \delta\rho$ then we can include that: for any $\frac{T}{2} \le T - \hbar \le t \le T$, the following estimate holds

$$\frac{\int_\Omega \left|v^0(x)\right|^2 dx}{\int_{\Omega\cap B(x_0,(1+\delta)R)} \left|v(x,t)\right|^2 dx} \le e^{\frac{(1+\delta)\delta R^2}{2\hbar}}. \quad (1.226)$$

Here

$$\frac{1}{\hbar} = \frac{2}{(\delta R)^2} \ln\left(e^{\frac{R^2}{2}(1+\frac{2}{T})} \frac{2\int_\Omega |v^0(x)|^2 dx}{\int_{\Omega\cap B(x_0,R)} |v(x,T)|^2 dx} \right).$$

This completes the proof of Lemma 1.3.

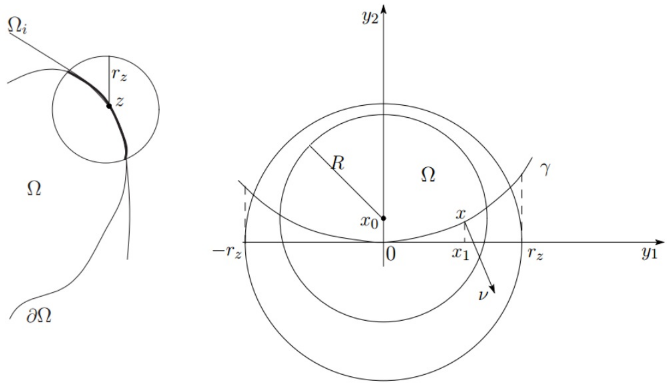


Figure 1.1 – Illustration for case 2D

1.2.5.4 Proof of Lemma 1.4

For simplicity, we will prove Lemma 1.4 in two dimensions $n = 2$. First of all, let us remind the C^2 boundary definition.

Definition 1.2. *(see [HaT, De.A.3, p.246])*
Let Ω be open and bounded in $\mathbb{R}^2$. We say $\partial\Omega$ is C^2 if $\partial\Omega$ can be covered finitely by many open balls $B(z_i, r_i)$ in $\mathbb{R}^2$ (i=1,2,..,N) for $z_i \in \partial\Omega$ and $r_i > 0$ such that

$$\Omega \cap B(z_i, r_i) = B(z_i, r_i) \cap \Omega_i \quad \textit{with} \quad i = 1, 2, ..., N, \tag{1.227}$$

where Ω_i are rotations of suitable special C^2 domains in $\mathbb{R}^2$.

Fix $z \in \partial\Omega$ and $r_z > 0$. We have the illustrated situation where one may assume that $\partial\Omega \cap B(z, r_z)$ can be presented in local coordinates by

$$y_2 = \gamma(y_1) \quad \text{with} \quad \gamma \in C^2([-r_z, r_z]), \gamma(0) = 0 \quad \text{and} \quad \gamma'(0) = 0. \tag{1.228}$$

In these local coordinates, we have $z = (0, 0)$ and

$$\Omega \cap B(z, r_z) = \{y = (y_1, y_2) \in B(0, r_z) : y_2 > \gamma(y_1)\}. \tag{1.229}$$

Step 1: Prove that: there exists $K > 0$ such that

$$|x\gamma'(x) - \gamma(x)| \leq K|x|^2 \quad \forall x \in [-r_z, r_z]. \tag{1.230}$$

Consider the following function:

$$\begin{aligned} g : [-r_z, r_z] &\to \mathbb{R} \\ x &\mapsto x\gamma'(x) - \gamma(x). \end{aligned} \tag{1.231}$$

Thanks to the fact that $\gamma \in C^2([-r_z, r_z])$, we get $g \in C^1([-r_z, r_z])$ with $g'(x) = x\gamma''(x)$. Since γ'' is continuous on $[-r_z, r_z]$, there exists $K > 0$ such that

$$|\gamma''(x)| \leq K \quad \forall x \in [-r_z, r_z]. \tag{1.232}$$

It implies that

$$|g'(x)| \leq K|x| \quad \forall x \in [-r_z, r_z]. \tag{1.233}$$

On the other hand, by using the mean value theorem, one has: For any $x \in [-r_z, r_z]$, there exists $t \in (0,1)$ such that

$$xg'(tx) = g(x) - g(0) = g(x). \tag{1.234}$$

Combining (1.233) and (1.234), we get

$$|g(x)| \leq |g'(tx)||x| \leq K|tx||x| \leq K|x|^2. \tag{1.235}$$

This completes the proof of Step 1.

Step 2: Choose $x_0 \in \Omega$ and $R > 0$.
Now, take $R := \min\{\frac{1}{1+K}, \frac{r_z}{2}\}$ and $x_0 := (0, KR^2)$.
Firstly, since $KR^2 = KRR \leq K\frac{1}{1+K}R < R$, one has $0 \in B(x_0, R)$.
Secondly, take $x = (x_1, x_2) \in \partial\Omega \cap B(x_0, R)$ $(|x_1| < R)$. We claim that $x \in \partial\Omega \cap B(0, r_z)$. Indeed, one has

$$|x| \leq |x - x_0| + |x_0| < R + KR^2 = (1 + KR)R \leq \left(1 + \frac{K}{1+K}\right) R < 2R \leq r_z. \tag{1.236}$$

Thus, we can write $x_2 = \gamma(x_1)$. The unit outward normal vector to x is computed as

$$\nu = \frac{1}{\sqrt{1 + |\gamma'(x_1)|^2}} (\gamma'(x_1) \quad -1). \tag{1.237}$$

Therefore, one has

$$(x - x_0)\nu = \frac{1}{\sqrt{1 + |\gamma'(x_1)|^2}} \left(x_1\gamma'(x_1) - \gamma(x_1) + KR^2\right). \tag{1.238}$$

Applying the result from Step 1, we get

$$|x_1\gamma'(x_1) - \gamma(x_1)| \leq K|x_1|^2 < KR^2. \tag{1.239}$$

It implies from (1.239) that

$$x_1\gamma'(x_1) - \gamma(x_1) > -KR^2. \tag{1.240}$$

Thus $(x - x_0)\nu > 0$. This completes the proof of Lemma 1.4.

1.2.5.5 Proof of Lemma 1.5

It implies from Theorem 1.8 that: There exist $\mathcal{K}_1 > 0$, $\mathcal{K}_2 > 0$ and $\mu \in (0,1)$ depending on Ω and ω such that

$$\|v(\cdot,T)\|^2_{L^2(\Omega)} \leq \left(\mathcal{K}_1 e^{\frac{\mathcal{K}_2}{T}}\right)^2 \|v(\cdot,T)\|^{2\mu}_{L^2(\omega)} \|v^0\|^{2(1-\mu)}_{L^2(\Omega)}. \tag{1.241}$$

Let ε be a positive number. Applying the Young's inequality $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ for the right-hand side of (1.241) with

$$a = \left(\left(\mathcal{K}_1 e^{\frac{\mathcal{K}_2}{T}}\right)^{\frac{1}{\mu}} \|v(\cdot,T)\|_{L^2(\omega)} \frac{1}{\varepsilon^{\frac{1-\mu}{\mu}}} (1-\mu)^{\frac{1-\mu}{2\mu}}\right)^{2\mu}; \tag{1.242}$$

$$b = \left(\varepsilon \left(\frac{1}{1-\mu}\right)^{\frac{1}{2}} \|v^0\|_{L^2(\Omega)}\right)^{2(1-\mu)}; \tag{1.243}$$

$$p = \frac{1}{\mu} \quad ; \quad q = \frac{1}{1-\mu}; \tag{1.244}$$

we obtain that the following estimate holds for any $\varepsilon > 0$

$$\|v(\cdot,T)\|^2_{L^2(\Omega)} \leq \left(\frac{\left(\mathcal{K}_1 e^{\frac{\mathcal{K}_2}{T}}\right)^{\frac{1}{\mu}} (1-\mu)^{\frac{1-\mu}{2\mu}}}{\varepsilon^{\frac{1-\mu}{\mu}}}\right)^2 \mu \|v(\cdot,T)\|^2_{L^2(\omega)} + \varepsilon^2 \|v^0\|^2_{L^2(\Omega)}. \tag{1.245}$$

Therefore, we get our desired estimate (1.152) with

$$\mathcal{M}_1 := \mathcal{K}_1^{\frac{1}{\mu}}(1-\mu)^{\frac{1-\mu}{2\mu}}\mu^{\frac{1}{2}}; \quad \mathcal{M}_2 := \frac{\mathcal{K}_2}{\mu} \quad ; \quad \theta := \frac{(1-\mu)}{\mu}. \tag{1.246}$$

This completes the proof of Lemma 1.5.

1.3 Controllability

The controllability of partial differential equations is an important area of research and has been the subject of many papers, such as [Co], [FeG], [Fe2], [FuI], [Li2], [Li3], [Mi3], [Zu1], [Zu2], [Zu3].... In this section, we present some of the recent progresses done on the problem of controllability of the heat equation. Roughly speaking, it consists in analyzing whether the solution of the (HP) can be driven to a given final target by means of a control applied on a subdomain of the domain in which the equation evolves. On one hand, when control is added during the time from 0 until T, we concern two problems: The null approximate controllability (see Subsection 1.3.1) where the solution at final time T gets null approximately and the null controllability (see Subsection 1.3.2) where the solution at final time T reaches zero exactly. These preliminary results are the key point for our study of null controllability of semilinear heat equation, which will be presented in Chapter 2. On the other hand, when control is only added at one point of time T (called impulse control), we study the null approximate impulse controllability (see 1.3.3) where the solution at final time $2T$ approximates to 0. Such result holds an important role in the topic of local backward heat problem, which is concerned in Chapter 3.

1.3.1 Null approximate controllability

1.3.1.1 Introduction and main result

Let ω be a nonempty, open subset of Ω. Now, we analyze the *null approximate controllability* problem. Consider the following control system

$$\begin{cases} \partial_t\varphi - \Delta\varphi = \mathbb{1}_\omega f & \text{in} \quad \Omega\times(0,T)\ , \\ \varphi = 0 & \text{on} \quad \partial\Omega\times(0,T)\ , \\ \varphi(\cdot,0) = \varphi^0 \in L^2(\Omega)\ , & \end{cases} \tag{1.247}$$

where T denotes a positive constant, $\mathbb{1}_\omega$ denotes the characteristic function of ω and $f \in L^2(\omega\times(0,T))$ denotes the control function acting only on the set $\omega\times(0,T)$. It is well-known that (1.247) possesses a unique solution $\varphi \in C(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))$ (see [Co, Th.2.63, p.77], [Br, Th.10.9, p.341] or [LiM]). Let us start with the definition of null approximate controllable property:

Definition 1.3. *(see [Co, De.2.40, p.55])*
System (1.247) will be said to be null approximate controllable at time T if, for any $\varepsilon > 0$, for any $\varphi^0 \in L^2(\Omega)$, there exists a control $f \in L^2(\omega\times(0,T))$ such that the associated state satisfies $\|\varphi(\cdot,T)\|_{L^2(\Omega)} \le \varepsilon\|\varphi^0\|_{L^2(\Omega)}$.

This means that for every $\varepsilon > 0$, for every $\varphi^0 \in L^2(\Omega)$, the set

$$C_{T,\varphi^0,\varepsilon} := \{f \in L^2(\omega\times(0,T)) : \text{the solution of (1.247) satisfies } \|\varphi(\cdot,T)\|_{L^2(\Omega)} \le \varepsilon\|\varphi^0\|_{L^2(\Omega)}\}. \tag{1.248}$$

is nonempty. It leads us to the definition of the cost of null approximate control

Definition 1.4. *(see [FeZ2])*
The quantity $K(T,\varepsilon) := \sup_{\|\varphi^0\|_{L^2(\Omega)}=1} \inf_{f\in C_{T,\varphi^0,\varepsilon}} \|f\|_{L^2(\omega\times(0,T))}$ is called the cost of null approximate control at time T.

Now, we state a main result of approximate null controllability for the heat equation.

Theorem 1.9. *(see [Mi3, Le.3.4, p.11])*
The system (1.247) is null approximate controllable at any time $T > 0$. Moreover, for any $\varepsilon > 0$ and any $\varphi^0 \in L^2(\Omega)$, there exist positive constants $\mathcal{C}_1, \mathcal{C}_2$ depending on Ω and ω such that the following estimate holds

$$\frac{1}{\left(\mathcal{C}_1 e^{\frac{\mathcal{C}_2}{T}}\right)^2} \int_0^T \int_\omega |f(x,t)|^2 dxdt + \frac{1}{\varepsilon^2} \int_\Omega |\varphi(x,T)|^2 dx \leq \|\varphi^0\|^2_{L^2(\Omega)}. \tag{1.249}$$

According to Definition 1.4, the cost of null approximate control satisfies $K(T) \leq \mathcal{C}_1 e^{\frac{\mathcal{C}_2}{T}}$. Here the constants $\mathcal{C}_1$ and $\mathcal{C}_2$ in (1.249) come from the estimate (1.9) of Theorem 1.7. There are several possible proofs for the null approximate controllable property, such as: Using the Hahn-Banach theorem (see [Zu2, Th.2.5.2, p.127]) or the minimization of a functional (see also [Zu2, p.129] or [FeZ2, Th.1.1, p.3]).

1.3.1.2 Construction of the null approximate control function

The control can be built by minimizing a suitable quadratic functional defined on the class of solutions of the adjoint system. For instance, according to Zuazua, (see [Zu2] or [Zu3]), the control function is constructed as $f = \tilde{v}$ where $\tilde{v}$ is the solution of the following system

$$\begin{cases} \partial_t v - \Delta v = 0 & \text{in } \Omega \times (0,T), \\ v = 0 & \text{on } \partial\Omega \times (0,T), \\ v(\cdot, 0) = v^0 \in L^2(\Omega) \end{cases} \tag{1.250}$$

corresponding to the initial data $\tilde{v}^0$. Here, $\tilde{v}^0$ is the unique minimizer of the following functional $J_\varepsilon : L^2(\Omega) \to \mathbb{R}$ such that

$$J_\varepsilon(v^0) = \frac{1}{2} \int_0^T \int_\omega |v(x, T-t)|^2 dxdt + \varepsilon \|v^0\|_{L^2(\Omega)} + \int_\Omega v(x,T)\varphi^0(x)dx. \tag{1.251}$$

In (1.251), v is the solution of (1.250) corresponding to v^0.

In [Mi3], the author changes the functional by defining

$$J_\varepsilon(v^0) = \frac{\left(\mathcal{C}_1 e^{\frac{\mathcal{C}_2}{T}}\right)^2}{2} \int_0^T \int_\omega |v(x,t)|^2 dxdt + \frac{\varepsilon^2}{2} \|v^0\|^2_{L^2(\Omega)} + \int_\Omega v(x,T)\varphi^0(x)dx. \tag{1.252}$$

Here, $\mathcal{C}_1$ and $\mathcal{C}_2$ are the constants from the observability estimate (1.9). Let $\tilde{v}_0$ be the minimizer of the functional J_ε and $\tilde{v}$ be the corresponding solution of (1.250). Then the null approximate control function is constructed as $f(x,t) = (\mathcal{C}_1 e^{\frac{\mathcal{C}_2}{T}})^2 \tilde{v}(x, T-t)$. In this section, we will focus on the second way with detailed proof below.

1.3.1.3 Proof of Theorem 1.9

Before starting the proof of Theorem 1.9, let us recall the following fundamental result whose proof is the basis of the so called Direct Method of the Calculus of Variations.

Lemma 1.6. *(see [Zu2, Th.1.5.1, p.30])*
If H is a Hilbert space with norm $\|\cdot\|_H$ and the function $J : H \to \mathbb{R}$ is continuous, convex and coercive in H, i.e. it satisfies $J(v) \to \infty$ as $\|v\|_H \to \infty$. Then J attains its minimum at some point $\tilde{v} \in H$. If, moreover, J is strictly convex, this point is unique. If, addition, J is a C^1 function, any minimizer $\tilde{v}$ necessarily satisfies

$$J'(\tilde{v})\zeta = 0, \qquad \forall \zeta \in H. \tag{1.253}$$

Now, we start the proof of Theorem 1.9.

Step 1: Construct a control function.

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the constants from Theorem 1.7. We consider the functional $J_\varepsilon : L^2(\Omega) \to \mathbb{R}$ such that

$$J_\varepsilon(v^0) = \frac{\left(\mathcal{C}_1 e^{\frac{\mathcal{C}_2}{T}}\right)^2}{2} \int_0^T \int_\omega |v(x,t)|^2 dx dt + \frac{\varepsilon^2}{2} \|v^0\|^2_{L^2(\Omega)} + \int_\Omega v(x,T)\varphi^0(x)dx. \tag{1.254}$$

Notice that J_ε is continuous, C^1, strictly convex and coercive. Thus, applying Lemma 1.6 with $H := L^2(\Omega)$ and $J = J_\epsilon$, we can conclude that J_ε has a unique minimizer $\tilde{v}^0$. Moreover, the assertion that $J'_\varepsilon(\tilde{v}^0)\zeta^0 = 0 \;\; \forall \zeta^0 \in H$ implies that the following equality holds for all $\zeta^0 \in L^2(\Omega)$

$$\left(\mathcal{C}_1 e^{\frac{\mathcal{C}_2}{T}}\right)^2 \int_0^T \int_\omega \tilde{v}(x,t)\zeta(x,t) dx dt + \varepsilon^2 \int_\Omega \tilde{v}^0(x)\zeta^0(x)dx + \int_\Omega \zeta(x,T)\varphi^0(x)dx = 0. \tag{1.255}$$

Here, $\tilde{v}$ and ζ are respectively the solution of (1.250) corresponding to initial data $\tilde{v}^0$ and ζ^0. Now, multiplying $\partial_t\varphi - \Delta\varphi = \mathbb{1}_\omega f$ by $\zeta(\cdot, T-t)$ and integrating over Ω, we get

$$\frac{d}{dt}\int_\Omega \varphi(x,t)\zeta(x,T-t)dx = \int_\omega f(x,t)\zeta(x,T-t)dx. \tag{1.256}$$

On one hand, integrating (1.256) over $(0,T)$, we obtain

$$\int_0^T \int_\omega f(x,t)\zeta(x,T-t)dx dt - \int_\Omega \varphi(x,T)\zeta^0(x)dx + \int_\Omega \varphi^0(x)\zeta(x,T)dx = 0. \tag{1.257}$$

Notice that $\int_0^T \int_\omega \tilde{v}(x,t)\zeta(x,t)dxdt = \int_0^T \int_\omega \tilde{v}(x,T-t)\zeta(x,T-t)dxdt$. Thus from (1.255) and (1.257), if we choose $f(x,t) = \left(\mathcal{C}_1 e^{\frac{\mathcal{C}_2}{T}}\right)^2 \tilde{v}(x,T-t)$ then

$$\int_\Omega \left(\varphi(x,T) + \varepsilon^2\tilde{v}^0(x)\right)\zeta^0(x)dx = 0 \;\; \forall \zeta^0 \in L^2(\Omega). \tag{1.258}$$

Hence

$$\varphi(x,T) = -\varepsilon^2 \tilde{v}^0(x). \tag{1.259}$$

Moreover, if we take $\zeta^0 \equiv \tilde{v}^0$, then by uniqueness property of (HP), we get $\zeta \equiv \tilde{v}$. It follows from (1.255) that

$$\left(\mathcal{C}_1 e^{\frac{\mathcal{C}_2}{T}}\right)^2 \int_0^T \int_\omega |\tilde{v}(x,t)|^2 dx dt + \varepsilon^2 \int_\Omega |\tilde{v}^0(x)|^2 dx + \int_\Omega \tilde{v}(x,T)\varphi^0(x)dx = 0. \tag{1.260}$$

Using the Cauchy-Schwarz inequality, that is $\left|\int_\Omega \tilde{v}(x,T)\varphi^0(x)dx\right| \leq \|\tilde{v}(\cdot,T)\|_{L^2(\Omega)}\|\varphi^0\|_{L^2(\Omega)}$, we get

$$\left(\mathcal{C}_1 e^{\frac{\mathcal{C}_2}{T}}\right)^2 \|\tilde{v}\|^2_{L^2(\omega\times(0,T))} + \varepsilon^2\|\tilde{v}^0\|^2_{L^2(\Omega)} \leq \|\tilde{v}(\cdot,T)\|_{L^2(\Omega)}\|\varphi^0\|_{L^2(\Omega)}. \tag{1.261}$$

Applying the result of observability estimate in Theorem 1.7, one gets

$$\|\tilde{v}(\cdot,T)\|^2_{L^2(\Omega)} \leq \left(\mathcal{C}_1 e^{\frac{\mathcal{C}_2}{T}}\right)^2 \|\tilde{v}\|^2_{L^2(\omega\times(0,T))}. \tag{1.262}$$

This estimate implies that

$$\|\tilde{v}(\cdot,T)\|^2_{L^2(\Omega)} \leq \left(\mathcal{C}_1 e^{\frac{\mathcal{C}_2}{T}}\right)^2 \|\tilde{v}\|^2_{L^2(\omega\times(0,T))} + \varepsilon^2\|\tilde{v}^0\|^2_{L^2(\Omega)}. \tag{1.263}$$

Combining (1.261) and (1.263), we obtain

$$\left(\mathcal{C}_1 e^{\frac{\mathcal{C}_2}{T}}\right)^2 \|\tilde{v}\|^2_{L^2(\omega\times(0,T))} + \varepsilon^2\|\tilde{v}^0\|^2_{L^2(\Omega)} \le \|\varphi^0\|_{L^2(\Omega)} \left(\left(\mathcal{C}_1 e^{\frac{\mathcal{C}_2}{T}}\right)^2 \|\tilde{v}\|^2_{L^2(\omega\times(0,T))} + \varepsilon^2\|\tilde{v}^0\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}. \tag{1.264}$$

This is equivalent to

$$\left(\mathcal{C}_1 e^{\frac{\mathcal{C}_2}{T}}\right)^2 \|\tilde{v}\|^2_{L^2(\omega\times(0,T))} + \varepsilon^2\|\tilde{v}^0\|^2_{L^2(\Omega)} \le \|\varphi^0\|^2_{L^2(\Omega)}. \tag{1.265}$$

Let us remind that $f(x,t) = \left(\mathcal{C}_1 e^{\frac{\mathcal{C}_2}{T}}\right)^2 \tilde{v}(x, T-t)$ and $\varphi(x,T) = -\varepsilon^2\tilde{v}^0(x)$. Furthermore, notice that

$$\|\tilde{v}\|^2_{L^2(\omega\times(0,T))} = \|\tilde{v}(\cdot, T-t)\|^2_{L^2(\omega\times(0,T))}. \tag{1.266}$$

Hence, it implies from (1.265) that

$$\frac{1}{\left(\mathcal{C}_1 e^{\frac{\mathcal{C}_2}{T}}\right)^2}\|f\|^2_{L^2(\omega\times(0,T))} + \frac{1}{\varepsilon^2}\|\varphi(\cdot,T)\|^2_{L^2(\Omega)} \le \|\varphi^0\|^2_{L^2(\Omega)}. \tag{1.267}$$

This completes the proof of Theorem 1.9.

1.3.2 Null controllability

1.3.2.1 Introduction and main result

Let us start by the definition of null controllable property.

Definition 1.5. *(see [Co, De.2.39, p.55])*
System (1.247) will be said to be null controllable at time T if, for any $\varphi^0 \in L^2(\Omega)$, there exists a control $f \in L^2(\omega \times (0,T))$ such that the associated state satisfies $\varphi(\cdot, T) = 0$.

This means that for every $\varphi^0 \in L^2(\Omega)$, the set

$$C_{T,\varphi^0} := \{f \in L^2(\omega\times(0,T)) : \text{the solution of (1.247) satisfies } \varphi(\cdot,T) = 0\} \tag{1.268}$$

is nonempty. It leads us to the definition of the cost of null control.

Definition 1.6. *(see [Mi1, De.1.1, p.2])*
The quantity $K(T) := \sup_{\|\varphi^0\|_{L^2(\Omega)}=1} \inf_{f\in C_{T,\varphi^0}} \|f\|_{L^2(\omega\times(0,T))}$ is called the cost of null control at time T.

The following theorem asserts the null controllability at any time $T > 0$ of the system (1.247).

Theorem 1.10. *(see [Co, Th.2.66, p.79])*
The system (1.247) is null controllable at any time $T > 0$.

The cost of null control at time T satisfies $K(T) \le \mathcal{C}_1 e^{\frac{\mathcal{C}_2}{T}}$. Here, the constants $\mathcal{C}_1$ and $\mathcal{C}_2$ are the same in observability estimate (1.9) from Theorem 1.7. It is reasonable from the well-known fact that the null controllability problem for system (1.247) is equivalent to the observability estimate (1.9) for the adjoint system (1.250) (see [Li2] or [Ru]). The null controllability of the heat equation has been extensively investigated for several decades by lots of method. The first result on null controllability of heat equation in one dimension have been obtained by Fattorini and Russell (see [FaR]) by using the moment method. Then the duality approach combined with Carleman estimates has been initiated by the works of Fursikov and Imanuvilov (see [FuI]) and of Lebeau and Robbiano (see [LeR]). Another method is based on the transmutation, which relates the null controllability of the heat equation to the exact controllability of the wave equation (see [Mi2]). Recently, Coron and Nguyen use the backstepping approach to deal with the heat equation with variable coefficients in space in one dimension (see [CoN]). The proof of Theorem 1.10 can be found in references therein. Here, we will only remind the construction of a null control function.

1.3.2.2 Construction of the null control function

One way to construct a null control function is due to Zuazua (see [Zu3]). Firstly, the author defines the following Hilbert space:

$$\mathbb{H} = \left\{ v^0 : \text{the corresponding solution } v \text{ of (1.250) satisfies } \int_0^T \int_\omega |v(x,t)|^2 dxdt < \infty \right\} \quad (1.269)$$

endowed with its norm

$$\|v^0\|_{\mathbb{H}} = \left(\int_0^T \int_\omega |v(x,t)|^2 dxdt \right)^{\frac{1}{2}}. \quad (1.270)$$

Secondly, the author constructs a functional on this Hilbert space, which is

$$\begin{aligned} J : \mathbb{H} &\to \mathbb{R} \\ v^0 &\mapsto \frac{1}{2} \int_0^T \int_\omega |v(x,t)|^2 dxdt + \int_\Omega v(x,0)\varphi^0(x)dx. \end{aligned} \quad (1.271)$$

Here, v is the solution of (1.250) corresponding to v^0.
Thirdly, the control function is constructed based on the minimizer of J over $\mathbb{H}$. Precisely, let $\tilde{v}^0$ be the minimizer of J over $\mathbb{H}$ and $\tilde{v}$ be the solution of (1.250) corresponding to the initial data $\tilde{v}^0$. Then the null control function is constructed as $f = \tilde{v}$.

Another way to construct the null control function, which can avoid working in the space $\mathbb{H}$ (see also [Zu3] or [BuP, p. 23]), is based on null approximate controllability. Precisely, they build a sequence of null approximate controls f_ε depending on arbitrary $\varepsilon > 0$ (see Subsection 1.3.1.2). In more detail, for any $\varepsilon > 0$, let $\tilde{v}^0_\varepsilon$ be the minimizer of the following functional

$$\begin{aligned} J_\varepsilon : L^2(\Omega) &\to \mathbb{R} \\ v^0 &\mapsto \frac{\left(\mathcal{C}_1 e^{\frac{\mathcal{C}_2}{T}}\right)^2}{2} \int_0^T \int_\omega |v(x,t)|^2 dxdt + \frac{\varepsilon^2}{2}\|v^0\|^2_{L^2(\Omega)} + \int_\Omega v(x,T)\varphi^0(x)dx. \end{aligned} \quad (1.272)$$

Moreover, thanks to the observability estimate, the sequence $\{f_\varepsilon\}$ is uniformly bounded in $L^2(\omega \times (0,T))$. Hence, by extracting subsequences, we have f_ε converges weakly to f in $L^2(\omega \times (0,T))$. The limit control f fullfils the null controllability requirement.

1.3.3 Null approximate impulse controllability

1.3.3.1 Introduction and main result

Now we study another issue of control theory, *null approximate impulse controllability*, where the control function also acts on a subdomain ω but at one point of time $\tau \in (0,T)$ (see more in [MiR] or [QiW]). Consider the following system

$$\begin{cases} \partial_t \psi - \Delta \psi = 0, & \text{in } \Omega \times (0,T) \setminus \{\tau\}, \\ \psi = 0, & \text{on } \partial\Omega \times (0,T), \\ \psi(\cdot,0) = \psi^0, & \text{in } \Omega, \\ \psi(\cdot,\tau) = \psi(\cdot,\tau^-) + \mathbb{1}_\omega h, & \text{in } \Omega, \end{cases} \quad (1.273)$$

where $\psi(\cdot,\tau^-)$ denotes the left limit of the function ψ at time τ.

Definition 1.7. *(see [QiW, De.1.2, p.3])*
System (1.273) will be said to be null approximate impulse controllable at time T if, for any $\varepsilon > 0$, any $\psi^0 \in L^2(\Omega)$, there exists a control $h \in L^2(\omega)$ such that the associated state at final time satisfies $\|\psi(\cdot,T)\|_{L^2(\Omega)} \le \varepsilon \|\psi^0\|_{L^2(\Omega)}$.

This means that for every $\varepsilon > 0$, for every $\varphi^0 \in L^2(\Omega)$, the set

$$C_{T,\psi^0,\varepsilon} := \{h \in L^2(\omega) : \text{the solution of (1.273) satisfies } \|\psi(\cdot,T)\|_{L^2(\Omega)} \leq \varepsilon\|\psi^0\|_{L^2(\Omega)}\} \quad (1.274)$$

is nonempty. It leads us to the definition of the cost of null approximate impulse control

Definition 1.8. *The quantity* $K(T,\varepsilon) := \sup_{\|\psi^0\|_{L^2(\Omega)}=1} \inf_{f \in C_{T,\psi^0,\varepsilon}} \|h\|_{L^2(\omega)}$ *is called the cost of null approximate impulse control at time* T.

Now, we state a main result of null approximate impulse controllability for the system (1.247).

Theorem 1.11. *(see [PhWX, Th.3.1, p.5021])*
The system (1.273) is null approximate impulse controllable at any time $T > 0$.

Moreover, for any $\varepsilon > 0$, the cost of null approximate impulse control function at time T satisfies $K(T,\varepsilon) \leq \frac{\mathcal{M}_1 e^{\frac{\mathcal{M}_2}{T-\tau}}}{\varepsilon^\theta}$. Here, the positive constants $\mathcal{M}_1$, $\mathcal{M}_2$ and θ are from the estimate (1.152) of Lemma 1.5 (see Subsection 1.2.4).

1.3.3.2 Proof of main result

Fix $\varepsilon > 0$, put $\kappa := \frac{\mathcal{M}_1 e^{\frac{\mathcal{M}_2}{T-\tau}}}{\varepsilon^\theta}$ where the constants $\mathcal{M}_1$, $\mathcal{M}_2$ and θ are from Lemma 1.5. We consider the functional $J_\varepsilon : L^2(\Omega) \to \mathbb{R}$ such that

$$J(v^0) = \frac{\kappa^2}{2}\|v(\cdot,T-\tau)\|^2_{L^2(\omega)} + \frac{\varepsilon^2}{2}\|v^0\|^2_{L^2(\Omega)} + \int_\Omega \psi^0(x)v(x,T)dx; \quad (1.275)$$

where $v(x,t)$ is the solution of the following system

$$\begin{cases} \partial_t v - \Delta v = 0 & \text{in } \Omega \times (0,T), \\ v = 0 & \text{on } \partial\Omega \times (0,T), \\ v(\cdot,0) = v^0 \in L^2(\Omega). \end{cases} \quad (1.276)$$

Notice that J is a strictly convex, C^1 and coercive, i.e $J(v^0) \to \infty$ when $\|v^0\|_{L^2(\Omega)} \to \infty$. Therefore, thanks to Lemma 1.6, J has a unique minimizer $\tilde{v}^0 \in L^2(\Omega)$ such that $J(\tilde{v}^0) = \min_{v^0 \in L^2(\Omega)} J(v^0)$. It implies that $J'(\tilde{v}^0)\zeta^0 = 0$ for any $\zeta^0 \in L^2(\Omega)$, i.e the following estimate holds for any ζ^0

$$\kappa^2 \int_\omega \tilde{v}(x,T-\tau)\zeta(x,T-\tau)dx + \varepsilon^2 \int_\Omega \tilde{v}^0(x)\zeta^0(x)dx + \int_\Omega \psi^0(x)\zeta(x,T)dx = 0; \quad (1.277)$$

where $\tilde{v}$ and ζ are respectively the solution of (1.276) corresponding to $\tilde{v}^0 := \tilde{v}(\cdot,0)$ and $\zeta^0 := \zeta(\cdot,0)$. Multiplying $\partial_t\psi - \Delta\psi = 0$ by $\zeta(\cdot,T-t)$ and integrating over Ω, one gets

$$\frac{d}{dt}\int_\Omega \psi(x,t)\zeta(x,T-t)dx = 0. \quad (1.278)$$

Integrating (1.278) over $(0,\tau)$ gives us

$$\int_\Omega \psi(x,0)\zeta(x,T)dx = \int_\Omega \psi(x,\tau^-)\zeta(x,T-\tau)dx. \quad (1.279)$$

Integrating (1.278) over (τ,T) gives us

$$\int_\Omega \psi(x,\tau)\zeta(x,T-\tau)dx = \int_\Omega \psi(x,T)\zeta^0(x)dx. \quad (1.280)$$

Combining (1.279), (1.280) and the fact $\psi(\cdot,\tau) = \psi(\cdot,\tau^-) + \mathbb{1}_\omega h_i$, one obtains

$$\int_\Omega \psi(x,T)\zeta^0(x)dx = \int_\Omega \psi^0(x)\zeta(x,T)dx + \int_\omega h(x)\zeta(x,T-\tau)dx. \quad (1.281)$$

The estimate (1.281) can be written as

$$\int_\omega h(x)\zeta(x,T-\tau)dx - \int_\Omega \psi(x,T)\zeta^0(x)dx + \int_\Omega \psi^0(x)\zeta(x,T)dx = 0. \tag{1.282}$$

Thus from (1.277) and (1.282), if we choose $h(x) = \kappa^2\tilde{v}(x,T-\tau)$ then

$$\int_\Omega \left(\psi(x,T) + \varepsilon^2\tilde{v}^0(x)\right)\zeta^0(x)dx = 0 \quad \forall\zeta^0 \in L^2(\Omega).$$

Hence, $\psi(x,T) = -\varepsilon^2\tilde{v}^0(x)$. Moreover, with $\zeta^0 \equiv \tilde{v}^0$, using the Cauchy-Schwarz inequality, it implies from (1.277) that

$$\kappa^2\|\tilde{v}(\cdot,T-\tau)\|^2_{L^2(\omega)} + \varepsilon^2\|\tilde{v}^0\|^2_{L^2(\Omega)} \leq \|\psi^0\|_{L^2(\Omega)}\|\tilde{v}(\cdot,T)\|_{L^2(\Omega)}. \tag{1.283}$$

Thanks to the energy estimate for the adjoint system (1.276), which is

$$\|\tilde{v}(\cdot,T)\|_{L^2(\Omega)} \leq \|\tilde{v}(\cdot,T-\tau)\|_{L^2(\Omega)}, \tag{1.284}$$

we obtain

$$\kappa^2\|\tilde{v}(\cdot,T-\tau)\|^2_{L^2(\omega)} + \varepsilon^2\|\tilde{v}^0\|^2_{L^2(\Omega)} \leq \|\psi^0\|_{L^2(\Omega)}\|\tilde{v}(\cdot,T-\tau)\|_{L^2(\Omega)}. \tag{1.285}$$

Applying the result in Lemma 1.5, which is

$$\|\tilde{v}(\cdot,T-\tau)\|^2_{L^2(\Omega)} \leq \kappa^2\|\tilde{v}(\cdot,T-\tau)\|^2_{L^2(\omega)} + \varepsilon^2\|\tilde{v}(\cdot,0)\|^2_{L^2(\Omega)}, \tag{1.286}$$

one has

$$\kappa^2\|\tilde{v}(\cdot,T-\tau)\|^2_{L^2(\omega)} + \varepsilon^2\|\tilde{v}^0\|^2_{L^2(\Omega)} \leq \|\psi^0\|^2_{L^2(\Omega)}. \tag{1.287}$$

This is equivalent to

$$\frac{1}{\kappa^2}\|h\|^2_{L^2(\omega)} + \frac{1}{\varepsilon^2}\|\psi(\cdot,T)\|^2_{L^2(\Omega)} \leq \|\psi^0\|^2_{L^2(\Omega)}. \tag{1.288}$$

This completes the proof of Theorem 1.11.

Chapter 2

Null controllability for cubic semilinear heat equation

In this Chapter, we consider the null controllability problem for the cubic semilinear heat equation in bounded domains Ω of $\mathbb{R}^3$, with Dirichlet boundary conditions for small initial data. A constructive way to compute a control function acting on any nonempty open subset ω of Ω is given such that the corresponding solution of the cubic semilinear heat equation can be driven to zero at a given final time T. Furthermore, we provide a quantitative estimate for the smallness of the size of the initial data with respect to T that ensures the null controllability property. The structure of this Chapter is given as below:

Section 2.1: We introduce our problem with locally well-posedness and blow up phenomenon (see Subsection 2.1.1). Then, we prove two main results (see Subsection 2.1.2): One is the locally null controllability for a blow up system under the smallness of initial data in $H_0^1(\Omega)$ (see Theorem 2.1); The other one is the locally null controllability for a non blow up system under the smallness of initial data in $L^2(\Omega)$ (see Corollary 2.1) . Finally, we remind some of relevant works for semilinear null controllability (see Subsection 2.1.3).

Section 2.2: We study the null controllability for a linear system with an outside force for two cases (see Subsection 2.2.1): The initial data belongs to $L^2(\Omega)$ (Theorem 2.2) and the initial data belongs to $H_0^1(\Omega)$ (Corollary 2.2). The method is based on the iterative algorithm, whose idea comes from [LiTT]. This method is completely different from the previous works based on Carleman estimate and can be applied for other nonlinear parabolic systems. The readers can see Subsection 2.2.2 for the detailed proof.

Section 2.3: We focus on the proof of the main results: The proof of Theorem 2.1 is given in Subsection 2.3.1 and the proof of Corollary 2.1 is presented in Subsection 2.3.2.

Section 2.4: We recall some preliminary results which are used for our main proofs, such as: The Sobolev embedding (see Subsection 2.4.1); The Banach fixed point theorem (see Subsection 2.4.2) or The classical estimates (see Subsection 2.4.3).

2.1 Introduction and main results

2.1.1 Problem

Let Ω be an open bounded domain in $\mathbb{R}^3$ with a boundary $\partial\Omega$ of class C^2 and $T > 0$. We consider the cubic semilinear heat equation complemented with initial and Dirichlet boundary condition, which has the following form

$$\begin{cases} \partial_t y - \Delta y + \gamma y^3 = \mathbb{1}_\omega f & \text{in} \quad \Omega \times (0,T) \ , \\ y = 0 & \text{on} \quad \partial\Omega \times (0,T) \ , \\ y(\cdot,0) = y^0 & \text{in} \quad \Omega \ , \end{cases} \tag{2.1}$$

where $\gamma \in \{1,-1\}$, $\mathbb{1}_\omega$ denotes the characteristic function of ω and f denotes the control function acting on $\omega \times (0,T)$.
Now we consider the well-posedness of the uncontrolled system.

When $\gamma = 1$, i.e we consider the following system

$$\begin{cases} \partial_t y - \Delta y = -y^3 & \text{in} \quad \Omega \times (0,T) \ , \\ y = 0 & \text{on} \quad \partial\Omega \times (0,T) \ , \\ y(\cdot,0) = y^0 & \text{in} \quad \Omega. \end{cases} \tag{2.2}$$

It is well-known (see [Li1, p.6]) that if $y^0 \in H_0^1(\Omega)$, there exists a unique solution $y \in L^2(0,T;H^2(\Omega))$ such that $\partial_t y \in L^2(\Omega \times (0,T))$ and y satisfies the system (2.2).

When $\gamma = -1$, we consider the following system

$$\begin{cases} \partial_t y - \Delta y = y^3 & \text{in} \quad \Omega \times (0,T) \ , \\ y = 0 & \text{on} \quad \partial\Omega \times (0,T) \ , \\ y(\cdot,0) = y^0 & \text{in} \quad \Omega. \end{cases} \tag{2.3}$$

It is well-known ([BrC1, Th.3.1.1, p.1]) that if $y^0 \in L^\infty(\Omega)$, there exists a unique solution y of (2.3), defined on a maximal time interval $[0,T_m)$, i.e $y \in L^\infty(\Omega \times (0,T))$ for all $T < T_m$. Moreover, we have the blow up in finite time phenomenon, i.e $T_m < +\infty$ and $\lim\limits_{t \to T_m} \|y(\cdot,t)\|_{L^\infty(\Omega)} = \infty$. The question what happens if $y_0 \notin L^\infty(\Omega)$ is considered by Brezis and Cazenave (see [BrC2]). Let us assume that $y_0 \in L^q(\Omega)$ for some $1 \le q < \infty$. The existence and uniqueness of solutions depend on the relationship between q and the dimension of domain Ω. Precisely

1. If $q \ge 3$ then there exists time $T(y^0) > 0$ and a unique solution $y \in C([0,T(y^0)],L^q(\Omega))$ satisfying the system (2.3) (see [BrC1, Th.1, p.278]).
2. If $q = 1$ or $q = 2$ then the well-posednesss of system (2.3) is still an open problem.

On the other hand, Brezis and Cazenave also prove that: When y^0 is small enough, the problem is global well-posed by energy method (see [BrC1, Th.3.4.1]) or by comparasion method (see [BrC1, Th.3.4.5]). In contrast, [BrC1, Th.3.6.1] also says that the solution will blow up in finite time when y^0 is big enough. Here, under a smallness condition on initial data in $H_0^1(\Omega)$, we will provide a null controllability result for the control system (2.2) when $\gamma = 1$ (non blow up case) or $\gamma = -1$ (blow up case) (see Theorem 2.1). Furthermore, in case non blow up $\gamma = 1$, we also can get the same result but with a weaker assumption on initial data, that is $y^0 \in L^2(\Omega)$ (see Corollary 2.1).

2.1.2 Main results

Firstly, let us state our first main result which asserts the local null controllability for system (2.1).

Theorem 2.1. *For any $T > 0$, suppose that $y^0 \in H_0^1(\Omega)$ satisfies*

$$\|y^0\|^2_{H_0^1(\Omega)} < \max_{(0,T]} \frac{1}{G(1+\sqrt{t})^{10} e^{\frac{G}{t}}} \ , \tag{2.4}$$

for some constant $G > 1$. Then there exists a control function $f \in L^2(\omega \times (0,T))$ such that the solution of (2.1) corresponding to y^0 satisfies $y(\cdot,T) = 0$. Furthermore, the control can be computed explicitly and the construction of the control is given below.

Remark 2.1. *1/ Theorem 2.1 ensures the local null controllability of (2.1) for any control set ω, any small enough initial data $y^0 \in H_0^1(\Omega)$, at any time T. It is well-known that the system (2.1) without control function blows up in finite time for the case $\gamma = -1$. But thanks to an appropriate control function, Theorem 2.1 affirms that the blow-up phenomena can be prevented for very specific initial data.*

2/ An important achievement of our result is that we can construct the control function. An outline of the construction is described as follows: Firstly, we remind the construction of the null approximate control for the linear heat equation with an estimate of the cost (see Subsection 1.3.1.2 in Chapter 1); Secondly, from the previous result, we do similarly when adding an outside force using the method of Y. Liu, T. Takahashi and M. Tucsnak in [LiTT], the solution will be forced to be null at time T by adding an exponential weight function; Lastly, thanks to an appropriate iterative fixed point process and linearization by replacing the outside force by cubic function, the desired control is constructed, but the result is only local, i.e. the initial condition must be small enough. The precise construction of the control function is found in the proof of Theorem 2.1.

3/ Another main achievement of our result is to give a quantitative estimate for the smallness of the size of the initial condition with respect to the control time T. The upper bound of initial data is a function with respect to the final control time T, which obviously increases to a certain value and then keeps to be a constant until T tends to ∞.

Another interesting problem is to study the case where the blow-up phenomena will not occur (see [AnT]), for example when $\gamma = 1$. Our method gives the following result:

Corollary 2.1. *For any $T > 0$, suppose that $y^0 \in L^2(\Omega)$ satisfying*

$$\|y^0\|^2_{L^2(\Omega)} < \max_{(0,T]} \frac{T}{G(1+\sqrt{t})^{10} e^{\frac{G}{t}}} \ , \tag{2.5}$$

for some constant $G > 1$. Then there exists a control function $f \in L^2(\omega \times (0,T))$ such that the solution of (2.1) with $\gamma = 1$ corresponding to y^0 satisfies $y(\cdot,T) = 0$.

2.1.3 State of art

We now review the achievements of controllability for the nonlinear heat equation which has been intensively studied in the past. Let Ω be an open, bounded domain in $\mathbb{R}^n$ $(n \geq 1)$. We consider the heat equation in the following form:

$$\begin{cases} \partial_t y - \Delta y = \mathbb{F}(y) + \mathbb{1}_\omega f & \text{in} \quad \Omega \times (0,T) \ , \\ y = 0 & \text{on} \quad \partial\Omega \times (0,T) \ , \\ y(\cdot,0) = y^0 & \text{in} \quad \Omega \ . \end{cases} \tag{2.6}$$

- For linear case, i.e $\mathbb{F} \equiv 0$, this issue is considered in Chapter 1 (see Section 1.3.2).
- For sublinear case, i.e $|\mathbb{F}(s)| \leq C(1+|s|)$ $\forall s \in \mathbb{R}$ for some $C > 0$, the system (2.6) is global null controllable. The first writing on this issue derives from A. Fursikov and O. Imanuvilov (see [FuI] or [Em]). Their method is based on the Schauder's fixed point theorem.

- For superlinear case, for example $\mathbb{F}(s) = |s|^p s$ with $p > 1$, we consider two cases when the blow-up phenomena occur or not:
  - ☐ For dissipative semilinear case when there is no blow-up phenomena, the system (2.6) is local null controllability. In [Ba], the author studies on the case when
$$|\mathbb{F}(s)| \leq C|s|(1 + |s|^\alpha) \tag{2.7}$$
where $\alpha > 0$ if $n = 1, 2$ and $\alpha = \frac{1}{n-1}$ if $n > 2$. The author can prove the system (2.6) is null controllable, under some assumption on initial data, which depends on C, α and T. In addition, S. Anita and D.Tataru [AnT] consider the system (2.6) with $\mathbb{F}$ has a good sign, i.e $s\mathbb{F}(s) \geq 0 \quad \forall s \in \mathbb{R}$. The authors can provide sharp estimates for the controllability time in terms of the size of the initial data.
  - ☐ For blowing-up semilinear heat equation, E. Fernández-Cara and E. Zuazua establish the first result in the literature on the null controllability of system (2.6) (see [FeZ1] or [Fe1]). In detail, they prove that the system (2.6) is global null controllable at any time provided if the nonlinear term $\mathbb{F}(s)$ grows slower than $|s|\log^{\frac{3}{2}}(1 + |s|)$ as $|s| \to \infty$, i.e
$$\lim_{|s| \to +\infty} \frac{|\mathbb{F}(s)|}{|s| \ln^{\frac{3}{2}}(1 + |s|)} = 0 \tag{2.8}$$
Furthermore, they observe that it is not possible to obtain a global controllability result for a cubic nonlinear term. Generally, for some functions that behave at infinity like $|s|\ln^p(1 + |s|)$ with $p > 2$, the null controllability does not hold (see also [Fe2] or [FuI]).

Recently, in [LiTT], Liu, Takahashi and Tucsnak introduce a new methodology which can be used for studying the null controllability of nonlinear parabolic systems. This method is as follows: Firstly, they construct a new iterative algorithm for the null controllability of linear parabolic equations in the presence of source terms; Secondly, by using a fixed point method, they obtain the null controllability for a nonlinear system. In previous works ([FuI] or [Em]), the authors use a linearized problem with time dependent coefficients, which is solved by using the global Carleman estimates. Being independent on this techniques, the linearized problem in [LiTT] is with constant coefficients. Hence, the spectral calculators can be used and their method can be applied for other nonlinear systems.

2.2 Null controllability for linear case with outside force

2.2.1 Main results

We consider the linear heat equation with the outside force, which has the following form
$$\begin{cases} \partial_t \phi - \Delta \phi = g + \mathbb{1}_\omega f & \text{in} \quad \Omega \times (0, T) \ , \\ \phi = 0 & \text{on} \quad \partial\Omega \times (0, T) \ , \\ \phi(\cdot, 0) = \phi^0 & \text{in} \quad \Omega \ . \end{cases} \tag{2.9}$$
For the moment, we choose $\phi^0 \in L^2(\Omega)$ and $g \in L^2(\Omega \times (0, T))$. In this section, our target is constructing a control function $f \in L^2(\omega \times (0, T))$ such that the solution of system (2.9) satisfies $\phi(\cdot, T) = 0$. By using the iterative algorithm in [LiTT], we divide our time into small intervals. On each divided interval of time, we again divide our problem (2.9) into two problems: The first one is the linear system with outside force but without control (see (2.11)); The second one is the basic linear system without outside force but with control (see (2.13)) such that the final data of one problem is the initial data of the other. The first problem is known well-posed and the second problem is the null approximate controllability for linear heat equation, which has already been studied in Section 1.3.1. Now, let us introduce some notations before we state the main Theorem in this Section.

Let $\{T_k\}_{k \geq 0}$ be the sequence of real positive numbers given by
$$T_k = T - \frac{T}{a^k} \ , \tag{2.10}$$

where $a > 1$ will be chosen later. Put $g_k = \mathbb{1}_{[T_k,T_{k+1}]}g$. We start to describe the algorithm to construct the control: We initiate with $\chi_0 = \phi^0$ and $\varphi_{-1} = 0$. Define the sequences $\{z_k\}_{k\geq 0}$, $\{\varphi_k\}_{k\geq 0}$ and $\{\phi_k\}_{k\geq 0}$ as follows:

1/ Let z_k be the solution of

$$\begin{cases} \partial_t z_k - \Delta z_k = g_k & \text{in} \quad \Omega \times (T_k, T_{k+1}) \ , \\ z_k = 0 & \text{on} \quad \partial\Omega \times (T_k, T_{k+1}) \ , \\ z_k(\cdot, T_k) = \varphi_{k-1}(\cdot, T_k) & \text{in} \quad \Omega \ . \end{cases} \tag{2.11}$$

It is well-known that with $\varphi_{k-1}(\cdot, T_k) \in L^2(\Omega)$, (2.11) is well-posed (see [Br] or [LiM]). Hence, let us introduce

$$\chi_{k+1} = z_k(\cdot, T_{k+1}) \ . \tag{2.12}$$

2/ Let φ_k be the solution of

$$\begin{cases} \partial_t \varphi_k - \Delta \varphi_k = \mathbb{1}_\omega f_k & \text{in} \quad \Omega \times (T_k, T_{k+1}) \ , \\ \varphi_k = 0 & \text{on} \quad \partial\Omega \times (T_k, T_{k+1}) \ , \\ \varphi_k(\cdot, T_k) = \chi_k & \text{in} \quad \Omega \ . \end{cases} \tag{2.13}$$

Theorem 1.9 says that the system (2.13) is null approximate controllable at any time T_{k+1}. Moreover, for any $\varepsilon_k > 0$, any $\chi_k \in L^2(\Omega)$, there exists $f_k \in L^2(\omega \times (T_k, T_{k+1}))$ such that

$$\frac{1}{\left(Ce^{\frac{C}{T_{k+1}-T_k}}\right)^2} \int_{T_k}^{T_{k+1}} \int_\omega |f_k(x,t)|^2 dxdt + \frac{1}{\varepsilon_k^2} \int_\Omega |\varphi_k(x, T_{k+1})|^2 dx \leq \|\chi_k\|^2_{L^2(\Omega)} \tag{2.14}$$

for some positive constant C. Precisely, the control function f_k is constructed as below (see more in Subsection 1.3.1.2)

$$f_k(x,t) = \left(Ce^{\frac{C}{T_{k+1}-T_k}}\right)^2 \tilde{v}_k(x, T_{k+1} + T_k - t). \tag{2.15}$$

Here $\tilde{v}_k$ is the solution of the following system

$$\begin{cases} \partial_t v_k - \Delta v_k = 0 & \text{in} \quad \Omega \times (T_k, T_{k+1}) \ , \\ v_k = 0 & \text{on} \quad \partial\Omega \times (T_k, T_{k+1}) \ , \\ v_k(\cdot, T_k) = v_k^0 & \text{in} \quad \Omega \ , \end{cases} \tag{2.16}$$

corresponding to the initial data $\tilde{v}_k^0$, which is the unique minimizer of the following functional depending on $\varepsilon_k > 0$: $J_{\varepsilon_k} : L^2(\Omega) \to \mathbb{R}$ such that

$$J_{\varepsilon_k}(v_k^0) = \frac{\left(Ce^{\frac{C}{T_{k+1}-T_k}}\right)^2}{2} \int_{T_k}^{T_{k+1}} \int_\omega |v_k(x,t)|^2 dxdt + \frac{\varepsilon_k^2}{2} \int_\Omega |v_k^0(x)|^2 dx + \int_\Omega v_k(x, T_{k+1}) \chi_k(x) dx.$$

Here, v_k is the solution of (2.16) corresponding to the initial data v_k^0. Furthermore, we also have (see (1.259) in Subsection 1.3.1.3)

$$\varphi_k(x, T_{k+1}) = -\varepsilon_k^2 \tilde{v}_k^0(x). \tag{2.17}$$

3/ Let $\phi_k = z_k + \varphi_k$, then it solves

$$\begin{cases} \partial_t \phi_0 - \Delta \phi_0 = g_0 + \mathbb{1}_\omega f_0 & \text{in} \quad \Omega \times (T, T_1) \ , \\ \phi_0 = 0 & \text{on} \quad \partial\Omega \times (T, T_1) \ , \\ \phi_0(\cdot, 0) = \phi^0 & \text{in} \quad \Omega \end{cases} \tag{2.18}$$

and

$$\begin{cases} \partial_t \phi_{k+1} - \Delta \phi_{k+1} = g_{k+1} + \mathbb{1}_\omega f_{k+1} & \text{in} \quad \Omega \times (T_{k+1}, T_{k+2}) \ , \\ \phi_{k+1} = 0 & \text{on} \quad \partial\Omega \times (T_{k+1}, T_{k+2}) \ , \\ \phi_{k+1}(\cdot, T_{k+1}) = \varphi_k(\cdot, T_{k+1}) + \chi_{k+1} & \text{in} \quad \Omega \ . \end{cases} \tag{2.19}$$

Notice that $\phi_k(\cdot, T_{k+1}) = \phi_{k+1}(\cdot, T_{k+1})$, therefore the functions $\phi = \sum_{k\geq 0} \mathbb{1}_{[T_k, T_{k+1}]}\phi_k$ is continous on $[0, T]$.

Our main results below will assert that the function $f := \sum_{k\geq 0} \mathbb{1}_{[T_k, T_{k+1}]} f_k$ leads the solution of the system (2.9) from any given ϕ^0 at time 0 to null at time T. Now we are able to state our result.

Theorem 2.2. *For any $\phi^0 \in L^2(\Omega)$, any $a > 1$, any g satisfying $ge^{\frac{2a^2C}{a-1}\frac{1}{T-t}} \in L^2(\Omega \times (0, T))$ for some positive constant C, there exists a control $f \in L^2(\omega \times (0, T))$ such that the solution of (2.9) corresponding to ϕ^0 satisfies $\phi(\cdot, T) = 0$. Furthermore, there exists a positive constant $K > 1$ such that the following estimate holds:*

$$\begin{aligned} & \|\phi e^{\frac{C}{a-1}\frac{1}{T-t}}\|_{C([0,T];L^2(\Omega))} + \|f e^{\frac{C}{a-1}\frac{1}{T-t}}\|_{L^2(\omega\times(0,T))} \\ \leq \quad & K\,(1+T)\left[e^{\frac{2C}{a-1}\frac{1}{T}}\|\phi^0\|_{L^2(\Omega)} + \|ge^{\frac{2a^2C}{a-1}\frac{1}{T-t}}\|_{L^2(\Omega\times(0,T))}\right] . \end{aligned} \tag{2.20}$$

Here, $f = \sum_{k\geq 0} \mathbb{1}_{[T_k, T_{k+1}]} f_k$ where f_k is constructed in (2.15).

When $\phi^0 \in H_0^1(\Omega)$, we have a corollary from Theorem 2.2 as below:

Corollary 2.2. *For any $\phi^0 \in H_0^1(\Omega)$, any g satisfying $ge^{\frac{3D}{T-t}} \in L^2(\Omega \times (0, T))$ for some positive constant D, there exists a control $f \in L^2(\omega \times (0, T))$ such that the solution of (2.9) corresponding to ϕ^0 satisfies $\phi(\cdot, T) = 0$. Furthermore, there exists a positive constant $K > 1$ such that the following estimate holds:*

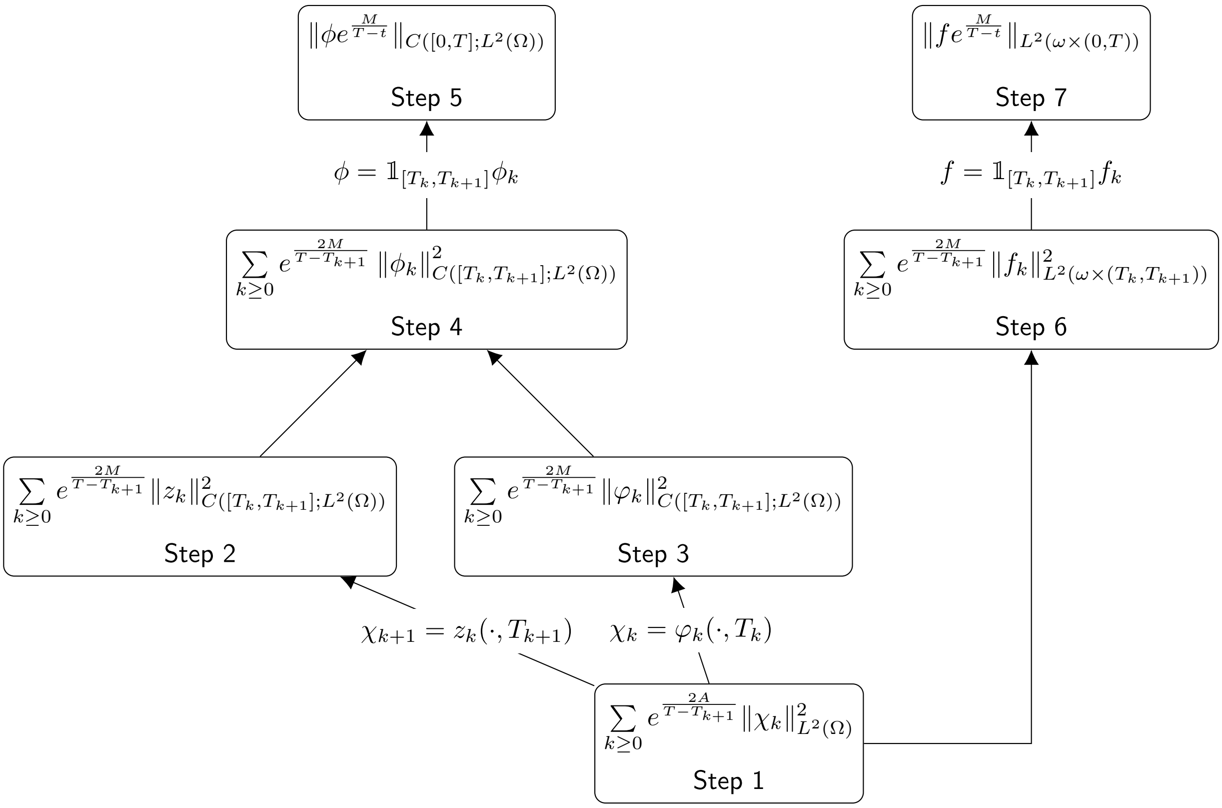
$$\begin{aligned} & \|\nabla\phi e^{\frac{D}{T-t}}\|_{C([0,T];L^2(\Omega))} + \|f e^{\frac{D}{T-t}}\|_{L^2(\omega\times(0,T))} \\ \leq \quad & K\left(1+\sqrt{T}\right)^3\left[e^{\frac{3D}{T}}\|\nabla\phi^0\|_{L^2(\Omega)} + \|ge^{\frac{3D}{T-t}}\|_{L^2(\Omega\times(0,T))}\right] . \end{aligned} \tag{2.21}$$

Here, the control function f comes from Theorem 2.2.

2.2.2 Proof of main results

2.2.2.1 Proof of Theorem 2.2

Sketch of proof of Theorem 2.2



Proof of Theorem 2.2

Step 1: Estimate $\sum_{k\geq 0} e^{\frac{2A}{T-T_{k+1}}}\|\chi_k\|^2_{L^2(\Omega)}$ for any $A > 0$.
Remind that $\chi_0 := \phi^0$ and $\chi_{k+1} := z_k(\cdot, T_{k+1})$ for $k \geq 0$.

Step 1.1: Estimate $\|z_k\|^2_{C([T_k,T_{k+1}];L^2(\Omega))}$ for $k \geq 0$.
Applying energy estimate (see Theorem 2.5) for the system (2.11), one has

$$\|z_0\|^2_{C([T_0,T_1];L^2(\Omega))} \leq T\|g_0\|^2_{L^2(\Omega\times(T_0,T_1))} \tag{2.22}$$

and

$$\|z_{k+1}\|^2_{C([T_{k+1},T_{k+2}];L^2(\Omega))} \leq 2T\|g_{k+1}\|^2_{L^2(\Omega\times(T_{k+1},T_{k+2}))} + 2\|\varphi_k(\cdot,T_{k+1})\|^2_{L^2(\Omega)} \quad \forall k \geq 0. \tag{2.23}$$

On the other hand, it implies from (2.14) for the system (2.13) that

$$\|\varphi_k(\cdot,T_{k+1})\|^2_{L^2(\Omega)} \leq \varepsilon_k^2\|\chi_k\|^2_{L^2(\Omega)} \quad \forall k \geq 0. \tag{2.24}$$

Combining (2.23) and (2.24) gives us

$$\|z_{k+1}\|^2_{C([T_{k+1},T_{k+2}];L^2(\Omega))} \leq 2T\|g_{k+1}\|^2_{L^2(\Omega\times(T_{k+1},T_{k+2}))} + 2\varepsilon_k^2\|\chi_k\|^2_{L^2(\Omega)} \quad \forall k \geq 0. \tag{2.25}$$

Step 1.2: Estimate $\|\chi_k\|^2_{L^2(\Omega)}$ for $k \geq 0$.
We have

$$\|\chi_0\|^2_{L^2(\Omega)} = \|\phi^0\|^2_{L^2(\Omega)}, \tag{2.26}$$

$$\|\chi_1\|^2_{L^2(\Omega)} \leq T\|g_0\|^2_{L^2(\Omega\times(0,T_1))} \tag{2.27}$$

and

$$\|\chi_{k+2}\|^2_{L^2(\Omega)} \leq 2T\|g_{k+1}\|^2_{L^2(\Omega\times(T_{k+1},T_{k+2}))} + 2\varepsilon_k^2\|\chi_k\|^2_{L^2(\Omega)} \quad \forall k \geq 0\ . \tag{2.28}$$

Step 1.3: Estimate $\sum_{k\geq 0} e^{\frac{2A}{T-T_{k+1}}}\|\chi_k\|^2_{L^2(\Omega)}$ for any $A > 0$.

For any constant $A > 0$, we get

$$\begin{aligned}
&\sum_{k\geq 0} e^{\frac{2A}{T-T_{k+1}}}\|\chi_k\|^2_{L^2(\Omega)}\\
= \quad & e^{\frac{2A}{T-T_1}}\|\chi_0\|^2_{L^2(\Omega)} + e^{\frac{2A}{T-T_2}}\|\chi_1\|^2_{L^2(\Omega)} + \sum_{k\geq 0} e^{\frac{2A}{T-T_{k+3}}}\|\chi_{k+2}\|^2_{L^2(\Omega)}\\
\leq \quad & e^{\frac{2A}{T-T_1}}\|\phi^0\|^2_{L^2(\Omega)} + e^{\frac{2A}{T-T_2}}T\|g_0\|^2_{L^2(\Omega\times(0,T_1))}\\
& +2T\sum_{k\geq 0} e^{\frac{2A}{T-T_{k+3}}}\|g_{k+1}\|^2_{L^2(\Omega\times(T_{k+1},T_{k+2}))} + 2\sum_{k\geq 0} e^{\frac{2A}{T-T_{k+3}}}\varepsilon_k^2\|\chi_k\|^2_{L^2(\Omega)}\\
\leq \quad & e^{\frac{2A}{T-T_1}}\|\phi^0\|^2_{L^2(\Omega)} + 2T\sum_{k\geq 0} e^{\frac{2A}{T-T_{k+2}}}\|g_k\|^2_{L^2(\Omega\times(T_k,T_{k+1}))} + 2\sum_{k\geq 0} e^{\frac{2A}{T-T_{k+3}}}\varepsilon_k^2\|\chi_k\|^2_{L^2(\Omega)}.
\end{aligned} \tag{2.29}$$

Using the fact that $T - T_{k+2} = \frac{T-T_{k+1}}{a}$ and $T - T_{k+3} = \frac{T-T_{k+1}}{a^2}$, one gets

$$\begin{aligned}
&\sum_{k\geq 0} e^{\frac{2A}{T-T_{k+1}}}\|\chi_k\|^2_{L^2(\Omega)}\\
\leq \quad & e^{\frac{2A}{T-T_1}}\|\phi^0\|^2_{L^2(\Omega)} + 2T\sum_{k\geq 0} e^{\frac{2aA}{T-T_{k+1}}}\|g_k\|^2_{L^2(\Omega\times(T_k,T_{k+1}))} + 2\sum_{k\geq 0} e^{\frac{2a^2A}{T-T_{k+1}}}\varepsilon_k^2\|\chi_k\|^2_{L^2(\Omega)}\ .
\end{aligned} \tag{2.30}$$

In order to get

$$2\sum_{k\geq 0} e^{\frac{2a^2A}{T-T_{k+1}}}\varepsilon_k^2\|\chi_k\|^2_{L^2(\Omega)} = \frac{1}{2}\sum_{k\geq 0} e^{\frac{2A}{T-T_{k+1}}}\|\chi_k\|^2_{L^2(\Omega)}, \tag{2.31}$$

we choose

$$\varepsilon_k = \frac{1}{2}e^{-\frac{A\left(a^2-1\right)}{T-T_{k+1}}} \quad \forall k \geq 0\ . \tag{2.32}$$

With this choice of ε_k, (2.30) becomes

$$\sum_{k\geq 0} e^{\frac{2A}{T-T_{k+1}}}\|\chi_k\|^2_{L^2(\Omega)} \leq 2e^{\frac{2A}{T-T_1}}\|\phi^0\|^2_{L^2(\Omega)} + 4T\sum_{k\geq 0} e^{\frac{2aA}{T-T_{k+1}}}\|g_k\|^2_{L^2(\Omega\times(T_k,T_{k+1}))}\ . \tag{2.33}$$

Step 2: Estimate $\sum_{k\geq 0} e^{\frac{2M}{T-T_{k+1}}}\|z_k\|^2_{C([T_k,T_{k+1}];L^2(\Omega))}$ for some $M > 0$.

For any constant $M > 0$, we get

$$\begin{aligned}
&\sum_{k\geq 0} e^{\frac{2M}{T-T_{k+1}}}\|z_k\|^2_{C([T_k,T_{k+1}];L^2(\Omega))}\\
= \quad & e^{\frac{2M}{T-T_1}}\|z_0\|^2_{C([T_0,T_1];L^2(\Omega))} + \sum_{k\geq 0} e^{\frac{2M}{T-T_{k+2}}}\|z_{k+1}\|^2_{C([T_{k+1},T_{k+2}];L^2(\Omega))}.
\end{aligned} \tag{2.34}$$

Combining (2.22), (2.25) and (2.34), one has

$$\begin{aligned}
&\sum_{k\geq 0} e^{\frac{2M}{T-T_{k+1}}}\|z_k\|^2_{C([T_k,T_{k+1}];L^2(\Omega))}\\
\leq \quad & e^{\frac{2M}{T-T_1}}T\|g_0\|^2_{L^2(\Omega\times(T_0,T_1))} + 2T\sum_{k\geq 0} e^{\frac{2M}{T-T_{k+2}}}\|g_{k+1}\|^2_{L^2(\Omega\times(T_{k+1},T_{k+2}))}\\
& +2\sum_{k\geq 0} e^{\frac{2M}{T-T_{k+1}}}\varepsilon_k^2\|\chi_k\|^2_{L^2(\Omega)}.
\end{aligned} \tag{2.35}$$

Thanks to (2.32), one yields

$$\begin{aligned}&\sum_{k\geq 0} e^{\frac{2M}{T-T_{k+1}}}\|z_k\|^2_{C([T_k,T_{k+1}];L^2(\Omega))}\\ \leq\ & 2T\sum_{k\geq 0} e^{\frac{2M}{T-T_{k+1}}}\|g_k\|^2_{L^2(\Omega\times(T_k,T_{k+1})}+\frac{1}{2}\sum_{k\geq 0} e^{\frac{2(M-A(a^2-1))}{T-T_{k+1}}}\|\chi_k\|^2_{L^2(\Omega)}.\end{aligned}\tag{2.36}$$

Under the condition that

$$M-A(a^2-1)\leq A,\tag{2.37}$$

one gets from (2.33) and (2.36)

$$\begin{aligned}&\sum_{k\geq 0} e^{\frac{2M}{T-T_{k+1}}}\|z_k\|^2_{C([T_k,T_{k+1}];L^2(\Omega))}\\ \leq\ & 2T\sum_{k\geq 0} e^{\frac{2M}{T-T_{k+1}}}\|g_k\|^2_{L^2(\Omega\times(T_k,T_{k+1}))}\\ &+e^{\frac{2A}{T-T_1}}\|\phi^0\|^2_{L^2(\Omega)}+2T\sum_{k\geq 0} e^{\frac{2aA}{T-T_{k+1}}}\|g_k\|^2_{L^2(\Omega\times(T_k,T_{k+1}))}\ .\end{aligned}\tag{2.38}$$

Under another condition that

$$M\leq aA,\tag{2.39}$$

we obtain

$$\begin{aligned}&\sum_{k\geq 0} e^{\frac{2M}{T-T_{k+1}}}\|z_k\|^2_{C([T_k,T_{k+1}];L^2(\Omega))}\\ \leq\ & e^{\frac{2A}{T-T_1}}\|\phi^0\|^2_{L^2(\Omega)}+4T\sum_{k\geq 0} e^{\frac{2aA}{T-T_{k+1}}}\|g_k\|^2_{L^2(\Omega\times(T_k,T_{k+1}))}\ .\end{aligned}\tag{2.40}$$

Step 3: Estimate $\sum\limits_{k\geq 0} e^{\frac{2M}{T-T_{k+1}}}\|\varphi_k\|^2_{C([T_k,T_{k+1}];L^2(\Omega))}$.
Applying again the energy estimate for the system (2.13), we also have

$$\|\varphi_k\|^2_{C([T_k,T_{k+1}];L^2(\Omega))}\leq 2T\|f_k\|^2_{L^2(\omega\times(T_k,T_{k+1}))}+2\|\chi_k\|^2_{L^2(\Omega)}\quad \forall k\geq 0\ .\tag{2.41}$$

It also implies from (2.14) that

$$\|f_k\|^2_{L^2(\omega\times(T_k,T_{k+1}))}\leq\left(Ce^{\frac{C}{T_{k+1}-T_k}}\right)^2\|\chi_k\|^2_{L^2(\Omega)}\quad \forall k\geq 0.\tag{2.42}$$

Combining (2.41) and (2.42) gives us

$$\|\varphi_k\|^2_{C([T_k,T_{k+1}];L^2(\Omega))}\leq 2\left(1+C^2Te^{\frac{2C}{T_{k+1}-T_k}}\right)\|\chi_k\|^2_{L^2(\Omega)}\quad \forall k\geq 0\ .\tag{2.43}$$

Thus, it deduces from (2.43) that

$$\sum_{k\geq 0} e^{\frac{2M}{T-T_{k+1}}}\|\varphi_k\|^2_{C([T_k,T_{k+1}];L^2(\Omega))}\leq 2\sum_{k\geq 0} e^{\frac{2M}{T-T_{k+1}}}\left(1+C^2Te^{\frac{2C}{T_{k+1}-T_k}}\right)\|\chi_k\|^2_{L^2(\Omega)}\ .\tag{2.44}$$

Using the fact that $T_{k+1}-T_k=\frac{T(a-1)}{a^{k+1}}=(a-1)(T-T_{k+1})$, one gets from (2.44) that

$$\sum_{k\geq 0} e^{\frac{2M}{T-T_{k+1}}}\|\varphi_k\|^2_{C([T_k,T_{k+1}];L^2(\Omega))}\leq 2(1+C^2)(1+T)\sum_{k\geq 0} e^{\frac{2(M+\frac{C}{a-1})}{T-T_{k+1}}}\|\chi_k\|^2_{L^2(\Omega)}\ .\tag{2.45}$$

Under the condition that

$$M + \frac{C}{a-1} \leq A, \tag{2.46}$$

it implies from (2.45) and (2.33) that

$$\begin{aligned} & \sum_{k\geq 0} e^{\frac{2M}{T-T_{k+1}}} \|\varphi_k\|^2_{C([T_k,T_{k+1}];L^2(\Omega))} \\ \leq \quad & 4(1+C^2)(1+T)e^{\frac{2A}{T-T_1}} \|\phi^0\|^2_{L^2(\Omega)} \\ & +8(1+C^2)T(1+T)\sum_{k\geq 0} e^{\frac{aA}{T-T_{k+1}}} \|g_k\|^2_{L^2(\Omega\times(T_k,T_{k+1}))} . \end{aligned} \tag{2.47}$$

Step 4: Estimate $\sum_{k\geq 0} e^{\frac{2M}{T-T_{k+1}}} \|\phi_k\|^2_{C([T_k,T_{k+1}];L^2(\Omega))}$.
Combining (2.40), (2.47) and the fact that $\phi_k = z_k + \varphi_k$, one gets

$$\begin{aligned} & \sum_{k\geq 0} e^{\frac{2M}{T-T_{k+1}}} \|\phi_k\|^2_{C([T_k,T_{k+1}];L^2(\Omega))} \\ \leq \quad & 2\sum_{k\geq 0} e^{\frac{2M}{T-T_{k+1}}} \|z_k\|^2_{C([T_k,T_{k+1}];L^2(\Omega))} + 2\sum_{k\geq 0} e^{\frac{2M}{T-T_{k+1}}} \|\varphi_k\|^2_{C([T_k,T_{k+1}];L^2(\Omega))} \\ \leq \quad & 10(1+C^2)(1+T)e^{\frac{2A}{T-T_1}} \|\phi^0\|_{L^2(\Omega)} + 24(1+C^2)(1+T)^2 \sum_{k\geq 0} e^{\frac{2aA}{T-T_{k+1}}} \|g_k\|^2_{L^2(\Omega\times(T_k,T_{k+1}))} . \end{aligned} \tag{2.48}$$

Using the claim that

$$\sum_{k\geq 0} e^{\frac{2B}{T-T_k}} \|g_k\|^2_{L^2(\Omega\times(T_k,T_{k+1}))} \leq \|e^{\frac{B}{T-t}} g\|^2_{L^2(\Omega\times(0,T))} \quad \forall B > 0, \tag{2.49}$$

one obtains

$$\begin{aligned} & \sum_{k\geq 0} e^{\frac{2M}{T-T_{k+1}}} \|\phi_k\|^2_{C([T_k,T_{k+1}];L^2(\Omega))} \\ \leq \quad & 10(1+C^2)(1+T)e^{\frac{2A}{T-T_1}} \|\phi^0\|^2_{L^2(\Omega)} + 24(1+C^2)(1+T)^2 \|e^{\frac{a^2A}{T-t}} g\|^2_{L^2(\Omega\times(0,T))} . \end{aligned} \tag{2.50}$$

The rest of this step is proving the claim (2.49). We have

$$\begin{aligned} \left\| g e^{\frac{B}{T-t}} \right\|^2_{L^2(\Omega\times(0,T))} &= \int_0^T \|e^{\frac{B}{T-t}} g(\cdot,t)\|^2_{L^2(\Omega)} dt \\ &= \sum_{k\geq 0} \int_{T_k}^{T_{k+1}} \|e^{\frac{B}{T-t}} g_k(\cdot,t)\|^2_{L^2(\Omega)} dt \\ &\geq \sum_{k\geq 0} e^{\frac{2B}{T-T_k}} \int_{T_k}^{T_{k+1}} \|g_k(\cdot,t)\|^2_{L^2(\Omega)} dt \\ &= \sum_{k\geq 0} e^{\frac{2B}{T-T_k}} \|g_k\|^2_{L^2(\Omega\times(T_k,T_{k+1}))}. \end{aligned} \tag{2.51}$$

Step 5: Estimate $\|\phi e^{\frac{M}{T-t}}\|^2_{C([0,T];L^2(\Omega))}$.
Using the following claim

$$\left\| \phi e^{\frac{B}{T-t}} \right\|^2_{C([0,T];L^2(\Omega))} \leq \sum_{k\geq 0} e^{\frac{2B}{T-T_{k+1}}} \|\phi_k\|^2_{C([T_k,T_{k+1}];L^2(\Omega))} \quad \forall B > 0, \tag{2.52}$$

we obtain from (2.50) that

$$\begin{aligned}
&\|\phi e^{\frac{M}{T-t}}\|^2_{C([0,T];L^2(\Omega))}\\
\leq\ & 10(1+C^2)(1+T)\left(e^{\frac{2A}{T-T_1}}\|\phi^0\|^2_{L^2(\Omega)}\right)+24(1+C^2)(1+T)^2\|ge^{\frac{a^2A}{T-t}}\|^2_{L^2(\Omega\times(0,T))}\\
\leq\ & 24(1+C^2)(1+T)^2\left(e^{\frac{2A}{T-T_1}}\|\phi^0\|^2_{L^2(\Omega)}+\|ge^{\frac{a^2A}{T-t}}\|^2_{L^2(\Omega\times(0,T))}\right). \qquad (2.53)
\end{aligned}$$

Now, we give the proof for claim (2.52). We have

$$\begin{aligned}
\left\|\phi e^{\frac{B}{T-t}}\right\|^2_{C([0,T];L^2(\Omega))} &= \left(\sup_{t\in[0,T]}\|\phi(\cdot,t)e^{\frac{B}{T-t}}\|_{L^2(\Omega)}\right)^2\\
&= \sup_{t\in[0,T]}\|\phi(\cdot,t)e^{\frac{B}{T-t}}\|^2_{L^2(\Omega)}\\
&\leq \sum_{k\geq0}\sup_{t\in[T_k,T_{k+1}]}\|\phi_k(\cdot,t)e^{\frac{B}{T-t}}\|^2_{L^2(\Omega)}\\
&\leq \sum_{k\geq0}e^{\frac{2B}{T-T_{k+1}}}\sup_{t\in[T_k,T_{k+1}]}\|\phi_k(\cdot,t)\|^2_{L^2(\Omega)}\\
&= \sum_{k\geq0}e^{\frac{2B}{T-T_{k+1}}}\left(\sup_{t\in[T_k,T_{k+1}]}\|\phi_k(\cdot,t)\|_{L^2(\Omega)}\right)^2\\
&= \sum_{k\geq0}e^{\frac{2B}{T-T_{k+1}}}\|\phi_k\|^2_{C([T_k,T_{k+1}];L^2(\Omega))}. \qquad (2.54)
\end{aligned}$$

Step 6: Estimate $\sum_{k\geq0}e^{\frac{2M}{T-T_{k+1}}}\|f_k\|^2_{L^2(\omega\times(T_k,T_{k+1}))}$.

It implies from (2.42) that

$$\begin{aligned}
\sum_{k\geq0}e^{\frac{2M}{T-T_{k+1}}}\|f_k\|^2_{L^2(\omega\times(T_k,T_{k+1}))} &\leq \sum_{k\geq0}e^{\frac{2M}{T-T_{k+1}}}C^2e^{\frac{2C}{T_{k+1}-T_k}}\|\chi_k\|^2_{L^2(\Omega)} \qquad (2.55)\\
&= C^2\sum_{k\geq0}e^{2\left(M+\frac{C}{(a-1)}\right)\frac{1}{T-T_{k+1}}}\|\chi_k\|^2_{L^2(\Omega)}. \qquad (2.56)
\end{aligned}$$

Under the condition (2.46), which is

$$M+\frac{C}{(a-1)}\leq A, \qquad (2.57)$$

we get from (2.56) and (2.33) that

$$\begin{aligned}
\sum_{k\geq0}e^{\frac{2M}{T-T_{k+1}}}\|f_k\|^2_{L^2(\omega\times(T_k,T_{k+1}))} &\leq 2C^2e^{\frac{2A}{T-T_1}}\|\phi^0\|^2_{L^2(\Omega)}\\
&\quad +4C^2T\sum_{k\geq0}e^{\frac{2aA}{T-T_{k+1}}}\|g_k\|^2_{L^2(\Omega\times(T_k,T_{k+1}))}. \qquad (2.58)
\end{aligned}$$

Using the claim (2.49), one has

$$\sum_{k\geq0}e^{\frac{2M}{T-T_{k+1}}}\|f_k\|^2_{L^2(\omega\times(T_k,T_{k+1}))} \leq 2C^2e^{\frac{2A}{T-T_1}}\|\phi^0\|^2_{L^2(\Omega)}+4C^2T\left\|ge^{\frac{a^2A}{T-t}}\right\|^2_{L^2(\Omega\times(0,T))}. \qquad (2.59)$$

Step 7: Estimate $\|fe^{\frac{M}{T-t}}\|^2_{L^2(\omega\times(0,T))}$.

Using the following claim

$$\|fe^{\frac{B}{T-t}}\|^2_{L^2(\omega\times(0,T))}\leq\sum_{k\geq0}e^{\frac{2B}{T-T_{k+1}}}\|f_k\|^2_{L^2(\omega\times(T_k,T_{k+1}))}\quad \forall B>0, \qquad (2.60)$$

one gets from (2.59) that

$$\begin{aligned} \|fe^{\frac{M}{T-t}}\|^2_{L^2(\omega\times(0,T))} &\leq 2C^2 e^{\frac{2A}{T-T_1}}\|\phi^0\|^2_{L^2(\Omega)} + 4C^2T\|ge^{\frac{a^2A}{T-t}}\|^2_{L^2(\Omega\times(0,T))} \\ &\leq 4C^2(1+T)\left(e^{\frac{2A}{T-T_1}}\|\phi^0\|^2_{L^2(\Omega)} + \|ge^{\frac{a^2A}{T-t}}\|^2_{L^2(\Omega\times(0,T))}\right) . \end{aligned} \tag{2.61}$$

Let us move to the proof of claim (2.60). We have

$$\begin{aligned} \left\|fe^{\frac{B}{T-t}}\right\|^2_{L^2(\Omega\times(0,T))} &= \int_0^T \|e^{\frac{B}{T-t}}g(\cdot,t)\|^2_{L^2(\Omega)}dt \\ &= \sum_{k\geq 0}\int_{T_k}^{T_{k+1}} \|e^{\frac{B}{T-t}}g_k(\cdot,t)\|^2_{L^2(\Omega)}dt \\ &\leq \sum_{k\geq 0} e^{\frac{2B}{T-T_{k+1}}}\int_{T_k}^{T_{k+1}} \|g_k(\cdot,t)\|^2_{L^2(\Omega)}dt \\ &= \sum_{k\geq 0} e^{\frac{2B}{T-T_{k+1}}}\|g_k\|^2_{L^2(\Omega\times(T_k,T_{k+1}))}. \end{aligned} \tag{2.62}$$

Step 8: Get conclusion.
Now, we assume our result from above steps: For any $A > 0$ and any $M > 0$ satisfying (2.37), (2.39) and (2.46), we get the following results (thanks to the fact that $a^2 + b^2 \leq (a+b)^2$):
On one hand, it follows from (2.53) that

$$\begin{aligned} &\|\phi e^{\frac{2M}{T-t}}\|_{C([0,T];L^2(\Omega))} \\ \leq\ & 2\sqrt{6(1+C^2)}(1+T)\left(e^{\frac{A}{T-T_1}}\|\phi^0\|_{L^2(\Omega)} + \|ge^{\frac{a^2A}{T-t}}\|_{L^2(\Omega\times(0,T))}\right) . \end{aligned} \tag{2.63}$$

On the other hand, it follows from (2.61) that

$$\|fe^{\frac{M}{T-t}}\|_{L^2(\omega\times(0,T))} \leq 2C\sqrt{1+T}\left(e^{\frac{A}{T-T_1}}\|\phi^0\|_{L^2(\Omega)} + \|ge^{\frac{a^2A}{T-t}}\|_{L^2(\Omega\times(0,T))}\right) . \tag{2.64}$$

Now, with $M = \frac{C}{(a-1)}$ and $A = \frac{2C}{a-1}$, all the conditions (2.37), (2.39) and (2.46) are satisfied. Hence, we conclude from (2.63) and (2.64) that: There exists a positive constant $K > 1$ such that

$$\begin{aligned} &\|\phi e^{\frac{C}{(a-1)}\frac{1}{T-t}}\|_{C([0,T];L^2(\Omega))} + \|fe^{\frac{C}{(a-1)}\frac{1}{T-t}}\|_{L^2(\omega\times(0,T))} \\ \leq\ & K(1+T)\left[e^{\frac{2C}{a-1}\frac{1}{T}}\|\phi^0\|_{L^2(\Omega)} + \|ge^{\frac{2a^2C}{a-1}\frac{1}{T-t}}\|_{L^2(\Omega\times(0,T))}\right]. \end{aligned} \tag{2.65}$$

This completes the proof of Theorem 2.2.

2.2.2.2 Proof of Corollary 2.2

We turn now to the case $\phi^0 \in H^1_0(\Omega)$. For any constant $D > 0$, put $q = q(t) = e^{\frac{D}{T-t}}$ and $\zeta = q\phi$ then ζ satisfies the following system

$$\begin{cases} \partial_t\zeta - \Delta\zeta = q'\phi + q(\mathbb{1}_\omega f + g) & \text{in} \quad \Omega\times(0,T) , \\ \zeta = 0 & \text{on} \quad \partial\Omega\times(0,T) , \\ \zeta(\cdot,0) = e^{\frac{D}{T}}\phi^0 & \text{in} \quad \Omega . \end{cases}$$

Applying the regularity estimate (see Theorem 2.5), one has

$$\|\nabla\zeta\|_{C([0,T];L^2(\Omega))} \leq e^{\frac{D}{T}}\|\nabla\phi^0\|_{L^2(\Omega)} + \|q'\phi\|_{L^2(\Omega\times(0,T))} + \|qf\|_{L^2(\omega\times(0,T))} + \|qg\|_{L^2(\Omega\times(0,T))}. \tag{2.66}$$

We claim that: For any $\rho \in (1, 3/2)$, there exists a constant $K_\rho > 1$ such that

$$\|q'\phi\|_{L^2(\Omega\times(0,T))} \leq K_\rho\|e^{\frac{\rho D}{T-t}}\phi\|_{L^2(\Omega\times(0,T))}. \tag{2.67}$$

Furthermore, we also have

$$\begin{aligned}\|\phi e^{\frac{\rho D}{T-t}}\|_{L^2(\Omega\times(0,T))} &= \left(\int_0^T \|\phi(\cdot,t)e^{\frac{\rho D}{T-t}}\|^2_{L^2(\Omega)}dt\right)^{\frac{1}{2}} \\ &\leq \sqrt{T}\left(\sup_{t\in[0,T]}\|\phi(\cdot,t)e^{\frac{\rho D}{T-t}}\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} \\ &= \sqrt{T}\sup_{t\in[0,T]}\|\phi(\cdot,t)e^{\frac{\rho D}{T-t}}\|_{L^2(\Omega)} \\ &= \sqrt{T}\|\phi e^{\frac{\rho D}{T-t}}\|_{C([0,T];L^2(\Omega))}. \end{aligned} \tag{2.68}$$

Thus, (2.66) can be written as

$$\begin{aligned}\|\nabla\phi e^{\frac{D}{T-t}}\|_{C([0,T];L^2(\Omega))} &\leq e^{\frac{D}{T}}\|\nabla\phi^0\|_{L^2(\Omega)} + K_\rho\sqrt{T}\|\phi e^{\frac{\rho D}{T-t}}\|_{C([0,T];L^2(\Omega))} \\ &\quad +\|fe^{\frac{D}{T-t}}\|_{L^2(\omega\times(0,T))} + \|ge^{\frac{D}{T-t}}\|_{L^2(\Omega\times(0,T))}\ .\end{aligned} \tag{2.69}$$

Thanks to the fact that $\rho > 1$ and $K_\rho > 1$, one obtains

$$\begin{aligned}\|\nabla\phi e^{\frac{D}{T-t}}\|_{C([0,T];L^2(\Omega))} &\leq e^{\frac{D}{T}}\|\nabla\phi^0\|_{L^2(\Omega)} + \|ge^{\frac{3D}{T-t}}\|_{L^2(\Omega\times(0,T))} \\ &\quad +K_\rho(1+\sqrt{T})\left[\|\phi e^{\frac{\rho D}{T-t}}\|_{C([0,T];L^2(\Omega))} + \|fe^{\frac{\rho D}{T-t}}\|_{L^2(\omega\times(0,T))}\right]\ .\end{aligned} \tag{2.70}$$

Take

$$a = \sqrt{\frac{3}{2\rho}} \text{ and } D = \frac{C}{\rho\left(\sqrt{\frac{3}{2\rho}}-1\right)}\ , \tag{2.71}$$

in order that $a > 1$, $\rho D = \frac{C}{(a-1)}$ and $\frac{2a^2C}{a-1} = 3D$. Then, Theorem 2.2 says: There exists a positive constant $K > 1$ such that

$$\begin{aligned}&\|\phi e^{\frac{\rho D}{T-t}}\|_{C([0,T];L^2(\Omega))} + \|fe^{\frac{\rho D}{T-t}}\|_{L^2(\omega\times(0,T))} \\ &\leq K\,(1+T)\left[e^{\frac{2\rho D}{T}}\|\phi^0\|_{L^2(\Omega)} + \|ge^{\frac{3D}{T-t}}\|_{L^2(\Omega\times(0,T))}\right].\end{aligned} \tag{2.72}$$

Combining (2.70), (2.72) and the fact $\rho < \frac{3}{2}$, one obtains

$$\|\nabla\phi e^{\frac{D}{T-t}}\|_{C([0,T];L^2(\Omega))} \leq K\left(1+\sqrt{T}\right)^3\left[e^{\frac{3D}{T}}\|\nabla\phi^0\|_{L^2(\Omega)} + \|ge^{\frac{3D}{T-t}}\|_{L^2(\Omega\times(0,T))}\right], \tag{2.73}$$

for another constant $K > 1$. The rest is the proof of our claim: For any $\rho \in (1, 3/2)$, there exists a constant $K_\rho > 1$ such that

$$\|q'\phi\|_{L^2(\Omega)} \leq K_\rho\|e^{\frac{\rho D}{T-t}}\phi\|_{L^2(\Omega\times(0,T))}. \tag{2.74}$$

Indeed, we have

$$\|q'\phi\|_{L^2(\Omega\times(0,T))} = \left\|\frac{D}{(T-t)^2}e^{\frac{D}{T-t}}\phi\right\|_{L^2(\Omega\times(0,T))}. \tag{2.75}$$

Next, we use the following argument

$$\frac{D}{(T-t)^2} = \frac{1}{\beta^2 D}\left(\frac{\beta D}{T-t}\right)^2 \leq \frac{1}{\beta^2 D}e^{\frac{2\beta D}{T-t}} \quad \forall\beta > 0. \tag{2.76}$$

Combining (2.75) and (2.76), one gets

$$\|q'\phi\|_{L^2(\Omega\times(0,T))} \leq \frac{1}{\beta^2 D}\|e^{\frac{(1+2\beta)D}{T-t}}\phi\|_{L^2(\Omega\times(0,T))}. \tag{2.77}$$

Thus, we get our claim with $\rho = 1+2\beta$ and $K_\rho = 1 + \frac{4}{D(\rho-1)^2} > 1$. This completes the proof of Corollary 2.2.

2.3 Proof of main results

2.3.1 Proof of Theorem 2.1

Sketch of proof of Theorem 2.1

Based on the idea of the proof of the Banach fixed point theorem (see Theorem 2.4), we divide the proof for Theorem 2.1 into four main steps as below.

Step 1: Choose y_0 such that $y_0^3 e^{\frac{3D}{T-t}} \in L^2(\Omega \times (0,T))$ and construct a sequence $\{y_m\}_{m\geq 1}$ and $\{f_m\}_{m\geq 1}$ satisfying

$$\begin{cases} \partial_t y_m - \Delta y_m + \gamma y_{m-1}^3 = \mathbb{1}_\omega f_m & \text{in} \quad \Omega \times (0,T) \ , \\ y_m = 0 & \text{on} \quad \partial\Omega \times (0,T) \ , \\ y_m(\cdot,0) = y^0 & \text{in} \quad \Omega \ . \end{cases} \tag{2.78}$$

The existence of $\{y_m\}_{m\geq 1}$ and $\{f_m\}_{m\geq 1}$ are based on the Corollary 2.2.

Step 2: Give assumption on the initial data $\|y^0\|_{H_0^1(\Omega)}$ in order to get

$$q = q(\|y^0\|_{H_0^1(\Omega)}) < 1 \tag{2.79}$$

satisfying

$$\|\nabla(y_{m+1} - y_m) e^{\frac{D}{T-t}}\|_{C([0,T];L^2(\Omega))} \leq q \|\nabla(y_m - y_{m-1}) e^{\frac{D}{T-t}}\|_{C([0,T];L^2(\Omega))} \quad \forall m \geq 1. \tag{2.80}$$

In order to get (2.80), the boundedness of $\|\nabla y_m e^{\frac{D}{T-t}}\|_{C([0,T];L^2(\Omega))} \quad \forall m \geq 0$ is required.

Step 3: Using the argument that $\|x_{m+1} - x_m\|_X \leq q\|x_m - x_{m-1}\|_X \quad \forall m \geq 1$ for $0 < q < 1$ implies $\{x_m\}_{m\geq 1}$ is a Cauchy sequence in a metric space $(X, \|\cdot\|_X)$, we can conclude $\{y_m\}_{m\geq 1}$ is a Cauchy sequence in $C([0,T]; H_0^1(\Omega))$.

Step 4: Thanks to the Sobolev embedding (see Theorem 2.3) and the fact that $\{y_m\}_{m\geq 1}$ is a Cauchy sequence in $C([0,T]; H_0^1(\Omega))$, we get that $\{f_m\}_{m\geq 1}$ is also a Cauchy sequence in $L^2(\omega \times (0,T))$. Then $f := \lim_{m\to\infty} f_m$ and $y := \lim_{m\to\infty} y_m$ in the corresponding spaces satisfy the system (2.1). Moreover, the fact that $y_m(\cdot,T) = 0 \quad \forall m \geq 1$ implies $y(\cdot,T) = 0$.

Now, let us move to the detailed proof of Theorem 2.1.

Step 1: Construct $\{y_m\}_{m\geq 1}$ and $\{f_m\}_{m\geq 1}$.

Firstly, take $y_0 = e^{-\frac{D}{T-t}} e^{\frac{D}{T}} y^0$. Then, we have

$$\begin{aligned} \|y_0^3 e^{\frac{3D}{T-t}}\|_{L^2(\Omega\times(0,T))} &= \|(y^0)^3 e^{\frac{3D}{T}}\|_{L^2(\Omega\times(0,T))} \\ &= e^{\frac{3D}{T}} \left(\int_0^T \int_\Omega |y^0(x)|^6 dxdt \right)^{\frac{1}{2}} \\ &= \sqrt{T} e^{\frac{3D}{T}} \|y^0\|_{L^6(\Omega)}^3. \end{aligned} \tag{2.81}$$

By using Sobolev embedding (see Theorem 2.3), which is $\|y^0\|_{L^6(\Omega)} \leq c\|\nabla y^0\|_{L^2(\Omega)}$ for some positive constant c, we get

$$\|y_0^3 e^{\frac{3D}{T-t}}\|_{L^2(\Omega\times(0,T))} \leq c^3 \sqrt{T} e^{\frac{3D}{T}} \|\nabla y^0\|_{L^2(\Omega)}^3 < \infty. \tag{2.82}$$

Thus, applying Corollary 2.2 with $g := -\gamma y_0^3$, one has: There exists $f_1 \in L^2(\omega \times (0,T))$ such that the solution of the following system

$$\begin{cases} \partial_t y_1 - \Delta y_1 + \gamma y_0^3 = \mathbb{1}_\omega f_1 & \text{in} \quad \Omega \times (0,T) \ , \\ y_1 = 0 & \text{on} \quad \partial\Omega \times (0,T) \ , \\ y_1(\cdot,0) = y^0 & \text{in} \quad \Omega \end{cases} \tag{2.83}$$

satisfies

$$\begin{aligned} &\|\nabla y_1 e^{\frac{D}{T-t}}\|_{C([0,T];L^2(\Omega))} + \|f_1 e^{\frac{D}{T-t}}\|_{L^2(\omega\times(0,T))} \\ \leq\ & K\left(1+\sqrt{T}\right)^3 \left[e^{\frac{3D}{T}}\|\nabla y^0\|_{L^2(\Omega)} + \|y_0^3 e^{\frac{3D}{T-t}}\|_{L^2(\Omega\times(0,T))}\right] \end{aligned} \tag{2.84}$$

for some positive constants D and K.
Secondly, we will prove that $\|y_1^3 e^{\frac{3D}{T-t}}\|_{L^2(\Omega\times(0,T))} < \infty$. We have

$$\|y_1^3 e^{\frac{3D}{T-t}}\|_{L^2(\Omega\times(0,T))} = \left(\int_0^T \|y_1^3 e^{\frac{3D}{T-t}}\|^2_{L^2(\Omega)} dt\right)^{\frac{1}{2}} = \left(\int_0^T \|y_1 e^{\frac{D}{T-t}}\|^6_{L^6(\Omega)} dt\right)^{\frac{1}{2}}. \tag{2.85}$$

Using again the Sobolev embedding, which is

$$\|y_1 e^{\frac{D}{T-t}}\|_{L^6(\Omega)} \leq c\|\nabla y_1 e^{\frac{D}{T-t}}\|_{L^2(\Omega)}, \tag{2.86}$$

one obtains

$$\begin{aligned} \|y_1^3 e^{\frac{3D}{T-t}}\|_{L^2(\Omega\times(0,T))} &\leq \left(\int_0^T c^6\|\nabla y_1 e^{\frac{D}{T-t}}\|^6_{L^2(\Omega)} dt\right)^{\frac{1}{2}} \\ &\leq c^3\sqrt{T}\|\nabla y_1 e^{\frac{D}{T-t}}\|^3_{C([0,T];L^2(\Omega))}. \end{aligned} \tag{2.87}$$

Combining (2.84) and (2.87), we get

$$\begin{aligned} \|y_1^3 e^{\frac{3D}{T-t}}\|_{L^2(\Omega\times(0,T))} &\leq c^3\sqrt{T}\left[K\left(1+\sqrt{T}\right)^3\left(e^{\frac{3D}{T}}\|\nabla y^0\|_{L^2(\Omega)} + \|y_0^3 e^{\frac{3D}{T-t}}\|_{L^2(\Omega\times(0,T))}\right)\right]^3 \\ &< \infty. \end{aligned} \tag{2.88}$$

Thus, applying again Corollary 2.2 with $g := -\gamma y_1^3$, one has: There exists $f_2 \in L^2(\omega\times(0,T))$ such that the solution y_2 of the system (2.83) satisfies (2.84) where y_1 is replaced by y_2, y_0 is replaced by y_1 and f_1 is replaced by f_2. Iterating the same procedure, we can construct a sequence $\{y_m\}_{m\geq 1}$ in $C([0,T];L^2(\Omega))$ and a sequence $\{f_m\}_{m\geq 1}$ in $L^2(\omega\times(0,T))$ such that the solution of the following system

$$\begin{cases} \partial_t y_m - \Delta y_m + \gamma y_{m-1}^3 = \mathbb{1}_\omega f_m & \text{in} \quad \Omega\times(0,T)\ , \\ y_m = 0 & \text{on} \quad \partial\Omega\times(0,T)\ , \\ y_m(\cdot,0) = y^0 & \text{in} \quad \Omega\ . \end{cases} \tag{2.89}$$

satisfies

$$\begin{aligned} &\|\nabla y_m e^{\frac{D}{T-t}}\|_{C([0,T];L^2(\Omega))} + \|f_m e^{\frac{D}{T-t}}\|_{L^2(\omega\times(0,T))} \\ \leq\ & K\left(1+\sqrt{T}\right)^3 \left[e^{\frac{3D}{T}}\|\nabla y^0\|_{L^2(\Omega)} + \|y_{m-1}^3 e^{\frac{3D}{T-t}}\|_{L^2(\Omega\times(0,T))}\right] \end{aligned} \tag{2.90}$$

for some positive constants D and K.

Step 2: Find upper bound $\alpha > 0$ such that $\|\nabla y_m e^{\frac{D}{T-t}}\|_{C([0,T];L^2(\Omega))} \leq \alpha \quad \forall m \geq 0$.
Firstly, we have

$$\|\nabla y_0 e^{\frac{D}{T-t}}\|_{C([0,T];L^2(\Omega))} = e^{\frac{D}{T}}\|\nabla y^0\|_{L^2(\Omega)}. \tag{2.91}$$

Suppose that $\|\nabla y_{m-1} e^{\frac{D}{T-t}}\|_{C([0,T];L^2(\Omega))} \leq \alpha$ for some $\alpha \geq e^{\frac{D}{T}}\|\nabla y^0\|_{L^2(\Omega)}$ (α will be chosen later). We need to prove that

$$\|\nabla y_m e^{\frac{D}{T-t}}\|_{C([0,T];L^2(\Omega))} \leq \alpha. \tag{2.92}$$

First of all, we claim that

$$\|y_{m-1}^3 e^{\frac{3D}{T-t}}\|_{L^2(\Omega\times(0,T))} \leq c^3\sqrt{T}\|\nabla y_{m-1} e^{\frac{D}{T-t}}\|^3_{C([0,T];L^2(\Omega))} \quad \forall m \geq 1. \tag{2.93}$$

Then, thanks to (2.90) and (2.93), one yields

$$\begin{aligned}
&\|\nabla y_m e^{\frac{D}{T-t}}\|_{C([0,T];L^2(\Omega))} \\
\leq\ & K\left(1+\sqrt{T}\right)^3 e^{\frac{3D}{T}}\|\nabla y^0\|_{L^2(\Omega)} + c^3\sqrt{T}K\left(1+\sqrt{T}\right)^3 \|\nabla y_{m-1}e^{\frac{D}{T-t}}\|^3_{C([0,T];L^2(\Omega))} \\
\leq\ & K\left(1+\sqrt{T}\right)^3 e^{\frac{3D}{T}}\|\nabla y^0\|_{L^2(\Omega)} + c^3\sqrt{T}K\left(1+\sqrt{T}\right)^3 \alpha^3. \qquad (2.94)
\end{aligned}$$

We consider that $A+B\alpha^3 \leq \alpha$ holds if we choose $\alpha = 2A$ and $B \leq \frac{1}{8A^2}$. Therefore, if

$$c^3\sqrt{T}K\left(1+\sqrt{T}\right)^3 \leq \frac{1}{8\left(K\left(1+\sqrt{T}\right)^3 e^{\frac{3D}{T}}\|\nabla y^0\|_{L^2(\Omega)}\right)^2} \qquad (2.95)$$

then we can choose

$$\alpha = 2K\left(1+\sqrt{T}\right)^3 e^{\frac{3D}{T}}\|\nabla y^0\|_{L^2(\Omega)}. \qquad (2.96)$$

in order to get (2.92). Obviously, $\alpha \geq e^{\frac{D}{T}}\|\nabla y^0\|_{L^2(\Omega)}$ with $K>1$. In conclusion, under the first assumption on the initial data, which is

$$\|\nabla y^0\|^2_{L^2(\Omega)} \leq \frac{1}{8c^3\sqrt{T}K^3(1+\sqrt{T})^9 e^{\frac{6D}{T}}} \qquad (2.97)$$

then by induction, we have for any $m \geq 0$, the following estimate holds

$$\|\nabla y_m e^{\frac{D}{T-t}}\|_{C([0,T];L^2(\Omega))} \leq 2K\left(1+\sqrt{T}\right)^3 e^{\frac{3D}{T}}\|\nabla y^0\|_{L^2(\Omega)}. \qquad (2.98)$$

The rest of this step is proving the claim (2.93):
For $m=0$, it implies from (2.82) that

$$\begin{aligned}
\|y_0^3 e^{\frac{3D}{T-t}}\|_{L^2(\Omega\times(0,T))} &\leq c^3\sqrt{T}\|\nabla y^0 e^{\frac{D}{T}}\|^3_{L^2(\Omega)} \\
&= c^3\sqrt{T}\sup_{[0,T]}\|\nabla y^0 e^{\frac{D}{T}}\|^3_{L^2(\Omega)} \\
&= c^3\sqrt{T}\sup_{[0,T]}\|\nabla y_0 e^{\frac{D}{T-t}}\|^3_{L^2(\Omega)} \\
&= c^3\sqrt{T}\|\nabla y_0 e^{\frac{D}{T-t}}\|^3_{C([0,T];L^2(\Omega))}. \qquad (2.99)
\end{aligned}$$

For $m \geq 1$, using the same above technique for y_1 (see Step 1), we get

$$\|y_m^3 e^{\frac{3D}{T-t}}\|_{L^2(\Omega\times(0,T))} < c^3\sqrt{T}\|\nabla y_m e^{\frac{D}{T-t}}\|^3_{C([0,T];L^2(\Omega))}. \qquad (2.100)$$

Thus, we get our claim (2.93).

Step 3: Prove that $\{y_m\}_{m\geq 1}$ is a Cauchy sequence in $C([0,T];H_0^1(\Omega))$.
Step 3.1: Prove that $\|\nabla(y_{m+1}-y_m)e^{\frac{D}{T-t}}\|_{C([0,T];L^2(\Omega))} \leq C(T)\|(y_m^3-y_{m-1}^3)e^{\frac{3D}{T-t}}\|_{L^2(\Omega\times(0,T))}$.
Put $Y_{m+1} = y_{m+1}-y_m$ and $F_{m+1} = f_{m+1}-f_m$ for any $m \geq 1$ then Y_{m+1} is solution of

$$\begin{cases}
\partial_t Y_{m+1} - \Delta Y_{m+1} = -\gamma(y_m^3 - y_{m-1}^3) + \mathbb{1}_\omega F_{m+1} & \text{in} \quad \Omega\times(0,T)\ , \\
Y_{m+1} = 0 & \text{on} \quad \partial\Omega\times(0,T)\ , \\
Y_{m+1}(\cdot,0) = 0 & \text{in} \quad \Omega\ .
\end{cases}$$

Secondly, following the same computations than in the proof of Corollary 2.2 (see (2.73)), we obtain

$$\|\nabla Y_{m+1}e^{\frac{D}{T-t}}\|_{C([0,T];L^2(\Omega))} \leq K\left(1+\sqrt{T}\right)^3 \|\left(y_m^3-y_{m-1}^3\right)e^{\frac{3D}{T-t}}\|_{L^2(\Omega\times(0,T))}\ . \qquad (2.101)$$

Step 3.2: Prove that $\|(y_m^3 - y_{m-1}^3)e^{\frac{3D}{T-t}}\|_{L^2(\Omega\times(0,T))} \leq \|\nabla(y_m - y_{m-1})e^{\frac{D}{T-t}}\|_{C([0,T];L^2(\Omega))}$. Thanks to the fact that $|a^3 - b^3| \leq 2|a-b|(a+b)^2$, we have

$$\|(y_m^3 - y_{m-1}^3)(\cdot,t)\|_{L^2(\Omega)}^2 \leq 4\int_\Omega |(y_m - y_{m-1})(x,t)|^2 \, |(y_m + y_{m-1})(x,t)|^4 \, dx \quad \forall t \in [0,T]. \tag{2.102}$$

Thanks to Hölder inequality, which is $\|ab\|_{L^1(\Omega)} \leq \|a\|_{L^3(\Omega)}\|b\|_{L^{\frac{3}{2}}(\Omega)}$, one gets

$$\begin{aligned} & \|(y_m^3 - y_{m-1}^3)(\cdot,t)\|_{L^2(\Omega)}^2 \\ \leq\ & 4\left(\int_\Omega |(y_m - y_{m-1})(x,t)|^6 \, dx\right)^{\frac{1}{3}} \left(\int_\Omega |(y_m + y_{m-1})(x,t)|^6 \, dx\right)^{\frac{2}{3}} \\ =\ & 4\|(y_m - y_{m-1})(\cdot,t)\|_{L^6(\Omega)}^2 \|(y_m + y_{m-1})(\cdot,t)\|_{L^6(\Omega)}^4 \quad \forall t \in [0,T]. \end{aligned} \tag{2.103}$$

Using Sobolev embedding again, one has

$$\begin{aligned} & \|(y_m^3 - y_{m-1}^3)(\cdot,t)\|_{L^2(\Omega)}^2 \\ \leq\ & 4c^6 \|(\nabla y_m - \nabla y_{m-1})(\cdot,t)\|_{L^2(\Omega)}^2 \|(\nabla y_m + \nabla y_{m-1})(\cdot,t)\|_{L^2(\Omega)}^4 \\ \leq\ & 16c^6 \|\nabla Y_m(\cdot,t)\|_{L^2(\Omega)}^2 \left(\|\nabla y_m(\cdot,t)\|_{L^2(\Omega)}^2 + \|\nabla y_{m-1}(\cdot,t)\|_{L^2(\Omega)}^2\right)^2 \quad \forall t \in [0,T]\ . \end{aligned} \tag{2.104}$$

As a result,

$$\begin{aligned} & \|(y_m^3 - y_{m-1}^3)e^{\frac{3D}{T-t}}\|_{L^2(\Omega\times(0,T))} \\ =\ & \left(\int_0^T \|(y_m^3 - y_{m-1}^3)(\cdot,t)\|_{L^2(\Omega)}^2 e^{\frac{6D}{T-t}} dt\right)^{\frac{1}{2}} \\ \leq\ & \left(\int_0^T 16c^6 \|\nabla Y_m(\cdot,t)\|_{L^2(\Omega)}^2 (\|\nabla y_m(\cdot,t)\|_{L^2(\Omega)}^2 + \|\nabla y_{m-1}(\cdot,t)\|_{L^2(\Omega)}^2)^2 e^{\frac{6D}{T-t}} dt\right)^{\frac{1}{2}} \\ \leq\ & \left(\int_0^T 16c^6 \|\nabla Y_m(\cdot,t)e^{\frac{D}{T-t}}\|_{L^2(\Omega)}^2 (\|\nabla y_m(\cdot,t)e^{\frac{D}{T-t}}\|_{L^2(\Omega)}^2 + \|\nabla y_{m-1}(\cdot,t)e^{\frac{D}{T-t}}\|_{L^2(\Omega)}^2)^2 dt\right)^{\frac{1}{2}} \\ \leq\ & \left(\int_0^T 64c^6 \|\nabla Y_m(\cdot,t)e^{\frac{D}{T-t}}\|_{L^2(\Omega)}^2 \|\nabla y_m(\cdot,t)e^{\frac{D}{T-t}}\|_{L^2(\Omega)}^4\right)^{\frac{1}{2}} \\ \leq\ & 8c^3\sqrt{T}\|\nabla Y_m e^{\frac{D}{T-t}}\|_{C([0,T];L^2(\Omega))} \|\nabla y_m e^{\frac{D}{T-t}}\|_{C([0,T];L^2(\Omega))}^2. \end{aligned} \tag{2.105}$$

Using the result in Step 2, which is

$$\|\nabla y_m e^{\frac{D}{T-t}}\|_{C([0,T];L^2(\Omega))} \leq 2K\left(1+\sqrt{T}\right)^3 e^{\frac{3D}{T}} \|\nabla y^0\|_{L^2(\Omega)}, \tag{2.106}$$

we get

$$\begin{aligned} & \|(y_m^3 - y_{m-1}^3)e^{\frac{3D}{T-t}}\|_{L^2(\Omega\times(0,T))} \\ \leq\ & 8c^3\sqrt{T}\left(2K\left(1+\sqrt{T}\right)^3 e^{\frac{3D}{T}}\|\nabla y^0\|_{L^2(\Omega)}\right)^2 \|\nabla Y_m e^{\frac{D}{T-t}}\|_{C([0,T];L^2(\Omega))}\ . \end{aligned} \tag{2.107}$$

Step 3.3: Find $0 < q < 1$ such that $\|\nabla Y_{m+1}e^{\frac{D}{T-t}}\|_{C([0,T];L^2(\Omega))} \leq q\|\nabla Y_m e^{\frac{D}{T-t}}\|_{C([0,T];L^2(\Omega))}$. Gathering (2.101) and (2.107), yields

$$\begin{aligned} & \|\nabla Y_{m+1}e^{\frac{D}{T-t}}\|_{C([0,T];L^2(\Omega))} \\ \leq\ & K\left(1+\sqrt{T}\right)^3 \|\left(y_m^3 - y_{m-1}^3\right) e^{\frac{3D}{T-t}}\|_{L^2(\Omega\times(0,T))} \\ \leq\ & 8c^3 K\left(1+\sqrt{T}\right)^3 \sqrt{T}\left(2K\left(1+\sqrt{T}\right)^2 e^{\frac{3D}{T}}\|\nabla y^0\|_{L^2(\Omega)}\right)^2 \|\nabla Y_m e^{\frac{D}{T-t}}\|_{C([0,T];L^2(\Omega))}. \end{aligned} \tag{2.108}$$

Therefore, in order to get our target in this step, we need the following condition

$$8c^3K\left(1+\sqrt{T}\right)^3\sqrt{T}\left(2K\left(1+\sqrt{T}\right)^3e^{\frac{3D}{T}}\|\nabla y^0\|_{L^2(\Omega)}\right)^2<1. \tag{2.109}$$

It deduces the second assumption on the initial data, which is

$$\|\nabla y^0\|^2_{L^2(\Omega)}<\frac{1}{32c^3\sqrt{T}K^3(1+\sqrt{T})^9e^{\frac{6D}{T}}}. \tag{2.110}$$

Step 3.4: Prove $\{y_m\}_{m\geq 1}$ is a Cauchy sequence in $C([0,T];H^1_0(\Omega))$.
In this step, we will use the following lemma, whose proof can be found in Subsection 2.4.4.

Lemma 2.1. *Let $(X,\|\cdot\|_X)$ be a metric space and $\{x_m\}_{m\geq1}\subset X$ such that there exists a constant $0<q<1$ satisfying*

$$\|x_{m+1}-x_m\|_X\leq q\|x_m-x_{m-1}\|_X \quad \forall m\geq 1. \tag{2.111}$$

Then $\{x_m\}_{m\geq1}$ is a Cauchy sequence.

Applying Lemma 2.1 with $X:=C([0,T];H^1_0(\Omega))$ and $x_m:=y_me^{\frac{D}{T-t}}$, we get that $\{y_me^{\frac{D}{T-t}}\}_{m\geq1}$ is a Cauchy sequence in $C([0,T];H^1_0(\Omega))$. It also implies that $\{y_m\}_{m\geq1}$ is a Cauchy sequence in $C([0,T];H^1_0(\Omega))$.

Step 4: Prove $\{f_m\}_{m\geq1}$ is a Cauchy sequence in $L^2(\omega\times(0,T))$.
Step 4.1: Construct $\{F_{m+1}\}_{m\geq1}$.
Recall that the control function F_{m+1} is constructed by $F_{m+1}(x,t)=\sum_{k\geq0}(f_{m+1,k}-f_{m,k})(x,t)$. Here

$$(f_{m+1,k}-f_{m,k})(x,t)=\left(Ce^{\frac{C}{T_{k+1}-T_k}}\right)^2(v_{m+1,k}-v_{m,k})\,(x,T_{k+1}+T_k-t)\ ,$$

where $v_{m+1,k}-v_{m,k}$ solves

$$\begin{cases}\partial_t(v_{m+1,k}-v_{m,k})-\Delta(v_{m+1,k}-v_{m,k})=0 & \text{in}\quad \Omega\times(T_k,T_{k+1})\ ,\\ (v_{m+1,k}-v_{m,k})=0 & \text{on}\quad \partial\Omega\times(T_k,T_{k+1})\ ,\\ (v_{m+1,k}-v_{m,k})(\cdot,T_k)\in L^2(\Omega)\ . \end{cases} \tag{2.112}$$

Now, we will consider $\|F_{m+1}e^{\frac{D}{T-t}}\|_{L^2(\omega\times(0,T))}$. By using the claim (2.60), we get

$$\begin{aligned}&\|F_{m+1}e^{\frac{D}{T-t}}\|^2_{L^2(\omega\times(0,T))}\\ \leq\ &\sum_{k\geq0}e^{\frac{2D}{T-T_{k+1}}}\|f_{m+1,k}-f_{m,k}\|_{L^2(\omega\times(T_k,T_{k+1}))}\\ \leq\ &\sum_{k\geq0}\left(Ce^{\frac{C}{T_{k+1}-T_k}}\right)^4e^{\frac{2D}{T-T_{k+1}}}\|(v_{m+1,k}-v_{m,k})(\cdot,T_{k+1}+T_k-t)\|^2_{L^2(\omega\times(T_k,T_{k+1}))}\\ =\ &\sum_{k\geq0}\left(Ce^{\frac{C}{T_{k+1}-T_k}}\right)^4e^{\frac{2D}{T-T_{k+1}}}\|(v_{m+1,k}-v_{m,k})\|^2_{L^2(\omega\times(T_k,T_{k+1}))}. \end{aligned} \tag{2.113}$$

The last equality is obtained by changing variable.
Step 4.2: Estimate $\|v_{m+1,k}-v_{m,k}\|_{L^2(\omega\times(T_k,T_{k+1}))}$ for $k\geq0$.
We also have constructed the functions $\varphi_{m+1,k}$ and $\varphi_{m,k}$ by applying Corollary 2.2 with $g=-\gamma y_m^3$ and $g=-\gamma y^3_{m-1}$ respectively

$$\begin{cases}\partial_t(\varphi_{m+1,k}-\varphi_{m,k})(\cdot,t)-\Delta(\varphi_{m+1,k}-\varphi_{m,k})(\cdot,t)\\ \quad=\mathbb{1}_\omega\left(Ce^{\frac{C}{T_{k+1}-T_k}}\right)^2(v_{m+1,k}-v_{m,k})(\cdot,T_{k+1}+T_k-t) & \text{in}\quad\Omega\times(T_k,T_{k+1})\ ,\\ (\varphi_{m+1,k}-\varphi_{m,k})=0 & \text{on}\quad\partial\Omega\times(T_k,T_{k+1})\ ,\\ (\varphi_{m+1,k}-\varphi_{m,k})(\cdot,T_k)=\chi_{m+1,k}-\chi_{m,k} & \text{in}\quad\Omega\ ,\\ (\varphi_{m+1,k}-\varphi_{m,k})(\cdot,T_{k+1})=-\varepsilon_k^2(v_{m+1,k}-v_{m,k})(\cdot,T_k) & \text{in}\quad\Omega\ . \end{cases} \tag{2.114}$$

Multiplying both sides of the first equation in (2.114) by $(v_{m+1,k} - v_{m,k})(\cdot, T_{k+1} + T_k - t)$ and integrating over Ω, we get

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega (\varphi_{m+1,k} - \varphi_{m,k})(x,t)(v_{m+1,k} - v_{m,k})(x, T_{k+1} + T_k - t)dx \\
= &\left(Ce^{\frac{C}{T_{k+1}-T_k}}\right)^2 \int_\omega |(v_{m+1,k} - v_{m,k})(x, T_{k+1} + T_k - t)|^2 dx.
\end{aligned} \tag{2.115}$$

Integrating both sides of (2.115) over (T_k, T_{k+1}) and changing variable gives us

$$\begin{aligned}
&\left(Ce^{\frac{C}{T_{k+1}-T_k}}\right)^2 \int_{T_k}^{T_{k+1}} \int_\omega |(v_{m+1,k} - v_{m,k})(x,t)|^2 dxdt \\
= &\int_\Omega (\varphi_{m+1,k} - \varphi_{m,k})(x, T_{k+1})(v_{m+1,k} - v_{m,k})(x, T_k)dx \\
&- \int_\Omega (\varphi_{m+1,k} - \varphi_{m,k})(x, T_k)(v_{m+1,k} - v_{m,k})(x, T_{k+1})dx \\
= &-\varepsilon_k^2 \int_\Omega |\left(v_{m+1,k} - v_{m,k}\right)(x, T_k)|^2 dx - \int_\Omega (\chi_{m+1,k} - \chi_{m,k})(x)(v_{m+1,k} - v_{m,k})(x, T_{k+1})dx\ .
\end{aligned} \tag{2.116}$$

Therefore, it follows from (2.116) that

$$\begin{aligned}
&\left(Ce^{\frac{C}{T_{k+1}-T_k}}\right)^2 \|v_{m+1,k} - v_{m,k}\|^2_{L^2(\omega\times(T_k,T_{k+1}))} \\
\le &\int_\Omega |(\chi_{m+1,k} - \chi_{m,k})(x)(v_{m+1,k} - v_{m,k})(x, T_{k+1})|dx.
\end{aligned} \tag{2.117}$$

Using the Cauchy-Schwarz inequality, yields

$$\begin{aligned}
&\left(Ce^{\frac{C}{T_{k+1}-T_k}}\right)^2 \|v_{m+1,k} - v_{m,k}\|^2_{L^2(\omega\times(T_k,T_{k+1}))} \\
\le &\|\chi_{m+1,k} - \chi_{m,k}\|_{L^2(\Omega)} \|(v_{m+1,k} - v_{m,k})(\cdot, T_{k+1})\|_{L^2(\Omega)}.
\end{aligned} \tag{2.118}$$

Furthermore, it also implies from Theorem 1.7 that

$$\|(v_{m+1,k} - v_{m,k})(\cdot, T_{k+1})\|_{L^2(\Omega)} \le Ce^{\frac{C}{T_{k+1}-T_k}} \|v_{m+1,k} - v_{m,k}\|_{L^2(\omega\times(T_k,T_{k+1}))}. \tag{2.119}$$

It deduces from (2.118) and (2.119) that

$$\begin{aligned}
&\left(Ce^{\frac{C}{T_{k+1}-T_k}}\right)^2 \|v_{m+1,k} - v_{m,k}\|^2_{L^2(\omega\times(T_k,T_{k+1}))} \\
\le &\|\chi_{m+1,k} - \chi_{m,k}\|_{L^2(\Omega)} Ce^{\frac{C}{T_{k+1}-T_k}} \|(v_{m+1,k} - v_{m,k})\|_{L^2(\omega\times(T_k,T_{k+1}))}.
\end{aligned} \tag{2.120}$$

Therefore, we get

$$Ce^{\frac{C}{T_{k+1}-T_k}} \|v_{m+1,k} - v_{m,k}\|_{L^2(\omega\times(T_k,T_{k+1}))} \le \|\chi_{m+1,k} - \chi_{m,k}\|_{L^2(\Omega)}\ . \tag{2.121}$$

Step 4.3: Estimate $\|F_{m+1}e^{\frac{D}{T-t}}\|_{L^2(\omega\times(0,T))}$.
Combining (2.121) and (2.113), one gets

$$\begin{aligned}
&\|F_{m+1}e^{\frac{D}{T-t}}\|^2_{L^2(\omega\times(0,T))} \\
\le &\sum_{k\ge 0} C^2 e^{\frac{2C}{T_{k+1}-T_k} + \frac{2D}{T-T_{k+1}}} \|\chi_{m+1,k} - \chi_{m,k}\|^2_{L^2(\omega)}.
\end{aligned} \tag{2.122}$$

Using the fact that $T_{k+1} - T_k = (a-1)(T - T_{k+1})$, one has

$$\|F_{m+1}e^{\frac{D}{T-t}}\|^2_{L^2(\omega\times(0,T))} \le C^2 \sum_{k\ge 0} e^{2\left(D+\frac{C}{a-1}\right)\frac{1}{T-T_{k+1}}} \|\chi_{m+1,k} - \chi_{m,k}\|^2_{L^2(\omega)}\ . \tag{2.123}$$

Recall that the constants a and D were given in the proof of Corollary 2.2 (see (2.71)) and satisfy

$$D+\frac{C}{a-1}=\frac{C}{\rho\,(a-1)}+\frac{C}{a-1}\leq\frac{2C}{a-1}=A.$$

Following the same computations as in the proof of Theorem 2.2 (see (2.33)), we get

$$\begin{aligned}&\sum_{k\geq 0}e^{\frac{2A}{T-T_{k+1}}}\|\chi_{m+1,k}-\chi_{m,k}\|_{L^2(\Omega)}\\ \leq\;& 4T\sum_{k\geq 0}e^{\frac{2aA}{T-T_{k+1}}}\|y^3_{m,k}-y^3_{m-1,k}\|^2_{L^2(\Omega\times(T_k,T_{k+1}))}.\end{aligned}\tag{2.124}$$

Using the same argument in the claim (2.49), one yields

$$\sum_{k\geq 0}e^{\frac{2aA}{T-T_{k+1}}}\|y^3_{m,k}-y^3_{m-1,k}\|^2_{L^2(\Omega\times(T_k,T_{k+1}))}\leq\|\left(y^3_m-y^3_{m-1}\right)e^{\frac{3D}{T-t}}\|^2_{L^2(\Omega\times(0,T))}.\tag{2.125}$$

Gathering (2.124) and (2.125), we obtain

$$\begin{aligned}\sum_{k\geq 0}e^{\frac{2A}{T-T_{k+1}}}\|\chi_{m+1,k}-\chi_{m,k}\|_{L^2(\Omega)}&\leq 4T\|\left(y^3_m-y^3_{m-1}\right)e^{\frac{a^2A}{T-t}}\|^2_{L^2(\Omega\times(0,T))}\\ &\leq 4T\|\left(y^3_m-y^3_{m-1}\right)e^{\frac{3D}{T-t}}\|^2_{L^2(\Omega\times(0,T))}.\end{aligned}\tag{2.126}$$

Now, combining (2.123) and (2.126), it holds

$$\|F_{m+1}e^{\frac{D}{T-t}}\|_{L^2(\omega\times(0,T))}\leq 2C\sqrt{T}\|(y^3_m-y^3_{m-1})e^{\frac{3D}{T-t}}\|_{L^2(\Omega\times(0,T))}\;.\tag{2.127}$$

Thus, it deduces from (2.127) and (2.107) that

$$\begin{aligned}&\|F_{m+1}e^{\frac{D}{T-t}}\|_{L^2(\omega\times(0,T))}\\ \leq\;& 8CTc^3\left(2K\left(1+\sqrt{T}\right)^3e^{\frac{3D}{T}}\|\nabla y^0\|_{L^2(\Omega)}\right)^2\|\nabla Y_m e^{\frac{D}{T-t}}\|_{C([0,T];L^2(\Omega))}.\end{aligned}\tag{2.128}$$

Thanks to the result that $\{y_m e^{\frac{D}{T-t}}\}_{m\geq 1}$ is a Cauchy sequence in $C([0,T];H^1_0(\Omega))$ in Step 3 and the estimate (2.128), one obtains $\{f_m e^{\frac{D}{T-t}}\}_{m\geq 1}$ is a Cauchy sequence in $L^2(\omega\times(0,T))$. It implies that $\{f_m\}_{m\geq 1}$ is a Cauchy sequence in $L^2(\omega\times(0,T))$.

Step 5: Get conclusion.
Step 5.1: Assume above result.
In brief, combining the two conditions (2.97) and (2.110) on the initial data, we get: If

$$\|\nabla y^0\|^2_{L^2(\Omega)}<\frac{1}{G\left(1+\sqrt{T}\right)^{10}e^{\frac{G}{T}}},\tag{2.129}$$

with $G=\max\{32c^3K^3,6D,1\}>1$, then $\{y_m\}_{m\geq 1}$ is a Cauchy sequence in $C([0,T];H^1_0(\Omega))$ and $\{f_m\}_{m\geq 1}$ is a Cauchy sequence in $L^2(\Omega\times(0,T))$. Hence, there exists $f\in L^2(\omega\times(0,T))$ and $y\in C([0,T];H^1_0(\Omega))$ such that $f_m\to f$ and $y_m\to y$ in the corresponding spaces. Moreover, the fact $y_m(T)=0\;\;\forall m\geq 1$ implies that $y(T)=0$.
Step 5.2: Improve assumption on initial data.
For a fixed constant $G>1$, let us consider the function which expresses the smallness of the initial data depending on the time control T:

$$\begin{aligned}F:(0,+\infty)&\to(0,+\infty)\\ t&\mapsto\frac{1}{G\left(1+\sqrt{t}\right)^{10}e^{\frac{G}{t}}}.\end{aligned}\tag{2.130}$$

The function F is increasing from the beginning until $t = T_0$ and then decreasing until t reaches ∞. Here, instead of putting assumption that (called the old assumption):

$$\|\nabla y^0\|^2_{L^2(\Omega)} < F(T), \tag{2.131}$$

we make a tricky choice, which is (called the new assumption):

$$\|\nabla y^0\|^2_{L^2(\Omega)} < \max_{t\in(0,T]} F(t). \tag{2.132}$$

The new function $\max_{t\in(0,T]} F(t)$ with respect to T is nondecreasing on $(0, +\infty)$. Now, we will prove our new assumption is reasonable.
Case 1: When $T \le T_0$.
In this case, one has $\max_{t\in(0,T]} F(t) = F(T)$, so the old assumption is satisfied. Hence, we obtain our desire result.
Case 2: When $T > T_0$.
In this case, one has $\max_{t\in(0,T]} F(t) = F(T_0)$. Hence, we only need to control our system for $t \in (0, T_0)$ and take the control equal to zero for $t \in (T_0, T)$. Precisely, we consider two following systems:

$$\begin{cases} \partial_t\widehat{y} - \Delta\widehat{y} + \gamma\widehat{y}^3 = \mathbb{1}_\omega\widehat{f} & \text{in} \quad \Omega \times (0, T_0) \ , \\ \widehat{y} = 0 & \text{on} \quad \partial\Omega \times (0, T_0) \ , \\ \widehat{y}(\cdot, 0) = y^0 & \text{in} \quad \Omega \ . \end{cases} \tag{2.133}$$

and

$$\begin{cases} \partial_t\widetilde{y} - \Delta\widetilde{y} + \gamma\widetilde{y}^3 = 0 & \text{in} \quad \Omega \times (T_0, T) \ , \\ \widetilde{y} = 0 & \text{on} \quad \partial\Omega \times (T_0, T) \ , \\ \widetilde{y}(\cdot, T_0) = 0 & \text{in} \quad \Omega. \end{cases} \tag{2.134}$$

Under the new assumption $\|\nabla y^0\|^2_{L^2(\Omega)} \le \max_{t\in(0,T]} F(t)$, one has $\|\nabla y^0\|^2_{L^2(\Omega)} \le F(T_0)$. Hence, applying the result from Step 5.1, we obtain the null controllability at time T_0 for the system (2.133). It means there exists $\widehat{f} \in L^2(\omega \times (0, T_0))$ such that $\widehat{y}(\cdot, T_0) = 0$. Furthermore, thanks to the uniqueness of solution of system (2.134) with null initial data, we obtain $\widetilde{y}(\cdot, T) = 0$. Put

$$y(\cdot, t) = \begin{cases} \widehat{y}(\cdot, t) & \text{for } t \in [0, T_0) \ , \\ \widetilde{y}(\cdot, t) & \text{for } t \in [T_0, T] \ , \end{cases}$$

then y satisfies (2.1) with

$$f(\cdot, t) = \begin{cases} \widehat{f}(\cdot, t) & \text{for } t \in (0, T_0) \ , \\ 0 & \text{for } t \in (T_0, T), \end{cases}$$

and $y(\cdot, T) = 0$. This completes the proof of Theorem 2.1.

2.3.2 Proof of Corollary 2.1

Now, we prove Corollary 2.1. Consider the following system

$$\begin{cases} \partial_t\widehat{y} - \Delta\widehat{y} + \widehat{y}^3 = 0 & \text{in} \quad \Omega \times (0, T/2) \ , \\ \widehat{y} = 0 & \text{on} \quad \partial\Omega \times (0, T/2) \ , \\ \widehat{y}(\cdot, 0) = y^0 & \text{in} \quad \Omega \ . \end{cases}$$

Recall that no blow-up phenomena occurs. We can establish by classical regularity estimate that $\widehat{y}(\cdot, T/2) \in H^1_0(\Omega)$. Furthermore, one has

$$\|\widehat{y}\,(\cdot, T/2)\|^2_{H^1_0(\Omega)} \le \frac{1}{T}\|y^0\|^2_{L^2(\Omega)} < \max_{(0;T]} \frac{1}{G(1+\sqrt{t})^{10} e^{\frac{G}{t}}} \ .$$

Consequently, applying Theorem 2.1, we obtain the existence of $\widetilde{f} \in L^2(\omega \times (T/2, T))$ such that the solution of

$$\begin{cases} \partial_t \widetilde{y} - \Delta \widetilde{y} + \widetilde{y}^3 = \mathbb{1}_\omega \widetilde{f} & \text{in} \quad \Omega \times (T/2, T) \ , \\ \widetilde{y} = 0 & \text{on} \quad \partial\Omega \times (T/2, T) \ , \\ \widetilde{y}(\cdot, T/2) = \widehat{y}(\cdot, T/2) & \text{in} \quad \Omega \ , \end{cases}$$

satisfies $\widetilde{y}(\cdot, T) = 0$.
Put

$$y(\cdot, t) = \begin{cases} \widehat{y}(\cdot, t) & \text{for } t \in [0, T/2) \ , \\ \widetilde{y}(\cdot, t) & \text{for } t \in [T/2, T] \ , \end{cases}$$

then y satisfies (2.1) in case $\gamma = 1$ with

$$f(\cdot, t) = \begin{cases} 0 & \text{for } t \in (0, T/2) \ , \\ \widetilde{f}(\cdot, t) & \text{for } t \in (T/2, T), \end{cases}$$

and $y(\cdot, T) = 0$. This completes the proof of Corollary 2.1.

2.4 Appendix

2.4.1 Sobolev embedding

The general Sobolev embedding inequality is presented in the following theorem.

Theorem 2.3. *(see [Ad, Chapter 4, p.79] or [GiT, p.156])*
Let Ω be a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$. Let $1 \leq p < n$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$. Then there exists a positive constant c such that

$$\|f\|_{L^q(\Omega)} \leq c \|\nabla f\|_{L^p(\Omega)},$$

for any $f \in W_0^{1,p}(\Omega)$.

The one we use in this Chapter is the following: Let Ω be a bounded domain in $\mathbb{R}^3$. Then for any function $u \in H_0^1(\Omega)$, we have:

$$\|u\|_{L^6(\Omega)} \leq c \|\nabla u\|_{L^2(\Omega)} \tag{2.135}$$

where the constant c is independent of the domain; in fact $c = \frac{2^{\frac{2}{3}}}{3^{\frac{1}{2}} \pi^{\frac{2}{3}}}$.

2.4.2 Banach fixed point theorem

The Banach fixed point theorem guarantees the existence and uniqueness of fixed points of certain self-maps of metric spaces. Moreover, the proof provides a constructive method to find those fixed points.

Theorem 2.4. *(see [GrD, Th.1.1, p.10])*
Let $(X, \|\cdot\|_X)$ be a complete metric space and $\mathcal{T} : X \to X$ be a contraction mapping, i.e there exists $q \in (0, 1)$ such that

$$\|\mathcal{T}(x) - \mathcal{T}(y)\|_X \leq q \|x - y\|_X \quad \forall x, y \in X. \tag{2.136}$$

Then there exists $x^* \in X$ satisfying $\mathcal{T}(x^*) = x^*$.

2.4.3 Classical estimates

We will recall the energy estimate and regularity estimate for the following system

$$\begin{cases} \partial_t w - \Delta w = g & \text{in } \Omega \times (0, T), \\ w = 0 & \text{on } \partial\Omega \times (0, T), \\ w(\cdot, 0) = w^0 \in L^2(\Omega), \end{cases} \tag{2.137}$$

with given $g \in L^2(\Omega \times (0, T))$. Now, let us state some classical estimates for this system.

Theorem 2.5. *Let w be the solution of (2.137). Then the following estimates hold:*
i/ The energy estimate (see [CaH, Le.4.1.5, p.52])

$$\|w\|_{C([0,T];L^2(\Omega))} \leq \|w^0\|_{L^2(\Omega)} + \sqrt{T}\|g\|_{L^2(\Omega\times(0,T))}. \tag{2.138}$$

ii/ The regularity estimate (see [Ev, Th.5, p.360])

$$\|\nabla w\|_{C([0,T];L^2(\Omega))} \leq \|\nabla w^0\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega\times(0,T))}. \tag{2.139}$$

2.4.4 Proof of Lemma 2.1

Step 1: Prove that $\|x_{m+1} - x_m\|_X \leq q^m\|x_1 - x_0\|_X \quad \forall m \geq 1$.
Thanks to assumption (2.111), we have $\|x_2 - x_1\|_X \leq q\|x_1 - x_0\|_X$. Moreover, due to the induction hypothesis that $\|x_{m+1} - x_m\|_X \leq q^m\|x_1 - x_0\|_X$, one gets

$$\|x_{m+2} - x_{m+1}\|_X \leq q\|x_{m+1} - x_m\|_X \leq q^{m+1}\|x_1 - x_0\|_X. \tag{2.140}$$

Thus, by induction, we can conclude that

$$\|x_{m+1} - x_m\|_X \leq q^m\|x_1 - x_0\|_X \quad \forall m \geq 1. \tag{2.141}$$

Step 2: Prove $\|x_n - x_m\|_X \leq \frac{q^m}{1-q}\|x_1 - x_0\|_X \quad \forall 1 \leq m < n$.
For $1 \leq m < n$, by using triangle inequality, we get

$$\|x_n - x_m\|_X \leq \|x_n - x_{n-1}\|_X + \|x_{n-1} - x_{n-2}\|_X + ... + \|x_{m+1} - x_m\|_X. \tag{2.142}$$

Using the result in Step 1, one obtains

$$\begin{aligned}
\|x_n - x_m\|_X &\leq q^{n-1}\|x_1 - x_0\|_X + q^{n-2}\|x_1 - x_0\|_X + ... + q^m\|x_1 - x_0\|_X \\
&\leq q^m\|x_1 - x_0\|_X \sum_{k=0}^{n-m-1} q^k \\
&\leq q^m\|x_1 - x_0\|_X \sum_{k=0}^{\infty} q^k \\
&\leq \frac{q^m}{1-q}\|x_1 - x_0\|_X.
\end{aligned} \tag{2.143}$$

Step 3: Prove $\{x_m\}_{m\geq1}$ is a Cauchy sequence in $(X, \|\cdot\|_X)$.
For any $\varepsilon > 0$, take $n_\varepsilon \in \mathbb{N}$ such that $q^{n_\varepsilon} < \frac{\varepsilon(1-q)}{\|x_1-x_0\|_X}$. Then for any $n > m \geq n_\varepsilon$, we have

$$\begin{aligned}
\|x_n - x_m\|_X &\leq \frac{q^m}{1-q}\|x_1 - x_0\|_X \\
&\leq \frac{q^{n_\varepsilon}}{1-q}\|x_1 - x_0\|_X < \varepsilon.
\end{aligned} \tag{2.144}$$

Thus, $\{x_m\}_{m\geq1}$ is a Cauchy sequence in $(X, \|\cdot\|_X)$. This completes the proof of Lemma 2.1.

Chapter 3

Backward and Local backward for heat equation

In this chapter, we study an inverse problem which is reconstructing the initial data of a heat equation from an internal measurement of the solution on the whole domain (backward problem) or on a subdomain (local backward problem) at some time later. Such inverse problem is well-known to be an ill-posed problem, i.e even if a solution exists, it does not depend continuously on the given data. As a consequence, it creates some troubles for numerical simulations. Therefore, some special regularization methods are required. In this chapter, we study two different methods: one is the filtering method basing on a filter for the eigenfunctions decomposition of solution and another one is the Tikhonov method basing on a stability estimate. The structure of this Chapter is given as below:

Section 3.1: We introduce the formal definition of inverse and ill-posed problem. We also provide an example for the ill-posedness of backward heat problem (see Subsection 3.1.1). It requires a regularization method in order to construct an approximate solution which depends continuously on the given data. A review of some regularization methods is mentioned (see Subsection 3.1.2).

Section 3.2: We set up our main problems: The backward and The local backward problem (see Subsection 3.2.1). In order to state our main results, some preliminaries are necessary (see Subsection 3.2.2). Then in Subsection 3.2.3, our main results of the reconstruction formula and the convergence rate of the approximate solution for backward and local backward problem are released.

Section 3.3: We study the filtering method which is used by Seidman (see [Se1] or [Se2]). Under a priori condition on the initial data, the author reconstructs the solution for the backward heat problem at time $t > 0$, from the observation at some time later $T > t$ on the whole domain. The method is optimal in sense of Tautenhahn (see more in Section 3.6.1).

Section 3.4: We provide the detailed proof for our first main result: The result of backward problem.

Section 3.5: We provide the detailed proof for our second main result: The result of local backward problem.

Section 3.6: We provide some further comments about the backward problem and the local backward problem, such as:

i/ In subsection 3.6.1, we study the optimality of our regularization method in sense of Tautenhahn, which concerns the best possible case error for identifying the approximate solution. Some definitions (see Subsection 3.6.1.1) as well as the optimality results for Seidman problem (see Subsection 3.6.1.2), backward problem (see Subsection 3.6.1.3) and local backward problem (see Subsection 3.6.1.4) are presented;

ii/ In subsection 3.6.2, we solve the local backward problem by another well-known regularization method, the Tikhonov method. Moreover, some comments about the comparison between the filtering method and the Tikhonov method are also given.

iii/ In subsection 3.6.3, we consider the backward and local backward problems for a time dependent thermal conductivity heat equation.

Section 3.7: We complete our arguments by the proof of all preliminary lemmas which are used in our proof of main results.

3.1 Introduction

3.1.1 Inverse and ill-posed problem

In this subsection, we will provide a general definition of inverse problem and ill-posed problem. Then, we will consider an example of inverse problem and explain the ill-posedness of this problem.

Inverse and ill-posed problems (see [Is], [Pa], [Ke], [Ka1], [Ka2]) are the heart of scientific inquiry and technological development. They play a significant role in engineering applications, as well as several practical areas, such as image processing, mathematical finance, physics, etc. and, more recently, modelling in the life sciences. During the last ten years or so, there have been remarkable developments both in the mathematical theory and applications of inverse problems.

A very general definition of *inverse problem* is formulated by Keller (see [Ke, p.1]): "We call two problems inverses of one another if the formulation of each involves all a part of the solution of the other. Often, for the historical reasons, one of the two problems has been studied extensively for some time, while the other is newer and not so well understood. In such cases, the former is called the direct problem, while the later is called inverse problem."

According to Kabanikhin (see [Ka1, p.3]) "An *ill-posed problem* is a problem that either has no solutions in the desired class, or has many (two or more) solutions, or the solution procedure is unstable (i.e., arbitrarily small errors in the measurement data may lead to indefinitely large errors in the solutions)". From this point of view, it can be said that an ill-posed problem is a problem which is not well-posed, i.e one of three conditions (existence, uniqueness and continuously dependence) of the well-posed problem is not satisfied.

Here, we will give an example of inverse problem and consider the ill-posedness of this problem: Let Ω be an open bounded domain in $\mathbb{R}^n (n \geq 1)$ with a boundary $\partial\Omega$ of class C^2 and $T > 0$. We consider the heat equation under the Dirichlet boundary condition

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega \times (0, T), \\ u = 0 & \text{on } \partial\Omega \times (0, T). \end{cases} \tag{3.1}$$

Direct problem: given $u^0 \in L^2(\Omega)$, the task is to find $u(\cdot, T) \in L^2(\Omega)$ where u satisfies (3.1) and $u(\cdot, 0) = u^0$.
Inverse problem (IP): given $u^T \in L^2(\Omega)$, the task is to find $u(\cdot, 0) \in L^2(\Omega)$ where u satisfies (3.1) and $u(\cdot, T) = u^T$.
In Chapter 1, we already know that the direct problem is well-posed and the unique solution is given as (see Subsection 1.1.2)

$$u(\cdot, T) = \sum_{i \geq 1} e^{-\lambda_i T} \left(\int_\Omega u^0(x) e_i(x) dx \right) e_i. \tag{3.2}$$

Thus, a formal exact solution of inverse problem has the following form

$$u(\cdot, 0) = \sum_{i \geq 1} e^{\lambda_i T} \left(\int_\Omega u^T(x) e_i(x) dx \right) e_i. \tag{3.3}$$

Here, the fact that $e^{\lambda_i T} \to \infty$ when $i \to \infty$ creates the non existence of solution in $L^2(\Omega)$ of the inverse problem, i.e $u(\cdot, 0) \notin L^2(\Omega)$, unless the given data u^T is smooth

$$\sum_{i \geq 1} e^{2\lambda_i T} \left(\int_\Omega u^T(x) e_i(x) dx \right)^2 < \infty. \tag{3.4}$$

This smoothness condition is hardly satisfied in practical problems. Moreover, even if the solution exists, it does not depend continuously on the given data u^T. For instance, suppose

$$v^T := u^T + \frac{e_N}{\lambda_N}, \tag{3.5}$$

for some $N \in \mathbb{N}$. Then, in one hand

$$\|v^T - u^T\|_{L^2(\Omega)} = \frac{1}{\lambda_N} \to 0 \quad \text{when} \quad N \to \infty. \tag{3.6}$$

On the other hand, let u and v be solutions of (3.1) and respectively satisfy $u(\cdot, T) = u^T$ and $v(\cdot, T) = v^T$. Then the corresponding formal exact solutions of inverse problem (if exist) are

$$u(\cdot, 0) = \sum_{i \geq 1} e^{\lambda_i T} \left(\int_\Omega u^T(x) e_i(x) dx \right) e_i \tag{3.7}$$

and

$$v(\cdot, 0) = \sum_{i \geq 1} e^{\lambda_i T} \left(\int_\Omega u^T(x) e_i(x) dx \right) e_i + \frac{e^{\lambda_N T}}{\lambda_N} e_N. \tag{3.8}$$

Thus, one has

$$\|u(\cdot, 0) - v(\cdot, 0)\|_{L^2(\Omega)} = \frac{e^{\lambda_N T}}{\lambda_N} \to \infty \quad \text{when} \quad N \to \infty. \tag{3.9}$$

Hence, we can see from (3.6) and (3.9) that: Even if solution exists, the small perturbations of the observation data may be dramatically scaled up in the solution. It means that the inverse problem is ill-posed. As a consequence, instead of finding the exact solution for inverse problem, we will search for an approximate solution which depends continuously on the given data. It will be done thanks to a regularization method, which will be mentioned in the next subsection.

3.1.2 Regularization methods

Roughly speaking, a regularization method is a special method which regularize an ill-posed problem: Given an ill-posed problem, we define an approximate problem depending on a small positive parameter such that it is well-posed; Then, one wishes to show that the solution of this well-posed problem will converge to the solution of the ill-posed one as the parameter converges to zero in an appropriate fashion. There are many such regularization methods for solving ill-posed problems. Here, let us recall the main idea of some commonly used methods.

1. *Quasi-reversibility method*
   One method for approaching the inverse problem is quasi reversibility, introduced by Lattes and Lions (see [LaL]). The main idea of this method is adding a "corrector" into the original operator in order to get a well-posed problem, then use the solution of this new problem to construct the approximate solution. Precisely, in order to regularize the above example (IP), the authors solve the following (well-posed) problem:

$$\begin{cases} \partial_t u_\epsilon - \Delta u_\epsilon - \epsilon \Delta^2 u_\epsilon = 0 & \text{in } \Omega \times (0, T), \\ u_\epsilon = \Delta u_\epsilon = 0 & \text{on } \partial\Omega \times (0, T), \\ u_\epsilon(\cdot, T) = u^T & \text{in } \Omega. \end{cases} \tag{3.10}$$

   For each $\epsilon > 0$, the authors use the initial value $u_\epsilon(\cdot, 0)$ to solve the following problem:

$$\begin{cases} \partial_t \tilde{u}_\epsilon - \Delta \tilde{u}_\epsilon = 0 & \text{in } \Omega \times (0, +\infty), \\ \tilde{u}_\epsilon = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ \tilde{u}_\epsilon(\cdot, 0) = u_\epsilon(\cdot, 0) & \text{in } \Omega. \end{cases} \tag{3.11}$$

   It is proved that $\tilde{u}_\epsilon(\cdot, T)$ converges to u^T when $\epsilon \to 0$. Such result shows that the quasi solution $\tilde{u}_\epsilon$ is an approximation for the exact solution u.

   This method gives the stability magnitude is of order $e^{\frac{C}{\epsilon}}$, which is so large for small ϵ. Then, Miller (see [Mi]) improves this method by finding optimal perturbations of the original operator. His method, named stabilized quasi reversibility gets the stability magnitude is of order $\frac{C}{\epsilon}$, which is better than $e^{\frac{C}{\epsilon}}$.

From the numerical point of view, the fact that the order of the original operator is multiplied by two is a drawback of the original quasi reversibility method. One way to cope with this problem is using some mixed formulations of quasi reversibility (see [BeBFD]). Precisely, the authors introduce a novel unknown which enables us to replace a fourth order problem by two coupled second order problems. Recently, this method has been generalized and improved for the backward heat equation (see [BoR]).

2. *Quasi boundary value method*
Another way to approach the ill-posed problem is called the quasi boundary value method, which is suggested by Showalter (see [Sh]). This method improves the quasi reversibility method by putting "corrector" into the final data, instead of into the original operator. In detail, the author approximate (IP) with

$$\begin{cases} \partial_t u_\epsilon - \Delta u_\epsilon = 0 & \text{in } \Omega \times (0,T), \\ u_\epsilon = 0 & \text{on } \partial\Omega \times (0,T), \\ u_\epsilon(\cdot,T) + \epsilon u_\epsilon(\cdot,0) = u^T & \text{in } \Omega, \end{cases} \tag{3.12}$$

One advantage of this method is that there is no need to solve the forward problem. The new problem is well-posed for each $\epsilon > 0$. Moreover, u_ϵ converges to u^T when $\epsilon \to 0$. The explicit estimate for the convergence rate of the approximation is lately provided in [ClO]. In [DeB], Dence and Bessila perturb the final condition in another way, which contains a derivative of the same order than the equation, as follows:

$$u_\epsilon(\cdot,T) - \epsilon u'_\epsilon(\cdot,0) = u^T \quad \text{in } \Omega.$$

3. *Tikhonov regularization method*
The most well-known regularization method is introduced by the Russian mathematician A. N. Tikhonov, the Tikhonov regularization method (see [Ti1], [Ti2], [Ti3],...). The general idea of this method as follows: The (IP) may be equivalently reformulated as finding the minimum of the functional

$$\begin{aligned} \mathcal{J} : H_0^1(\Omega) &\to \mathbb{R} \\ \phi^0 &\mapsto \|\phi(\cdot,T) - u^T\|^2_{L^2(\Omega)}. \end{aligned} \tag{3.13}$$

Here, ϕ is the solution of the following system:

$$\begin{cases} \partial_t \phi - \Delta \phi = 0 & \text{in } \Omega \times (0,T), \\ \phi = 0 & \text{on } \partial\Omega \times (0,T), \\ \phi(\cdot,0) = \phi^0 & \text{in } \Omega. \end{cases} \tag{3.14}$$

The solution to this minimization problem again does not depend continuously on the given data. Hence, in order to restore stability, the author add a penalty term to the functional:

$$\mathcal{J}(\phi^0) = \|\phi(\cdot,T) - u^T\|^2_{L^2(\Omega)} + \epsilon\|\phi^0\|_{H_0^1(\Omega)}, \tag{3.15}$$

for some regularization parameter $\epsilon > 0$. It is proved that (see [Ho2, Th. 2.1, p.14]) $\mathcal{J}$ has a unique minimizer for all $\epsilon > 0$. This minimizer of the functional $\mathcal{J}$ is the approximation for the ill-posed problem (IP). This method can also be found in lots of documents, such as [Fr], [Ma], [Sc], [ZhM], [ItJ], etc.

4. *Truncation method*
It is a natural think to recover the stability of an ill-posed problem by removing the high frequency components in the eigenfunctions expansion of solution. This is the main idea of a regularization method, which is named the truncation method. In fact, one constructs the approximate solution by the following formula:

$$\sum_{i=1}^{N(\epsilon)} e^{\lambda_i T} \left(\int_\Omega u^T(x) e_i(x) dx \right) e_i$$

with a suitable cutting point $N(\epsilon)$ satisfying $N(\epsilon) \to \infty$ as $\epsilon \to 0$. This method is simple and effective for solving some inverse problems. For instance, Nam, Trong and Tuan (see [NaTT]) use the truncation method to solve an inhomogeneous backward heat problem in two dimensions; Zhang, Fu and Ma (see [ZhFM]) also use this method to tackle the time dependent thermal conductivity heat equation in one dimension and recently, Yang, Sun, Li and Ma (see [YaSLM]) improve this method for identifying the initial value of an inhomogeneous heat equation on a spherical symmetric domain in high dimension.

5. *Filtering method*
With the same idea of the truncation method, but instead of finding the cutting point, one uses a filter function for eliminating the high frequency noise, which called the filtering method. For example, an approximate solution for (IP) is constructed as:

$$\sum_{i=1}^{\infty} R(i,\epsilon)\left(\int_\Omega u^T(x)e_i(x)dx\right)e_i$$

where R is a bounded function and closed to $e^{\lambda_i T}$ when ϵ tends to 0. The above regularization methods can be correspond to a suitable filter function:

i/ Quasi boundary value method: The corresponding filter function for this method is

$$R(i,\epsilon) = \frac{1}{e^{-\lambda_i T}+\epsilon}. \tag{3.16}$$

ii/ Tikhonov method: According to [Ho2, Th. 2.1, p.14], we get the explicit formula for the approximation of (IP), based on the eigenfunction decomposition, as below

$$\sum_{i\geq 1} \frac{e^{-\lambda_i T}}{e^{-2\lambda_i T}+\epsilon}\left(\int_\Omega u^T(x)e_i(x)dx\right)e_i(x). \tag{3.17}$$

It corresponds to the following filter function:

$$R(i,\epsilon) = \frac{e^{-\lambda_i T}}{e^{-2\lambda_i T}+\epsilon}. \tag{3.18}$$

iii/ Truncation method: The filter function for this method is

$$R(i,\epsilon) = \begin{cases} e^{\lambda_i T} & \text{if } \lambda_i \leq \lambda_{N(\epsilon)}, \\ 0 & \text{if } \lambda_i > \lambda_{N(\epsilon)}. \end{cases} \tag{3.19}$$

In [Se1] or [Se2]), Seidman constructs a special filtering method, which gives the optimal result in sense of Tautenhahn (see Subsection 3.6.1). Precisely, he uses the following filter function:

$$R(i,\epsilon) = \min\left\{e^{\lambda_i T}, \frac{1}{\epsilon}\right\}. \tag{3.20}$$

In [TuKLT], the authors solve the backward heat equation in the multi-dimensional case by a new general filter regularization method. From this method, they can derive several regularization solutions by choosing a specific filter.

3.2 Main results

3.2.1 The backward problem and the local backward problem

Now, we move to the statement of our main problems. Firstly, we consider the following heat equation under the Dirichlet boundary condition

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega\times(0,T), \\ u = 0 & \text{on } \partial\Omega\times(0,T), \\ u(\cdot,T) = \mathbb{f} \in L^2(\Omega). \end{cases} \tag{3.21}$$

In reality, it is impossible to get the exact data $\mathbb{f}$, as the data are based on physical observations or the numerical methods. Hence, instead of exact data, a noisy data $\mathbb{f}_\delta$ and a noisy level δ are given such that

$$\|\mathbb{f} - \mathbb{f}_\delta\|_{L^2(\Omega)} \leq \delta. \tag{3.22}$$

Our target is constructing $\mathbb{g}_\delta$, based on $\mathbb{f}_\delta$ and δ, such that $\mathbb{g}_\delta$ approximate to $u(\cdot, 0)$. Such problem is called *the backward problem*, which is stated as below:

Backward problem: *Given $\delta > 0$ and $\mathbb{f}_\delta \in \boldsymbol{L^2(\Omega)}$ satisfying (3.22). Find $\mathbb{g}_\delta \in L^2(\Omega)$ such that the solution of (3.21) satisfies*

$$\|u(\cdot, 0) - \mathbb{g}_\delta\|_{L^2(\Omega)} \leq \epsilon(\delta) \ \text{ where } \ \epsilon(\delta) \xrightarrow{\delta \to 0} 0. \tag{3.23}$$

A slightly different (and less well-known) problem consists, for some non empty open subset $\omega \Subset \Omega$, in finding $u(\cdot, 0)$ such that

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega \times (0, T), \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(\cdot, T) = \mathbb{f} \in L^2(\omega). \end{cases} \tag{3.24}$$

This problem, named the local backward problem, might be more interesting than the backward problem from the point of view of applications, since the given data corresponds to measurements which might be accessible only on a subpart of the spatial domain. Precisely, the local backward problem is stated as below:

Local backward problem: *Let ω be a nonempty, open subset of Ω. Given $\delta > 0$ and $\mathbb{f}_\delta \in \boldsymbol{L^2(\omega)}$ satisfying*

$$\|\mathbb{f} - \mathbb{f}_\delta\|_{L^2(\omega)} \leq \delta. \tag{3.25}$$

Find $\mathbb{g}_\delta \in L^2(\Omega)$ such that the solution of (3.24) satisfies

$$\|u(\cdot, 0) - \mathbb{g}_\delta\|_{L^2(\Omega)} \leq \epsilon(\delta) \ \text{ where } \ \epsilon(\delta) \xrightarrow{\delta \to 0} 0. \tag{3.26}$$

Obviously, the ill-posedness of the local backward problem is even more severe than that of the backward problem.

3.2.2 Preliminaries

In order to state our main result, let us firstly introduce some functions. Let $a \in \left(\frac{5}{4}, \frac{6}{4}\right)$ be the unique solution of $e^a = 1 + 2a$ and $b := \sqrt{a}e^a \in \left(\frac{39}{10}, 4\right)$.

Firstly, consider the function

$$\begin{aligned} \mathcal{P} : [a, +\infty) &\to [1, +\infty) \\ x &\mapsto \frac{e^x}{1 + 2x}. \end{aligned} \tag{3.27}$$

The function $\mathcal{P}$ is increasing on $[a, +\infty)$. Moreover, it is bijective. Hence, the inverse function, denoted by $\mathcal{P}^{-1}$, is well-defined on $[1, +\infty)$.

Secondly, consider the function

$$\begin{aligned} \mathcal{Q} : [0, +\infty) &\to [0, +\infty) \\ x &\mapsto \sqrt{x}e^x. \end{aligned} \tag{3.28}$$

The function $\mathcal{Q}$ is also increasing on $[0, +\infty)$. Moreover, it is bijective. Hence, there exists an inverse function, denoted by $\mathcal{Q}^{-1} : [0, +\infty) \to [0, +\infty)$.

Thirdly, consider the function

$$\mathcal{P}\mathcal{Q}^{-1} : [b, +\infty) \to [1, +\infty). \tag{3.29}$$

The function $\mathcal{P}\mathcal{Q}^{-1}$ is increasing on $[b, +\infty)$. Moreover, we have $\mathcal{P}\mathcal{Q}^{-1}(x) \le x \quad \forall x \in [b, +\infty)$.

Now, dealing with the local backward problem where the observation is only available on a subdomain, a natural idea is connecting the information on the whole domain and on the subdomain. In order to get this connection, we will use a result of impulse controllability which will be presented in the following lemma.

Lemma 3.1. *Let T be a positive number and ω be a nonempty open subset of Ω. Then for any $\varepsilon > 0$, for any $i = 1, 2, ...$, there exists $h_i \in L^2(\omega)$ such that the solution of*

$$\begin{cases} \partial_t \psi_i - \Delta \psi_i = 0 & \text{in } \Omega \times (0, 2T) \setminus \{T\}, \\ \psi_i = 0 & \text{on } \partial\Omega \times (0, 2T), \\ \psi_i(\cdot, 0) = e_i & \text{in } \Omega, \\ \psi_i(\cdot, T) = \psi_i(\cdot, T^-) + \mathbb{1}_\omega h_i & \text{in } \Omega \end{cases} \tag{3.30}$$

satisfies $\|\psi_i(\cdot, 2T)\|_{L^2(\Omega)} \le \varepsilon$. Remind that $\{e_i\}_{i \ge 1}$ are the eigenfunctions of Laplacian under the Dirichlet boundary condition. Moreover, there exist positive constants $\mathcal{M}_1, \mathcal{M}_2$ and θ depending on Ω and ω, such that the following estimate holds

$$\|h_i\|_{L^2(\omega)} \le \frac{\mathcal{M}_1 e^{\frac{\mathcal{M}_2}{T}}}{\varepsilon^\theta} \quad \forall i \ge 1. \tag{3.31}$$

Lemma 3.1 is a direct corollary of Theorem 1.11 with $\psi^0 = e_i (i = 1, 2, ...)$. Now, we can state our two main results for the backward and the local backward problem, respectively.

3.2.3 Main results

Let us start by our first result for the backward problem.

Theorem 3.1. *Let u be the solution of (3.21) such that $M := \|u(\cdot, 0)\|_{H_0^1(\Omega)} < \infty$. Suppose $\delta > 0$ and $\mathbb{f}_\delta \in L^2(\Omega)$ are given such that*

$$\|\mathbb{f} - \mathbb{f}_\delta\|_{L^2(\Omega)} \le \delta. \tag{3.32}$$

Then there exists $\mathbb{g}_\delta \in L^2(\Omega)$ satisfying

$$\|u(\cdot, 0) - \mathbb{g}_\delta\|_{L^2(\Omega)} \le \epsilon(\delta) \quad \text{where} \quad \epsilon(\delta) \xrightarrow{\delta \to 0} 0. \tag{3.33}$$

Specially, when $\delta < 1$, the convergence rate is of order $\left(\ln \frac{1}{\delta}\right)^{-\frac{1}{2}}$, i.e there exists a positive constant $C > 0$ depending on T and M such that $\epsilon(\delta) = C \left(\ln \frac{1}{\delta}\right)^{-\frac{1}{2}}$. Furthermore, the reconstruction formula of the approximation and the error estimate are explicitly given below:

1. ***Reconstruction formula***
   The approximate solution $\mathbb{g}_\delta$ is constructed as below

$$\mathbb{g}_\delta := \begin{cases} 0 & \text{if } \delta \ge \frac{\sqrt{T}M}{b}, \\ \sum\limits_{i \ge 1} \min\{e^{\lambda_i T}, \alpha\} \left(\int_\Omega \mathbb{f}_\delta(x) e_i(x) dx \right) e_i & \text{if } \delta < \frac{\sqrt{T}M}{b}, \end{cases} \tag{3.34}$$

   Here

$$\alpha = \mathcal{P}\mathcal{Q}^{-1}\left(\frac{\sqrt{T}M}{\delta}\right), \tag{3.35}$$

   where the functions $\mathcal{P}$ and $\mathcal{Q}$ are respectively defined in (3.27) and (3.28).

2. ***Convergence rate***
The convergence of the approximate solution $\mathbb{g}_\delta$ *in (3.34) is estimated as*

$$\|u(\cdot,0)-\mathbb{g}_\delta\|_{L^2(\Omega)} \le \begin{cases} \frac{b\delta}{\sqrt{\lambda_1 T}} & \text{if } \delta \ge \frac{\sqrt{T}M}{b}, \\ \frac{\sqrt{T}M}{\sqrt{\mathcal{Q}^{-1}\left(\frac{\sqrt{T}M}{\delta}\right)}} & \text{if } \delta < \frac{\sqrt{T}M}{b}. \end{cases} \tag{3.36}$$

Remark 3.1. 1. *The approximate parameter* α*.*
Thanks to the increasing property of the function $\mathcal{P}\mathcal{Q}^{-1}$*, we have: Smaller the noisy level* δ *is, bigger the parameter* α *is. As a consequence, the approximation is more closed to the exact data. Furthermore, we get the bound for parameter* α *as below*

$$1 \le \alpha \le \frac{\sqrt{T}M}{\delta}.$$

2. *Upper bound of* $\|\mathbb{g}_\delta\|_{L^2(\Omega)}$*.*
Thanks to the upper bound of α *and the construction of* $\mathbb{g}_\delta$ *in (3.34), one gets*

$$\|\mathbb{g}_\delta\|_{L^2(\Omega)} \le \frac{\sqrt{T}M}{\delta}\|\mathbb{f}_\delta\|_{L^2(\Omega)}. \tag{3.37}$$

3. *Connection to the backward estimate.*
Let us remind the backward estimate (1.7) for the system (3.21): If $u(\cdot,0) \in H_0^1(\Omega)$ *and* $u(\cdot,0) \neq 0$ *then*

$$\|u(\cdot,0)\|_{L^2(\Omega)} \le e^{\frac{\|u(\cdot,0)\|^2_{H_0^1(\Omega)}}{\|u(\cdot,0)\|^2_{L^2(\Omega)}}T}\|u(\cdot,T)\|_{L^2(\Omega)}. \tag{3.38}$$

In order to make disappear the term $\|u(\cdot,0)\|_{L^2(\Omega)}$ *on the right-hand side of (3.38), one has*

$$\begin{aligned} \sqrt{T}\|u(\cdot,0)\|_{H_0^1(\Omega)} &\le \frac{\sqrt{T}\|u(\cdot,0)\|_{H_0^1(\Omega)}}{\|u(\cdot,0)\|_{L^2(\Omega)}} e^{\frac{T\|u(\cdot,0)\|^2_{H_0^1(\Omega)}}{\|u(\cdot,0)\|^2_{L^2(\Omega)}}}\|u(\cdot,T)\|_{L^2(\Omega)} \\ &= \mathcal{Q}\left(\frac{T\|u(\cdot,0)\|^2_{H_0^1(\Omega)}}{\|u(\cdot,0)\|^2_{L^2(\Omega)}}\right)\|u(\cdot,T)\|_{L^2(\Omega)}. \end{aligned} \tag{3.39}$$

Thanks to the increasing property of the function $\mathcal{Q}^{-1}$*, we obtain*

$$\frac{T\|u(\cdot,0)\|^2_{H_0^1(\Omega)}}{\|u(\cdot,0)\|^2_{L^2(\Omega)}} \ge \mathcal{Q}^{-1}\left(\frac{\sqrt{T}\|u(\cdot,0)\|_{H_0^1(\Omega)}}{\|u(\cdot,T)\|_{L^2(\Omega)}}\right). \tag{3.40}$$

This estimate is equivalent to

$$\|u(\cdot,0)\|_{L^2(\Omega)} \le \frac{\sqrt{T}\|u(\cdot,0)\|_{H_0^1(\Omega)}}{\sqrt{\mathcal{Q}^{-1}\left(\frac{\sqrt{T}\|u(\cdot,0)\|_{H_0^1(\Omega)}}{\|u(\cdot,T)\|_{L^2(\Omega)}}\right)}}. \tag{3.41}$$

Thus, when the noisy level is small, the error estimate (3.36) connects to the backward estimate (3.41). Furthermore, the convergence is optimal on $H_0^1(\Omega)$ *in sense of Tautenhahn (see Section 3.6.1.3).*

Let us move to the second main result for the local backward problem.

Theorem 3.2. *Let* u *be the solution of (3.24) such that* $M := \|u(\cdot,0)\|_{H_0^1(\Omega)} < \infty$*. Suppose* $\delta > 0$ *and* $\mathbb{f}_\delta \in L^2(\omega)$ *are given such that*

$$\|\mathbb{f} - \mathbb{f}_\delta\|_{L^2(\omega)} \le \delta. \tag{3.42}$$

Then there exists $\mathfrak{g}_\delta \in L^2(\Omega)$ *such that*

$$\|u(\cdot,0)-\mathfrak{g}_\delta\|_{L^2(\Omega)} \le \epsilon(\delta) \quad \textit{where} \quad \epsilon(\delta) \xrightarrow{\delta\to 0} 0. \tag{3.43}$$

Specially, when $\delta < 1$*, the convergence rate is of order* $\left(\ln \frac{1}{\delta}\right)^{-\frac{1}{2}}$*. Furthermore, the reconstruction formula of the approximation and the error estimate are given below:*

1. ***Reconstruction formula***
   The approximation solution $\mathfrak{g}_\delta$ *is constructed as below*

$$\mathfrak{g}_\delta := \begin{cases} 0 & \textit{if } \delta \ge \left(\frac{\sqrt{T}}{bC_1 e^{\frac{C_2}{T}}}\right)^{\frac{1}{\mu}} M, \\ \sum\limits_{i\ge 1} \min\{e^{3\lambda_i T}, \beta\} e^{-\lambda_i T} \left(\int_\omega \mathfrak{f}_\delta(x) h_i(x)dx\right) e_i & \textit{if } \delta < \left(\frac{\sqrt{T}}{bC_1 e^{\frac{C_2}{T}}}\right)^{\frac{1}{\mu}} M, \end{cases} \tag{3.44}$$

   for some positive constants C_1, C_2 *and* $\mu \in (0,1)$ *depending on* Ω *and* ω*. Here*

$$\beta = \mathcal{P}\mathcal{Q}^{-1}\left(\frac{\sqrt{T}}{C_1 e^{\frac{C_2}{T}}}\left(\frac{M}{\delta}\right)^{\mu}\right)$$

   where the functions $\mathcal{P}$ *and* $\mathcal{Q}$ *are respectively defined in (3.27) and (3.28). And* $h_i \in L^2(\omega)(i \ge 1)$ *comes from Lemma 3.1 with an explicit choice of* ε*.*

2. ***Convergence rate***
   The convergence of the approximate solution $\mathfrak{g}_\delta$ *in (3.176) is estimated as*

$$\|u(\cdot,0)-\mathfrak{g}_\delta\|_{L^2(\Omega)} \le \begin{cases} \left(\frac{bC_1 e^{\frac{C_2}{T}}}{\sqrt{T}}\right)^{\frac{1}{\mu}} \frac{\delta}{\sqrt{\lambda_1}} & \textit{if } \delta \ge \left(\frac{\sqrt{T}}{bC_1 e^{\frac{C_2}{T}}}\right)^{\frac{1}{\mu}} M, \\ \frac{\sqrt{3T}M}{\sqrt{\mathcal{Q}^{-1}\left(\frac{\sqrt{T}}{C_1 e^{\frac{C_2}{T}}}\left(\frac{M}{\delta}\right)^{\mu}\right)}} & \textit{if } \delta < \left(\frac{\sqrt{T}}{bC_1 e^{\frac{C_2}{T}}}\right)^{\frac{1}{\mu}} M. \end{cases} \tag{3.45}$$

Remark 3.2. 1. *The approximate parameter* β*.*

$$1 \le \beta \le \frac{\sqrt{T}}{C_1 e^{\frac{C_2}{T}}}\left(\frac{M}{\delta}\right)^{\mu}. \tag{3.46}$$

2. *Upper bound of* $\|\mathfrak{g}_\delta\|_{L^2(\Omega)}$*.*

$$\|\mathfrak{g}_\delta\|_{L^2(\Omega)} \le C\frac{\sqrt{T}M}{\delta}\|\mathfrak{f}_\delta\|_{L^2(\omega)}, \tag{3.47}$$

   for some positive constant C *only depending on* Ω *and* ω*. The proof of (3.47) can be found in the proof of Theorem 3.2 (see Section 3.5).*

3. *Connection to the backward estimate.*
   Under the assumption that $u(\cdot,0) \in H_0^1(\Omega)$ *and* $u(\cdot,0) \ne 0$*, we recall the backward estimate (3.41) for the system (3.24):*

$$\|u(\cdot,0)\|_{L^2(\Omega)} \le \frac{\sqrt{T}\|u(\cdot,0)\|_{H_0^1(\Omega)}}{\sqrt{\mathcal{Q}^{-1}\left(\frac{\sqrt{T}\|u(\cdot,0)\|_{H_0^1(\Omega)}}{\|u(\cdot,T)\|_{L^2(\Omega)}}\right)}}. \tag{3.48}$$

   Furthermore, we also have the observation estimate at one point of time for the system (3.24) (see Theorem 1.8): There exists $K_1 > 0$ *and* $K_2 > 0$ *such that*

$$\|u(\cdot,T)\|_{L^2(\Omega)} \le K_1 e^{\frac{K_2}{T}} \|u(\cdot,T)\|^{\mu}_{L^2(\omega)} \|u(\cdot,0)\|^{1-\mu}_{H_0^1(\Omega)}. \tag{3.49}$$

Combining (3.48) and (3.49), one gets the following estimate, named local backward estimate

$$\|u(\cdot,0)\|_{L^2(\Omega)} \leq \frac{\sqrt{T}\|u(\cdot,0)\|_{H_0^1(\Omega)}}{\sqrt{\mathcal{Q}^{-1}\left(\frac{\sqrt{T}}{K_1 e^{\frac{K_2}{T}}}\left(\frac{\|u(\cdot,0)\|_{H_0^1(\Omega)}}{\|u(\cdot,T)\|_{L^2(\omega)}}\right)^{\mu}\right)}}. \tag{3.50}$$

When the noisy data is acceptable, i.e the noisy level δ is small, this estimate connects to the error estimate (3.45) with the notice that $C_2 = K_2$.

3.3 Seidman problem

Let us recall the Seidman problem (see [Se2]). Let $T > 0$, we consider the following system:

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(\cdot,T) = \mathbb{f} \in L^2(\Omega). \end{cases} \tag{3.51}$$

Seidman problem: *Given $\delta > 0$ and $\mathbb{f}_\delta \in L^2(\Omega)$ satisfying*

$$\|\mathbb{f} - \mathbb{f}_\delta\|_{L^2(\Omega)} \leq \delta.$$

For $t \in (0,T)$, find $\mathbb{g}_\delta \in L^2(\Omega)$ such that the solution of (3.51) satisfies

$$\|u(\cdot,t) - \mathbb{g}_\delta\|_{L^2(\Omega)} \leq \epsilon(\delta) \quad \text{where} \quad \epsilon(\delta) \xrightarrow{\delta\to 0} 0. \tag{3.52}$$

The "exact backward representation" for solution of Seidman problem, corresponding to $\mathbb{f}_\delta$ is

$$\mathbb{g}_\delta(t) = \sum_{i\geq 1} e^{\lambda_i(T-t)}\left(\int_\Omega \mathbb{f}_\delta(x)e_i(x)dx\right)e_i. \tag{3.53}$$

The fact that $e^{\lambda_i(T-t)} \to \infty$ when $i \to \infty$ shows that the approximation (3.53) is useless. This is the essence of the ill-posedness of Seidman problem. Hence, a natural idea is replacing $e^{\lambda_i(T-t)}$ in the formula (3.53) by $p_i(t)$ such that $p_i(t)$ satisfies two following conditions:

1. $p_i(t)$ is bounded by some positive constant $\gamma = \gamma(t)$ for any $i \geq 1$,
2. $p_i(t)$ is as closed to $e^{\lambda_i(T-t)}$ as possible.

This construction of $p_i(t)$ corresponds to the following minimization problem: Fix i and consider

$$\text{minimize}\left|e^{\lambda_i(T-t)} - p_i\right| \quad \text{subject to} \quad |p_i| \leq \gamma. \tag{3.54}$$

When $\gamma > e^{\lambda_i(T-t)}$, the minimizer is $p_i = e^{\lambda_i(T-t)}$ (see Figure 3.1) and when $\gamma \leq e^{\lambda_i(T-t)}$, the minimizer is $p_i = \gamma$ (see Figure 3.2).

Hence, we can conclude that the minimizer for problem (3.54) is

$$p_i(t) = \min\{e^{\lambda_i(T-t)}, \gamma\}.$$

The approximate solution is constructed by the following form:

$$\mathbb{g}_\delta(t) = \sum_{i\geq 1} \min\{e^{\lambda_i(T-t)}, \gamma\}\left(\int_\Omega \mathbb{f}_\delta(x)e_i(x)dx\right)e_i. \tag{3.55}$$

for some positive parameter γ. Our target is finding a suitable parameter γ in order to get the minimal loss of resolution. Now, let us move to the main result of Seidman problem.

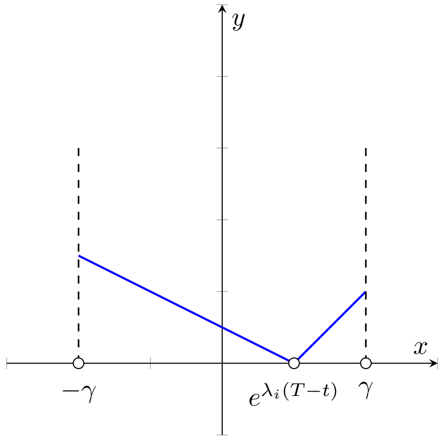


Figure 3.1 – Graph of function $y = \left|e^{\lambda_i(T-t)} - x\right|$ when $\gamma > e^{\lambda_i(T-t)}$.

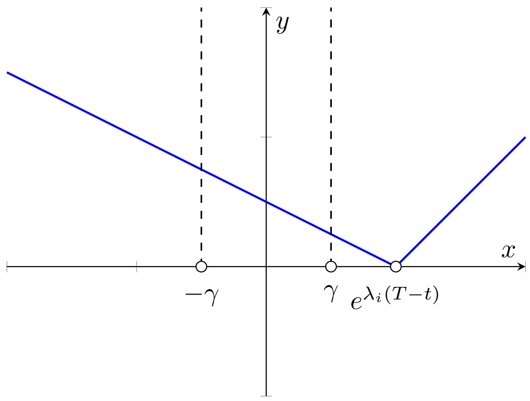


Figure 3.2 – Graph of function $y = \left|e^{\lambda_i(T-t)} - x\right|$ when $\gamma < e^{\lambda_i(T-t)}$.

Theorem 3.3. *(see [Se1, Th.3.1, p.166])*
Let u be the solution of (3.51) such that $M := \|u(\cdot,0)\|_{L^2(\Omega)} < \infty$. Let $0 < t < T$. Suppose $\delta > 0$ and $\mathbb{f}_\delta \in L^2(\Omega)$ are given such that

$$\|\mathbb{f} - \mathbb{f}_\delta\|_{L^2(\Omega)} \leq \delta. \tag{3.56}$$

Then there exists $\mathbb{g}_\delta \in L^2(\Omega)$ such that

$$\|u(\cdot,t) - \mathbb{g}_\delta\|_{L^2(\Omega)} \leq \epsilon(\delta) \quad \text{where} \quad \epsilon(\delta) \xrightarrow{\delta\to 0} 0. \tag{3.57}$$

1. ***Reconstruction formula***
   The approximation solution $\mathbb{g}_\delta$ is constructed as below

$$\mathbb{g}_\delta := \sum_{i\geq 1} \min\left\{e^{\lambda_i(T-t)}, \frac{t}{T}\left(\frac{M}{\delta}\right)^{1-\frac{t}{T}}\right\}\left(\int_\Omega \mathbb{f}_\delta(x)e_i(x)dx\right)e_i \quad . \tag{3.58}$$

2. ***Convergence rate***
   The convergence of the approximate solution $\mathbb{g}_\delta$ in (3.58) is estimated as

$$\|u(\cdot,t) - \mathbb{g}_\delta\|_{L^2(\Omega)} \leq M^{1-\frac{t}{T}}\delta^{\frac{t}{T}}. \tag{3.59}$$

Moreover, the convergence (3.59) is optimal in sense of Tautenhahn (see more in Section 3.6.1.2).

Proof of Theorem 3.3.
First of all, let us state two technical lemmas, whose proofs can be found in Section 3.7.

Lemma 3.2. *Let $0 < t < T$ and $\gamma > 0$. Consider the following function*

$$\begin{aligned} F_1 : \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto e^{-xt} - \gamma e^{-xT}. \end{aligned} \tag{3.60}$$

Then $\sup_{x\in\mathbb{R}} F_1(x) = F_1\left(\frac{\ln\frac{\gamma T}{t}}{T-t}\right) = \left(1 - \frac{t}{T}\right)\left(\frac{t}{\gamma T}\right)^{\frac{t}{T-t}}$.

Lemma 3.3. *Let A, B, s be positive real numbers. Consider the following function*

$$\begin{aligned} F_2 : (0,+\infty) &\rightarrow (0,+\infty) \\ x &\mapsto Ax^{-s} + Bx. \end{aligned} \tag{3.61}$$

Then $\inf_{x\in(0,\infty)} F_2(x) = F_2\left(\left(\frac{As}{B}\right)^{\frac{1}{1+s}}\right) = (As)^{\frac{1}{1+s}} B^{\frac{s}{1+s}}\left(1 + \frac{1}{s}\right)$.

Now, we can start the proof of Theorem 3.3. Let us define

$$\mathfrak{g}_\delta := \sum_{i\geq 1} \min\{e^{\lambda_i(T-t)}; \gamma\} \left(\int_\Omega \mathfrak{f}_\delta(x)e_i(x)dx\right) e_i \tag{3.62}$$

and

$$\mathfrak{g}_T := \sum_{i\geq 1} \min\{e^{\lambda_i(T-t)}; \gamma\} \left(\int_\Omega \mathfrak{f}(x)e_i(x)dx\right) e_i, \tag{3.63}$$

for some $\gamma > 0$ which will be chosen later. The error estimate is established by using the triangle inequality, i.e

$$\|u(\cdot,t) - \mathfrak{g}_\delta\|_{L^2(\Omega)} \leq \|u(\cdot,t) - \mathfrak{g}_T\|_{L^2(\Omega)} + \|\mathfrak{g}_T - \mathfrak{g}_\delta\|_{L^2(\Omega)}. \tag{3.64}$$

Step 1: Compute $\|\mathfrak{g}_T - \mathfrak{g}_\delta\|_{L^2(\Omega)}$.
On one hand, we have

$$\|\mathfrak{g}_T - \mathfrak{g}_\delta\|_{L^2(\Omega)} = \left\|\sum_{i\geq 1} \min\{e^{\lambda_i(T-t)}, \gamma\} \left(\int_\Omega (\mathfrak{f} - \mathfrak{f}_\delta)(x)e_i(x)dx\right) e_i\right\|_{L^2(\Omega)} \leq \gamma\delta. \tag{3.65}$$

Step 2: Compute $\|u(\cdot,t) - \mathfrak{g}_T\|_{L^2(\Omega)}$.
On the other hand, we also have

$$\begin{aligned} &\|u(\cdot,t) - \mathfrak{g}_T\|_{L^2(\Omega)} \\ =\ & \left\|\sum_{i\geq 1}\left(e^{-\lambda_i t} - \min\{e^{\lambda_i(T-t)}; \gamma\}e^{-\lambda_i T}\right)\int_\Omega u(x,0)e_i(x)dx e_i\right\|_{L^2(\Omega)} \\ \leq\ & M \sup\left\{\left|e^{-\lambda_i t} - \min\{e^{\lambda_i(T-t)}; \gamma\}e^{-\lambda_i T}\right| : i \geq 1\right\} \\ =\ & M\sup\{e^{-\lambda_i t} - \gamma e^{-\lambda_i T} : i \geq 1 \ \text{ with } \ e^{\lambda_i(T-t)} > \gamma\} \\ \leq\ & M\sup\{e^{-\lambda t} - \gamma e^{-\lambda T} : \lambda \in \mathbb{R}\}. \end{aligned} \tag{3.66}$$

In the first equality of (3.66), we use the following formula

$$\mathfrak{f} = u(\cdot,T) = \sum_{i\geq 1} e^{-\lambda_i T}\left(\int_\Omega u(x,0)e_i(x)dx\right) e_i \tag{3.67}$$

and

$$u(\cdot,t) = \sum_{i\geq 1} e^{-\lambda_i t}\left(\int_\Omega u(x,0)e_i(x)dx\right) e_i. \tag{3.68}$$

The second equality of (3.66) comes from the fact that all the terms corresponding to i such that $e^{\lambda_i(T-t)} \le \gamma$ equal 0. Now, applying Lemma 3.2, one obtains

$$\sup\{e^{-\lambda t} - \gamma e^{-\lambda T} : \lambda \in \mathbb{R}\} = \left(1 - \frac{t}{T}\right)\left(\frac{t}{\gamma T}\right)^{\frac{t}{T-t}}. \tag{3.69}$$

Thus, it implies from (3.66) that

$$\|u(\cdot, t) - \mathfrak{g}_T\|_{L^2(\Omega)} \le M\left(1 - \frac{t}{T}\right)\left(\frac{t}{\gamma T}\right)^{\frac{t}{T-t}}. \tag{3.70}$$

Step 3: Compute $\|u(\cdot, t) - \mathfrak{g}_\delta\|_{L^2(\Omega)}$.
Combining (3.65) and (3.70) and using the triangle inequality, one gets

$$\begin{aligned} \|u(\cdot, t) - \mathfrak{g}_\delta\|_{L^2(\Omega)} &\le \|u(\cdot, t) - \mathfrak{g}_T\|_{L^2(\Omega)} + \|\mathfrak{g}_T - \mathfrak{g}_\delta\|_{L^2(\Omega)} \\ &\le \gamma\delta + M\left(1 - \frac{t}{T}\right)\left(\frac{t}{\gamma T}\right)^{\frac{t}{T-t}}. \end{aligned} \tag{3.71}$$

In order to minimize the right-hand side of (3.71), we apply Lemma 3.3 with $A = M\left(1 - \frac{t}{T}\right)\left(\frac{t}{T}\right)^{\frac{t}{T-t}}$, $B = \delta$ and $s = \frac{t}{T-t}$. Then, the choice of γ is

$$\gamma = \left(\frac{M\left(1 - \frac{t}{T}\right)\left(\frac{t}{T}\right)^{\frac{t}{T-t}} \frac{t}{T-t}}{\delta}\right)^{1-\frac{t}{T}} = \frac{t}{T}\left(\frac{M}{\delta}\right)^{1-\frac{t}{T}}. \tag{3.72}$$

With this choice of γ, we get from (3.71) that

$$\begin{aligned} &\|u(\cdot, t) - \mathfrak{g}_\delta\|_{L^2(\Omega)} \\ \le\ & \left(M\left(1 - \frac{t}{T}\right)\left(\frac{t}{T}\right)^{\frac{t}{T-t}} \frac{t}{T-t}\right)^{\frac{1}{1+\frac{t}{T-t}}} \delta^{\frac{\frac{t}{T-t}}{1+\frac{t}{T-t}}}\left(1 + \frac{1}{\frac{t}{T-t}}\right) \\ =\ & \left(M\left(\frac{t}{T}\right)^{\frac{T}{T-t}}\right)^{1-\frac{t}{T}} \delta^{\frac{t}{T}} \frac{T}{t} \\ =\ & M^{1-\frac{t}{T}}\delta^{\frac{t}{T}}. \end{aligned} \tag{3.73}$$

This completes the proof of Theorem 3.3.

3.4 Proof of Theorem 3.1

In this section, we will reconstruct the initial solution $u(\cdot, 0)$ for the system (3.21) from the noisy data of $u(\cdot, T)|_\Omega$ by the optimal filtering method of Seidman which is mentioned in the previous section. Now, we will use the same idea with the filtering method of Seidman for solving the backward problem. Precisely, we construct the approximate solution at time 0 as below:

$$\mathfrak{g}_\delta := \sum_{i \ge 1} \min\{e^{\lambda_i T}, \alpha\}\left(\int_\Omega \mathfrak{f}_\delta(x) e_i(x) dx\right) e_i, \tag{3.74}$$

for some regularization parameter $\alpha > 0$ which will be chosen later. In progress of solving the minimization problem, we need $\alpha > 1$. This condition requires that the noisy level δ should be small enough. However, for the other case, i.e when δ is big, the backward problem is also solved with the approximate solution can be chosen by 0. Now, let us move to the detailed proof.
Case 1: When $\delta \ge \frac{\sqrt{T}M}{b}$.
We take $\mathfrak{g}_\delta = 0$, then

$$\|u(\cdot, 0) - \mathfrak{g}_\delta\|_{L^2(\Omega)} = \|u(\cdot, 0)\|_{L^2(\Omega)} \le \frac{M}{\sqrt{\lambda_1}} \le \frac{b\delta}{\sqrt{\lambda_1 T}}. \tag{3.75}$$

Case 2: When $\delta < \frac{\sqrt{T}M}{b}$.
First of all, we state the following classical technical lemma whose proof is mentioned in Subsection 3.7.

Lemma 3.4. *Given $\alpha \geq 1$. Then the following function*

$$\begin{aligned} F_\alpha : (0,+\infty) &\to (0,+\infty) \\ x &\mapsto \frac{1-\alpha e^{-x}}{\sqrt{x}} \end{aligned} \tag{3.76}$$

has the property that $\sup_{x\in(0,+\infty)} F_\alpha(x) = F_\alpha(\mathcal{P}^{-1}(\alpha))$ *where $\mathcal{P}$ is defined in (3.27).*

Now, we compute the error $\|u(\cdot,0) - \mathfrak{g}_\delta\|_{L^2(\Omega)}$ by splitting into the approximate error $\|u(\cdot,0) - \mathfrak{g}_T\|_{L^2(\Omega)}$ and the data error $\|\mathfrak{g}_T - \mathfrak{g}_\delta\|_{L^2(\Omega)}$. Here, $\mathfrak{g}_T$ is the approximate solution corresponding to exact data $u(\cdot,T)$, which is

$$\mathfrak{g}_T := \sum_{i\geq 1} \min\{e^{\lambda_i T},\alpha\} \left(\int_\Omega \mathfrak{f}(x) e_i(x)dx\right) e_i.$$

Step 1: Compute $\|\mathfrak{g}_\delta - \mathfrak{g}_T\|_{L^2(\Omega)}$.
The data error is estimated by

$$\|\mathfrak{g}_\delta - \mathfrak{g}_T\|_{L^2(\Omega)} \leq \left\| \sum_{i\geq 1} \min\{e^{\lambda_i T},\alpha\} \left(\int_\Omega (\mathfrak{f}-\mathfrak{f}_\delta)(x) e_i(x)dx\right) e_i \right\|_{L^2(\Omega)} \leq \alpha\delta. \tag{3.77}$$

Step 2: Compute $\|u(\cdot,0) - \mathfrak{g}_T\|_{L^2(\Omega)}$.
The approximate data is computed as below

$$\begin{aligned} &\|u(\cdot,0) - \mathfrak{g}_T\|_{L^2(\Omega)} \\ = &\left\| \sum_{i\geq 1} \left(1 - \min\{e^{\lambda_i T},\alpha\} e^{-\lambda_i T}\right) \left(\int_\Omega u(x,0) e_i(x)dx\right) e_i \right\|_{L^2(\Omega)} \\ \leq &\ M \sup\left\{ \left| 1 - \min\{e^{\lambda_i T},\alpha\} e^{-\lambda_i T} \frac{1}{\sqrt{\lambda_i}} \right| : i \geq 1 \right\} \\ = &\ \sqrt{T} M \sup\left\{ \frac{1-\alpha e^{-\lambda_i T}}{\sqrt{\lambda_i T}} : i \geq 1 \ \text{ with } \ e^{\lambda_i T} > \alpha \right\} \\ \leq &\ \sqrt{T} M \sup\left\{ \frac{1-\alpha e^{-\lambda T}}{\sqrt{\lambda T}} : \lambda > 0 \right\}. \end{aligned} \tag{3.78}$$

In the first equality, we use the following formula

$$\mathfrak{f} = u(\cdot,T) = \sum_{i\geq 1} e^{-\lambda_i T} \left(\int_\Omega u(x,0) e_i(x)dx\right) e_i. \tag{3.79}$$

The second equality comes from the fact that all the terms corresponding to i such that $e^{\lambda_i T} \leq \alpha$ equal 0. Now, suppose that $\alpha \geq 1$, applying Lemma 3.4, we get

$$\sup_{\lambda>0} \frac{(1-\alpha e^{-\lambda T})}{\sqrt{\lambda T}} = \frac{1-\alpha e^{-\mathcal{P}^{-1}(\alpha)}}{\sqrt{\mathcal{P}^{-1}(\alpha)}}. \tag{3.80}$$

It deduces from (3.78) and (3.80) that

$$\|u(\cdot,0) - \mathfrak{g}_T\|_{L^2(\Omega)} \leq \sqrt{T} M \frac{1-\alpha e^{-\mathcal{P}^{-1}(\alpha)}}{\sqrt{\mathcal{P}^{-1}(\alpha)}}. \tag{3.81}$$

Step 3: Compute $\|u(\cdot,0)-\mathfrak{g}_\delta\|_{L^2(\Omega)}$.
Combining (3.77), (3.81) and using the triangle inequality, one yields

$$\begin{aligned}\|u(\cdot,0)-\mathfrak{g}_\delta\|_{L^2(\Omega)} &\le \|u(\cdot,0)-\mathfrak{g}_T\|_{L^2(\Omega)}+\|\mathfrak{g}_T-\mathfrak{g}_\delta\|_{L^2(\Omega)}\\ &\le \alpha\delta+\frac{(1-\alpha e^{-\mathcal{P}^{-1}(\alpha)})}{\sqrt{\mathcal{P}^{-1}(\alpha)}}\sqrt{T}M\\ &= \alpha e^{-\mathcal{P}^{-1}(\alpha)}e^{\mathcal{P}^{-1}(\alpha)}\delta+(1-\alpha e^{-\mathcal{P}^{-1}(\alpha)})\frac{\sqrt{T}M}{\sqrt{\mathcal{P}^{-1}(\alpha)}}.\end{aligned} \tag{3.82}$$

In order to minimize the right-hand side of (3.82), we choose α such that

$$e^{\mathcal{P}^{-1}(\alpha)}\delta=\frac{\sqrt{T}M}{\sqrt{\mathcal{P}^{-1}(\alpha)}}. \tag{3.83}$$

It is equivalent to

$$e^{\mathcal{P}^{-1}(\alpha)}\sqrt{\mathcal{P}^{-1}(\alpha)}=\frac{\sqrt{T}M}{\delta}. \tag{3.84}$$

We can rewrite (3.84) as below

$$\mathcal{Q}\left(\mathcal{P}^{-1}(\alpha)\right)=\frac{\sqrt{T}M}{\delta}. \tag{3.85}$$

Under the assumption that $\delta<\frac{\sqrt{T}M}{b}$, one has $\frac{\sqrt{T}M}{\delta}>b$. Thus, (3.85) has a unique solution:

$$\alpha=\mathcal{P}\mathcal{Q}^{-1}\left(\frac{\sqrt{T}M}{\delta}\right)>1. \tag{3.86}$$

With this choice of α, it implies from (3.82) that

$$\|u(\cdot,0)-\mathfrak{g}_\delta\|_{L^2(\Omega)}\le\delta e^{\mathcal{P}^{-1}(\alpha)}=\frac{\sqrt{T}M}{\sqrt{\mathcal{P}^{-1}(\alpha)}}=\frac{\sqrt{T}M}{\sqrt{\mathcal{Q}^{-1}\left(\frac{\sqrt{T}M}{\delta}\right)}}. \tag{3.87}$$

Step 4: Make appear logarithm error.
Now, using the fact that $\sqrt{\xi x}\le e^{\frac{\xi x}{2}}\quad\forall\xi>0\quad\forall x>0$, one has

$$y:=\mathcal{Q}(x)=\sqrt{x}e^{x}\le\frac{1}{\sqrt{\xi}}e^{(1+\frac{\xi}{2})x}. \tag{3.88}$$

It implies that

$$x=\mathcal{Q}^{-1}(y)\ge\frac{\ln\left(\sqrt{\xi}y\right)}{1+\frac{\xi}{2}}\quad\forall\xi>0. \tag{3.89}$$

It follows from (3.87) and (3.89) that

$$\|u(\cdot,0)-\mathfrak{g}_\delta\|_{L^2(\Omega)}\le\frac{\sqrt{T}M\sqrt{\left(1+\frac{\xi}{2}\right)}}{\sqrt{\ln\left(\sqrt{\xi}\frac{\sqrt{T}M}{\delta}\right)}}\qquad\forall\xi>\frac{\delta^2}{TM^2}. \tag{3.90}$$

If $\delta<1$ then we choose $\xi=\frac{1}{TM^2}$ in order to get from (3.90) that

$$\|u(\cdot,0)-\mathfrak{g}_\delta\|_{L^2(\Omega)}\le\sqrt{TM^2+\frac{1}{2}}\left(\ln\frac{1}{\delta}\right)^{-\frac{1}{2}}. \tag{3.91}$$

This completes the proof of Theorem 3.1.

3.5 Proof of Theorem 3.2

In this section, we will present the proof of Theorem 3.2, a result for the local backward problem. In the previous section, we already proved that from the observation on the whole domain, we can recover the solution at the initial time (see Theorem 3.1). Hence, a natural idea to determine the solution from the observation on a subdomain is making a connection between the data on the whole domain and the data on the subdomain. For this purpose, we use a result of the impulse controllability (see Lemma 3.1). Precisely, from $u(\cdot,T)|_\omega$, thanks to Lemma 3.1, we construct $u(\cdot,3T)|_\Omega$, then we recover $u(\cdot,0)|_\Omega$ due to Theorem 3.1. First of all, let us consider the case when the noisy level is large:

Case 1: When $\delta \geq \left(\frac{\sqrt{T}}{bC_1 e^{\frac{C_2}{T}}}\right)^{\frac{1}{\mu}} M$.

We take $\mathbb{g}_\delta = 0$, then

$$\|u(\cdot,0) - \mathbb{g}_\delta\|_{L^2(\Omega)} = \|u(\cdot,0)\|_{L^2(\Omega)} \leq \frac{M}{\sqrt{\lambda_1}} \leq \left(\frac{bC_1 e^{\frac{C_2}{T}}}{\sqrt{T}}\right)^{\frac{1}{\mu}} \frac{\delta}{\sqrt{\lambda_1}}. \tag{3.92}$$

Case 2: When $\delta < \left(\frac{\sqrt{T}}{bC_1 e^{\frac{C_2}{T}}}\right)^{\frac{1}{\mu}} M$.

In this case, it is necessary to use a weight function $e^{-\lambda_i T}$ which has the following property:

Lemma 3.5. *(see [Ar, p.83])*
Let $\{\lambda_i\}_{i\geq 1}$ be eigenvalues of Laplacian under the Dirichlet boundary condition and T be a positive number. Then

$$\sum_{i\geq 1} e^{-2\lambda_i T} < \infty.$$

The proof of Lemma 3.5 can be found in Section 3.7. Furthermore, let us define the following function, which is the extension of the function $u(x,t)$ on $\Omega \times (0,+\infty)$:

$$\begin{aligned} \hat{u} : \Omega \times (0,+\infty) &\rightarrow \mathbb{R} \\ (x,t) &\mapsto \sum_{i\geq 1} e^{-\lambda_i t} \left(\int_\Omega u(x,0) e_i(x) dx\right) e_i(x). \end{aligned} \tag{3.93}$$

Thanks to the spectral theory, one has: $\hat{u}$ satisfies

$$\begin{cases} \partial_t \hat{u} - \Delta \hat{u} = 0 & \text{in } \Omega \times (0,+\infty), \\ \hat{u} = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ \hat{u}(\cdot,0) = u(\cdot,0) & \text{in } \Omega, \\ \hat{u}(\cdot,T) = \mathbb{f} & \text{in } \omega. \end{cases} \tag{3.94}$$

Now, we start the main steps of the proof of Theorem 3.2.

Step 1: Connect $\hat{u}(\cdot,2T)|_\Omega$ and $u(\cdot,T)|_\omega$.

For any $\varepsilon > 0$, for any $i = 1, 2...$, thanks to Lemma 3.1, there exists $h_i \in L^2(\omega)$ such that

$$\begin{cases} \partial_t \psi_i - \Delta \psi_i = 0 & \text{in } \Omega \times (0,2T) \setminus \{T\}, \\ \psi_i = 0 & \text{on } \partial\Omega \times (0,2T), \\ \psi_i(\cdot,0) = e_i & \text{in } \Omega, \\ \psi_i(\cdot,T) = \psi_i(\cdot,T^-) + \mathbb{1}_\omega h_i & \text{in } \Omega, \\ \|\psi_i(\cdot,2T)\|_{L^2(\Omega)} \leq \varepsilon. \end{cases} \tag{3.95}$$

Remind that $\{e_i\}_{i\geq 1}$ are the eigenfunctions of Laplacian under the Dirichlet boundary condition. Moreover, there exist positive constants $\mathcal{M}_1, \mathcal{M}_2$ and θ depending on Ω and ω, such that the following estimate holds

$$\|h_i\|_{L^2(\omega)} \leq \frac{\mathcal{M}_1 e^{\frac{\mathcal{M}_2}{T}}}{\varepsilon^\theta} \quad \forall i \geq 1. \tag{3.96}$$

Multiplying both sides of the equation $\partial_t \hat{u} - \Delta \hat{u} = 0$ by $\psi_i(\cdot, 2T - t)$ and integrating over Ω, one gets

$$\frac{d}{dt} \int_\Omega \hat{u}(x,t)\psi_i(x, 2T-t)dx = 0. \tag{3.97}$$

Integrating (3.97) over $(0,T)$ gives us

$$\int_\Omega \hat{u}(x,0)\psi_i(x,2T)dx = \int_\Omega \hat{u}(x,T)\psi_i(x,T^-)dx. \tag{3.98}$$

Integrating (3.97) over $(T, 2T)$ gives us

$$\int_\Omega \hat{u}(x,T)\psi_i(x,T)dx = \int_\Omega \hat{u}(x,2T)\psi_i(x,0)dx. \tag{3.99}$$

Combining (3.98), (3.99) and the fact $\psi_i(\cdot,T) = \psi_i(\cdot,T^-) + \mathbb{1}_\omega h_i$, one obtains

$$\int_\Omega \hat{u}(x,2T)\psi_i(x,0)dx = \int_\Omega \hat{u}(x,0)\psi_i(x,2T)dx + \int_\omega \hat{u}(x,T)h_i(x)dx. \tag{3.100}$$

Remind that $\psi_i(\cdot,0) = e_i$ and $\hat{u}(\cdot,T) = \mathbb{f}$, it follows from (3.101) that

$$\int_\Omega \hat{u}(x,2T)e_i(x)dx - \int_\omega \mathbb{f}(x)h_i(x)dx = \int_\Omega \hat{u}(x,0)\psi_i(x,2T)dx. \tag{3.101}$$

Step 2: Approximate $\hat{u}(\cdot,3T)|_\Omega$.
Let us remind that $\|h_i\|_{L^2(\omega)} \leq \frac{C(T)}{\varepsilon^\theta}$ and $\|\psi_i(\cdot,2T)\|_{L^2(\Omega)} \leq \varepsilon \;\; \forall i = 1, 2, ...$ Hence, if we take the infinite sum from $i=1$ to ∞ for getting information of $\hat{u}(\cdot,2T)$ on the whole domain Ω then we have a difficulty, that is: There is no $\varepsilon = \varepsilon(i) > 0$ such that $\sum_{i\geq 1} \varepsilon(i)^2 < \infty$ and $\sum_{i\geq 1} \frac{1}{\varepsilon(i)^{2\theta}} < \infty$ for some $\theta > 0$. To overcome this difficulty, we multiply both sides of (3.101) by a weight function $e^{-\lambda_i T}$ and take the sum from $i=1$ to ∞ in order to get

$$\hat{u}(\cdot,3T) - \sum_{i\geq 1} e^{-\lambda_i T}\left(\int_\omega \mathbb{f}(x)h_i(x)dx\right)e_i = \sum_{i\geq 1} e^{-\lambda_i T}\left(\int_\Omega \hat{u}(x,0)\psi_i(x,2T)dx\right)e_i. \tag{3.102}$$

Using Cauchy-Schwarz inequality and the fact that $\|\psi_i(\cdot,2T)\|_{L^2(\Omega)} \leq \varepsilon \;\; \forall i \geq 1$, yields

$$\left\| \hat{u}(\cdot,3T) - \sum_{i\geq 1} e^{-\lambda_i T}\left(\int_\omega \mathbb{f}(x)h_i(x)dx\right)e_i \right\|_{L^2(\Omega)} \leq \left(\sum_{i\geq 1} e^{-2\lambda_i T}\right)^{\frac{1}{2}} \frac{M\varepsilon}{\sqrt{\lambda_1}}. \tag{3.103}$$

In (3.103), we also use the following argument

$$\|\hat{u}(\cdot,0)\|_{L^2(\Omega)} = \|u(\cdot,0)\|_{L^2(\Omega)} \leq \frac{M}{\sqrt{\lambda_1}}. \tag{3.104}$$

Thanks to Lemma 3.5, one gets: There exists a positive constant S such that

$$\left(\sum_{i\geq 1} e^{-2\lambda_i T}\right)^{\frac{1}{2}} \leq S. \tag{3.105}$$

Gathering (3.103) and (3.105) gives us

$$\left\| \hat{u}(\cdot,3T) - \sum_{i\geq 1} e^{-\lambda_i T}\left(\int_\omega \mathbb{f}(x)h_i(x)dx\right)e_i \right\|_{L^2(\Omega)} \leq E_1 M\varepsilon, \tag{3.106}$$

where $E_1 := \frac{S}{\sqrt{\lambda_1}}$.

Step 3: Make appear $\mathbb{f}_\delta$.
Now, we will make appear $\mathbb{f}_\delta$ by using the following triangle inequality

$$
\begin{aligned}
&\left\| \hat{u}(\cdot,3T) - \sum_{i\geq 1} e^{-\lambda_i T}\left(\int_\omega \mathbb{f}_\delta(x)h_i(x)dx\right)e_i\right\|_{L^2(\Omega)} \\
\leq\ &\left\| \hat{u}(\cdot,3T) - \sum_{i\geq 1} e^{-\lambda_i T}\left(\int_\omega \mathbb{f}(x)h_i(x)dx\right)e_i\right\|_{L^2(\Omega)} \\
&+\left\| \sum_{i\geq 1} e^{-\lambda_i T}\left(\int_\omega \left(\mathbb{f}(x)-\mathbb{f}_\delta(x)\right)h_i(x)dx\right)e_i\right\|_{L^2(\Omega)}. \qquad (3.107)
\end{aligned}
$$

Using Cauchy-Schwarz inequality for the second term in (3.107), one gets

$$
\begin{aligned}
&\left\| \sum_{i\geq 1} e^{-\lambda_i T}\left(\int_\omega \left(\mathbb{f}(x)-\mathbb{f}_\delta(x)\right)h_i(x)dx\right)e_i\right\|_{L^2(\Omega)} \\
\leq\ &\left(\sum_{i\geq 1} e^{-2\lambda_i T}\|\mathbb{f}-\mathbb{f}_\delta\|^2_{L^2(\omega)}\|h_i\|^2_{L^2(\omega)}\right)^{\frac{1}{2}}. \qquad (3.108)
\end{aligned}
$$

Thanks to Lemma 3.5 and (3.96), one has:

$$
\left\| \sum_{i\geq 1} e^{-\lambda_i T}\left(\int_\omega \left(\mathbb{f}(x)-\mathbb{f}_\delta(x)\right)h_i(x)dx\right)e_i\right\|_{L^2(\Omega)} \leq \frac{E_2 e^{\frac{\mathcal{M}_2}{T}}\delta}{\varepsilon^\theta}, \qquad (3.109)
$$

where $E_2 := S\mathcal{M}_1$. Combining (3.106) and (3.107) and (3.109), yields

$$
\left\| \hat{u}(\cdot,3T) - \sum_{i\geq 1} e^{-\lambda_i T}\left(\int_\omega \mathbb{f}_\delta(x)h_i(x)dx\right)e_i\right\|_{L^2(\Omega)} \leq E_1 M\varepsilon + \frac{E_2 e^{\frac{\mathcal{M}_2}{T}}\delta}{\varepsilon^\theta}. \qquad (3.110)
$$

In order to minimize the right-hand side of (3.110), we apply Lemma 3.3 with $A = E_2 e^{\frac{\mathcal{M}_2}{T}}\delta$; $B = E_1 M$ and $s = \theta$. Then, we obtain

$$
\min_{\varepsilon>0}\left\{E_1 M\varepsilon + \frac{E_2 e^{\frac{\mathcal{M}_2}{T}}\delta}{\varepsilon^\theta}\right\} \text{ gets at } \varepsilon = \left(\frac{E_2 e^{\frac{\mathcal{M}_2}{T}}\delta\theta}{E_1 M}\right)^{\frac{1}{1+\theta}}. \qquad (3.111)
$$

Moreover, we also have

$$
\begin{aligned}
\min_{\varepsilon>0}\left\{E_1 M\varepsilon + \frac{E_2 e^{\frac{\mathcal{M}_2}{T}}\delta}{\varepsilon^\theta}\right\} &= \left(E_2 e^{\frac{\mathcal{M}_2}{T}}\delta\right)^{\frac{1}{1+\theta}}(E_1 M\theta)^{\frac{\theta}{1+\theta}}\left(1+\frac{1}{\theta}\right) \\
&= Ce^{\frac{\mu\mathcal{M}_2}{T}}M^{1-\mu}\delta^\mu, \qquad (3.112)
\end{aligned}
$$

with $C = E_1^{\frac{\theta}{1+\theta}}(E_2\theta)^{\frac{1}{1+\theta}}\left(1+\frac{1}{\theta}\right)$ and $\mu = \frac{1}{1+\theta}$. Then, the estimate (3.110) becomes

$$
\left\| \hat{u}(\cdot,3T) - \sum_{i\geq 1} e^{-\lambda_i T}\int_\omega \mathbb{f}_\delta(x)h_i(x)dx e_i\right\|_{L^2(\Omega)} \leq Ce^{\frac{\mu\mathcal{M}_2}{T}}M^{1-\mu}\delta^\mu. \qquad (3.113)
$$

Step 4: Apply the global backward result.
Theorem 3.1 says that: for any $\tau > 0$, any $\eta > 0$, any $\mathbb{f}_\eta \in L^2(\Omega)$ such that $\|\hat{u}(\cdot,\tau) - \mathbb{f}_\eta\|_{L^2(\Omega)} \leq \eta$

then there exists $\mathbb{g}_\eta \in L^2(\Omega)$ satisfying $\|\hat{u}(\cdot,0) - \mathbb{g}_\eta\|_{L^2(\Omega)} \leq \epsilon(\eta)$ where $\epsilon(\eta) \xrightarrow{\eta\to 0} 0$. From the facts that $h_i \in L^2(\omega)$, $\mathbb{f}_\delta \in L^2(\omega)$ and $\sum\limits_{i\geq 1} e^{-2\lambda_i T} < \infty$, one gets

$$\sum_{i\geq 1} e^{-\lambda_i T}\left(\int_\omega \mathbb{f}_\delta(x)h_i(x)dx\right)e_i \in L^2(\Omega). \tag{3.114}$$

Put $C_1 := \frac{C}{\sqrt{3}}$ and $C_2 := \mu\mathcal{M}_2$, then the assumptions in Theorem 3.1 are satisfied with

$$\tau = 3T,$$
$$\eta = \sqrt{3}C_1 e^{\frac{C_2}{T}} M^{1-\mu}\delta^\mu,$$
$$\mathbb{f}_\eta = \sum_{i\geq 1} e^{-\lambda_i T}\left(\int_\omega \mathbb{f}_\delta(x)h_i(x)dx\right)e_i.$$

As a consequence, there exists $\mathbb{g}_\delta \in L^2(\Omega)$ such that

$$\|\hat{u}(\cdot,0) - \mathbb{g}_\delta\|_{L^2(\Omega)} \leq \epsilon(\delta) \text{ where } \epsilon(\delta) \xrightarrow{\delta\to 0} 0. \tag{3.115}$$

The fact that $\hat{u}(\cdot,0) = u(\cdot,0)$ gives us

$$\|u(\cdot,0) - \mathbb{g}_\delta\|_{L^2(\Omega)} \leq \epsilon(\delta) \text{ where } \epsilon(\delta) \xrightarrow{\delta\to 0} 0. \tag{3.116}$$

Moreover, under the condition that

$$\delta < \left(\frac{\sqrt{T}}{bC_1 e^{\frac{C_2}{T}}}\right)^{\frac{1}{\mu}} M, \tag{3.117}$$

we get

$$\begin{aligned}
\eta &:= \sqrt{3}C_1 e^{\frac{C_2}{T}} M^{1-\mu}\delta^\mu \\
&\leq \sqrt{3}C_1 e^{\frac{C_2}{T}} M^{1-\mu}\frac{\sqrt{T}M^\mu}{bC_1 e^{\frac{C_2}{T}}} \\
&= \frac{\sqrt{\tau}M}{b}.
\end{aligned} \tag{3.118}$$

Hence, according to Theorem 3.1, we get the reconstruction formula for the approximate solution as below

$$\begin{aligned}
\mathbb{g}_\delta &= \sum_{i\geq 1}\min\{e^{\lambda_i\tau},\beta\}\left(\int_\Omega \mathbb{f}_\eta(x)e_i(x)dx\right)e_i \\
&= \sum_{i\geq 1}\min\{e^{3\lambda_i T},\beta\}\left(\int_\Omega\left(\sum_{j\geq 1}e^{-\lambda_j T}\left(\int_\omega \mathbb{f}_\delta(s)h_j(s)ds\right)e_j(x)\right)e_i(x)dx\right)e_i \\
&= \sum_{i\geq 1}\min\{e^{3\lambda_i T},\beta\}e^{-\lambda_i T}\left(\int_\omega \mathbb{f}_\delta(x)h_i(x)dx\right)e_i.
\end{aligned} \tag{3.119}$$

Here

$$\begin{aligned}
\beta &= \mathcal{P}\mathcal{Q}^{-1}\left(\frac{\sqrt{\tau}M}{\eta}\right) \\
&= \mathcal{P}\mathcal{Q}^{-1}\left(\frac{\sqrt{3T}M}{\sqrt{3}C_1 e^{\frac{C_2}{T}}M^{1-\mu}\delta^\mu}\right) \\
&= \mathcal{P}\mathcal{Q}^{-1}\left(\frac{\sqrt{T}}{C_1 e^{\frac{C_2}{T}}}\left(\frac{M}{\delta}\right)^\mu\right).
\end{aligned} \tag{3.120}$$

Furthermore, the convergence rate is

$$\begin{aligned}
\|u(\cdot,0)-\mathfrak{g}_\delta\|_{L^2(\Omega)} = \|\hat{u}(\cdot,0)-\mathfrak{g}_\delta\|_{L^2(\Omega)} &\leq \frac{\sqrt{\tau}M}{\sqrt{\mathcal{Q}^{-1}\left(\frac{\sqrt{\tau}M}{\eta}\right)}} \\
&= \frac{\sqrt{3T}M}{\sqrt{\mathcal{Q}^{-1}\left(\frac{\sqrt{3T}M}{\sqrt{3}C_1 e^{\frac{C_2}{T}} M^{1-\mu}\delta^\mu}\right)}} \\
&\leq \frac{\sqrt{3T}M}{\sqrt{\mathcal{Q}^{-1}\left(\frac{\sqrt{T}}{C_1 e^{\frac{C_2}{T}}}\left(\frac{M}{\delta}\right)^\mu\right)}}. \qquad (3.121)
\end{aligned}$$

Step 5: Bound of $\|\mathfrak{g}_\delta\|_{L^2(\Omega)}$ and logarithm error.
Bound of $\|\mathfrak{g}_\delta\|_{L^2(\Omega)}$.
Remind that

$$\mathfrak{g}_\delta = \sum_{i\geq 1} \min\{e^{3\lambda_i T}, \beta\} e^{-\lambda_i T}\left(\int_\omega \mathfrak{f}_\delta(x) h_i(x)dx\right) e_i, \qquad (3.122)$$

with

$$\beta = \mathcal{P}\mathcal{Q}^{-1}\left(\frac{\sqrt{T}}{C_1 e^{\frac{C_2}{T}}}\left(\frac{M}{\delta}\right)^\mu\right). \qquad (3.123)$$

The fact that $\mathcal{P}\mathcal{Q}^{-1}(x) \leq x \quad \forall x \in [b,+\infty)$ gives us

$$\beta \leq \frac{\sqrt{T}}{C_1 e^{\frac{C_2}{T}}}\left(\frac{M}{\delta}\right)^\mu. \qquad (3.124)$$

Moreover, thanks to (3.96) and (3.111), the following estimation for the cost of control function h_i holds for any $i \geq 1$

$$\begin{aligned}
\|h_i\|_{L^2(\omega)} &\leq \frac{\mathcal{M}_1 e^{\frac{\mathcal{M}_2}{T}}}{\varepsilon^\theta} \\
&= \mathcal{M}_1 e^{\frac{\mathcal{M}_2}{T}}\left(\frac{E_1 M}{E_2 e^{\frac{\mathcal{M}_2}{T}}\delta\theta}\right)^{\frac{\theta}{1+\theta}} \\
&\leq K e^{\frac{\mu\mathcal{M}_2}{T}}\left(\frac{M}{\delta}\right)^{1-\mu}, \qquad (3.125)
\end{aligned}$$

for some positive constant K. Thus, we obtain (notice that $C_2 = \mu\mathcal{M}_2$)

$$\begin{aligned}
\|\mathfrak{g}_\delta\|_{L^2(\Omega)} &\leq \frac{\sqrt{T}}{C_1 e^{\frac{C_2}{T}}}\left(\frac{M}{\delta}\right)^\mu\left(\sum_{i\geq 1} e^{-2\lambda_i T}\right)^{\frac{1}{2}} K e^{\frac{C_2}{T}}\left(\frac{M}{\delta}\right)^{1-\mu}\|\mathfrak{f}_\delta\|_{L^2(\omega)} \\
&\leq C\sqrt{T}\frac{M}{\delta}\|\mathfrak{f}_\delta\|_{L^2(\omega)}, \qquad (3.126)
\end{aligned}$$

for some constant $C > 0$ only depending on Ω and ω.
Logarithm error.
Using Remark 3.1 of Theorem 3.1, which is

$$\|u(\cdot,0)-\mathfrak{g}_\delta\|_{L^2(\Omega)} \leq \frac{\sqrt{\tau}M\sqrt{1+\frac{\xi}{2}}}{\sqrt{\ln\left(\sqrt{\xi}\frac{\sqrt{\tau}M}{\eta}\right)}} \qquad \forall \xi > \frac{\eta^2}{\tau M^2}, \qquad (3.127)$$

we get

$$\|u(\cdot,0)-\mathbb{g}_\delta\|_{L^2(\Omega)} \le \frac{\sqrt{3T}M\sqrt{1+\frac{\xi}{2}}}{\sqrt{\ln\left(\sqrt{\xi}\frac{\sqrt{T}}{C_1 e^{\frac{C_2}{T}}}\left(\frac{M}{\delta}\right)^\mu\right)}} \quad \forall \xi > \left(\frac{C_1 e^{\frac{C_2}{T}}}{\sqrt{T}}\left(\frac{\delta}{M}\right)^\mu\right)^2. \tag{3.128}$$

When $\delta < 1$, we choose $\xi = \left(\frac{C_1 e^{\frac{C_2}{T}}}{\sqrt{T}}\left(\frac{1}{M}\right)^\mu\right)^2$ in order to get

$$\|u(\cdot,0)-\mathbb{g}_\delta\|_{L^2(\Omega)} \le \frac{\sqrt{3T}M}{\sqrt{\mu}}\sqrt{1+\frac{1}{2}\left(\frac{C_1 e^{\frac{C_2}{T}}}{\sqrt{T}}\left(\frac{1}{M}\right)^\mu\right)^2}\left(\ln\frac{1}{\delta}\right)^{-\frac{1}{2}}. \tag{3.129}$$

This completes the proof of Theorem 3.2.

3.6 Further comments

3.6.1 Optimality

3.6.1.1 Introduction

In this section, we answer the question concerning the best possible worst case error for identifying the approximate solution from the noisy data. Roughly speaking, a regularization method is called *optimal* if it achieves the best worst case error and *order optimal* if it is optimal up to a multiplicative constant. This property is named the optimality of regularization methods, which is investigated by Vainikko (see [Va1] or [Va2]) and then by Tautenhahn (see [Ta1], [Ta2] or [TaS]) or Hohage (see [Ho1] or [Ho2]). In order to introduce some notifications, we would like to restate our problem in an abstract way: Let $S : L^2(\Omega) \to L^2(\Omega)$ be a linear bounded operator. We consider the problem of identifying the unknown solution $\mathbb{g} \in L^2(\Omega)$ of the ill-posed inverse problem $S\mathbb{g} = \mathbb{f}$, where instead of $\mathbb{f}$, a noisy data $\mathbb{f}_\delta$ is available with the noisy level δ. Let $\mathcal{M}$, called the source set, be a bounded set in $L^2(\Omega)$ which contains $\mathbb{g}$ satisfying the priori condition. In detail, in this Subsection, we will focus on the optimality of three problems: Seidman problem, backward problem and local backward problem.

Problem	Operator S	Known data $\mathbb{f}$	Unknown solution $\mathbb{g}$	Source set $\mathcal{M}$
Seidman	$e^{-\Delta(T-t)}$	$u(\cdot,T)\vert_\Omega$	$u(\cdot,t)\vert_\Omega(t \in (0,T))$	$\{u(\cdot,t) \in L^2(\Omega) : \Vert u(\cdot,0)\Vert_{L^2(\Omega)} = M\}$
Backward	$e^{-\Delta(T)}$	$u(\cdot,T)\vert_\Omega$	$u(\cdot,0)\vert_\Omega$	$\{u(\cdot,0) \in H_0^1(\Omega) : \Vert u(\cdot,0)\Vert_{H_0^1(\Omega)} = M\}$
Local backward		$u(\cdot,T)\vert_\omega$	$u(\cdot,0)\vert_\Omega$	$\{u(\cdot,0) \in H_0^1(\Omega) : \Vert u(\cdot,0)\Vert_{H_0^1(\Omega)} = M\}$

Now, we state some definitions:

1/ The modulus of continuity of the operator S^{-1} on the source set $\mathcal{M}$:

$$m(\delta, S, \mathcal{M}) := \sup\{\|\mathbb{g}\|_{L^2(\Omega)} : \mathbb{g} \in \mathcal{M} \text{ and } \|S\mathbb{g}\|_{L^2(\Omega)} \le \delta\}. \tag{3.130}$$

2/ The regularization method:
An arbitrarily mapping $\mathcal{R} : L^2(\Omega) \to L^2(\Omega)$ is called a regularization method for solving $S\mathbb{g} = \mathbb{f}$ on the source set $\mathcal{M}$ with the noisy data $\mathbb{f}_\delta$ and the noisy level δ if

$$\lim_{\delta\to 0}\sup\left\{\|\mathbb{g}-\mathcal{R}\mathbb{f}_\delta\|_{L^2(\Omega)} : \mathbb{g} \in \mathcal{M} \text{ and } \|S\mathbb{g}-\mathbb{f}_\delta\|_{L^2(\Omega)} \le \delta\right\} = 0 \tag{3.131}$$

3/ The "worst case error" for identifying $\mathbb{g}_\delta$ from $\mathbb{f}_\delta$ by a regularization method $\mathcal{R}$ under the assumption that $\|\mathbb{f} - \mathbb{f}_\delta\|_{L^2(\Omega)} \le \delta$ on $\mathcal{M}$ is defined as:

$$W_{\mathcal{R}}(\delta, S, \mathcal{M}) := \sup\left\{\|\mathbb{g} - \mathcal{R}\mathbb{f}_\delta\|_{L^2(\Omega)} : \mathbb{g} \in \mathcal{M} \text{ and } \|S\mathbb{g} - \mathbb{f}_\delta\|_{L^2(\Omega)} \le \delta\right\}. \tag{3.132}$$

4/ The "best possible worst case error" is defined as the infimum over all the mappings R

$$W(\delta, S, \mathcal{M}) := \inf_{\mathcal{R}} W_{\mathcal{R}}(\delta, S, \mathcal{M}). \tag{3.133}$$

It can be shown that the infimum in (3.133) is actually attained.

Lemma 3.6. *([Ho2, Th. 5.4, p.44])*
Let $m(\delta, S, \mathcal{M})$ and $W(\delta, S, \mathcal{M})$ are respectively defined in (3.130) and (3.133), then

$$W(\delta, S, \mathcal{M}) \ge m(\delta, S, \mathcal{M}). \tag{3.134}$$

The proof can be found in Subsection 3.7. The assertion in Lemma 3.6 leads us to the following definition.

Definition 3.1. *Let $\mathcal{R} : L^2(\Omega) \to L^2(\Omega)$ be a regularization method for solving $S\mathbb{g} = \mathbb{f}$ on the source set $\mathcal{M}$ with the noisy level δ. The convergence of the method $\mathcal{R}$ is called*

i/ "optimal" on $\mathcal{M}$ if $W_{\mathcal{R}}(\delta, S, \mathcal{M}) \le m(\delta, S, \mathcal{M})$,

ii/ "order optimal" on $\mathcal{M}$ if there exists a constant $C > 1$ such that $W_{\mathcal{R}}(\delta, S, \mathcal{M}) \le Cm(\delta, S, \mathcal{M})$.

3.6.1.2 Optimality of Seidman problem

From now, let $\sigma(-\Delta)$ denote the spectrum of Laplacian, under the Dirichlet boundary condition.

Theorem 3.4. *Under the same assumptions of Theorem 3.3, if there exists $\lambda_m \in \sigma(-\Delta)$ $(m \in \mathbb{N}^*)$ satisfying*

$$\lambda_m = \frac{\ln \frac{M}{\delta}}{T} \tag{3.135}$$

then the convergence in (3.59) is optimal on $L^2(\Omega)$.

Proof of Theorem 3.4
The regularization method we use in Theorem 3.3 is

$$\begin{aligned} \mathcal{R} : L^2(\Omega) &\to L^2(\Omega) \\ e_i &\mapsto \min\{e^{\lambda_i(T-t)}, \gamma\}e_i. \end{aligned} \tag{3.136}$$

The worst case error for identifying $\mathbb{g}_\delta$ by the method $\mathcal{R}$ is rewritten as

$$W_{\mathcal{R}}(\delta, S, \mathcal{M}) := \\ \sup\left\{\|u(\cdot,t) - \mathbb{g}_\delta\|_{L^2(\Omega)} : \|u(\cdot,0)\|_{L^2(\Omega)} = M < \infty \text{ and } \|u(\cdot,T) - \mathbb{f}_\delta\|_{L^2(\Omega)} \le \delta\right\}. \tag{3.137}$$

Thanks to the error estimate (3.59), we get

$$W_R(\delta, S, \mathcal{M}) \le M^{1-\frac{t}{T}}\delta^{\frac{t}{T}}. \tag{3.138}$$

Hence, according to Definition 3.1, we only need to prove that $m(\delta, S, \mathcal{M}) = M^{1-\frac{t}{T}}\delta^{\frac{t}{T}}$. On the other hand, the modulus of continuity is rewritten as

$$m(\delta, S, \mathcal{M}) := \sup\{\|u(\cdot,t)\|_{L^2(\Omega)} : \|u(\cdot,0)\|_{L^2(\Omega)} = M < \infty \text{ and } \|u(\cdot,T)\|_{L^2(\Omega)} \le \delta\}. \tag{3.139}$$

Thanks to the stability estimate in Theorem 1.6, which is

$$\|u(\cdot,t)\|_{L^2(\Omega)} \le \|u(\cdot,0)\|_{L^2(\Omega)}^{1-\frac{t}{T}} \|u(\cdot,T)\|_{L^2(\Omega)}^{\frac{t}{T}}, \tag{3.140}$$

one has

$$m(\delta, S, \mathcal{M}) \le M^{1-\frac{t}{T}} \delta^{\frac{t}{T}}. \tag{3.141}$$

Moreover, with $\tilde{u} = M e_m e^{-\lambda_m t}$, where e_m is the eigenfunction corresponding to the eigenvalue λ_m, we get

1. $\|\tilde{u}(\cdot,0)\|_{L^2(\Omega)} = M$,
2. $\|\tilde{u}(\cdot,T)\|_{L^2(\Omega)} = Me^{-\lambda_m T} = \delta$ (thanks to assumption (3.135)),
3. $\|\tilde{u}(\cdot,t)\|_{L^2(\Omega)} = Me^{-\lambda_m t} = \left(e^{-\lambda_m T} M\right)^{\frac{t}{T}} M^{1-\frac{t}{T}} = M^{1-\frac{t}{T}} \delta^{\frac{t}{T}}$.

Thus

$$m(\delta, S, \mathcal{M}) \ge \|\tilde{u}(\cdot,t)\|_{L^2(\Omega)} = M^{1-\frac{t}{T}} \delta^{\frac{t}{T}}. \tag{3.142}$$

Combining (3.141) and (3.142), one yields

$$m(\delta, S, \mathcal{M}) = M^{1-\frac{t}{T}} \delta^{\frac{t}{T}}. \tag{3.143}$$

This completes the proof of Theorem 3.4.

3.6.1.3 Optimality of Backward problem

In this Section, we only consider the optimality of backward problem when the noisy level is small enough.

Theorem 3.5. *Suppose all the assumptions of Theorem 3.1 are satisfied and* $\delta < \frac{\sqrt{T}M}{b}$. *Then if there exists* $\lambda_m \in \sigma(-\Delta)$ *satisfying*

$$\mathcal{Q}(\lambda_m) = \frac{\sqrt{T}M}{\delta} \tag{3.144}$$

with $\mathcal{Q}$ *defined in (3.28), the convergence in (3.36) is optimal on* $H_0^1(\Omega)$.

Proof of Theorem 3.5

Thanks to Theorem 3.1, we have that: The worst case error in this case is estimated as

$$W_R(\delta, S, \mathcal{M}) \le \frac{\sqrt{T}M}{\sqrt{\mathcal{Q}^{-1}\left(\frac{\sqrt{T}M}{\delta}\right)}}. \tag{3.145}$$

Hence, according to definition 3.1, we only need to prove that

$$m(\delta, S, \mathcal{M}) = \frac{\sqrt{T}M}{\sqrt{\mathcal{Q}^{-1}\left(\frac{\sqrt{T}M}{\delta}\right)}}.$$

Based on the definition of the modulus of continuity (3.130) and the backward estimate (3.41), we get

$$m(\delta, S, \mathcal{M}) \le \frac{\sqrt{T}M}{\sqrt{\mathcal{Q}^{-1}\left(\frac{\sqrt{T}M}{\delta}\right)}}. \tag{3.146}$$

Moreover, with $\tilde{u}(x,t) = \frac{M e_m e^{-\lambda_m t}}{\sqrt{\lambda_m}}$, where e_m is the eigenfunction corresponding to the eigenvalue λ_m, we get

1. $\|\tilde{u}(\cdot,0)\|_{H_0^1(\Omega)} = M$,
2. $\|\tilde{u}(\cdot,T)\|_{L^2(\Omega)} = \frac{Me^{-\lambda_m T}}{\sqrt{\lambda_m}} = \delta$ (thanks to assumption (3.144)),
3. $\|\tilde{u}(\cdot,0)\|_{L^2(\Omega)} = \frac{M}{\sqrt{\lambda_m}} = \frac{M\sqrt{T}}{\sqrt{\lambda_m T}} = \frac{\sqrt{T}M}{\sqrt{\mathcal{Q}^{-1}\left(\frac{\sqrt{T}M}{\delta}\right)}}$.

Thus

$$m(\delta, S, \mathcal{M}) \geq \|\tilde{u}(\cdot,0)\|_{L^2(\Omega)} = \frac{\sqrt{T}M}{\sqrt{\mathcal{Q}^{-1}\left(\frac{\sqrt{T}M}{\delta}\right)}}. \tag{3.147}$$

Combining (3.146) and (3.147), one yields

$$m(\delta, S, \mathcal{M}) = \frac{\sqrt{T}M}{\sqrt{\mathcal{Q}^{-1}\left(\frac{\sqrt{T}M}{\delta}\right)}}. \tag{3.148}$$

This completes the proof of Theorem 3.4.

3.6.1.4 Optimality of Local backward problem

The local backward estimate (3.50) is not sharp, i.e there does not exist $u \in H_0^1(\Omega)$ such that the equality in (3.50) occurs. Therefore, we can not provide any conclusion about the optimality of the convergence (3.45) for the local backward problem.

3.6.2 Tikhonov method

In this section, we will use Tikhonov regularization method for solving our local backward problem. Precisely, let us state the following theorem.

Theorem 3.6. *Let u be the solution of (3.24) such that $M := \|u(\cdot,0)\|_{H_0^1(\Omega)} < \infty$. Suppose $\delta > 0$ and $\mathbb{f}_\delta \in L^2(\omega)$ are given such that*

$$\|\mathbb{f} - \mathbb{f}_\delta\|_{L^2(\omega)} \leq \delta. \tag{3.149}$$

Then there exists $\mathbb{g}_\delta \in L^2(\Omega)$ satisfying

$$\|u(\cdot,0) - \mathbb{g}_\delta\|_{L^2(\Omega)} \leq \epsilon(\delta) \quad \text{where} \quad \epsilon(\delta) \xrightarrow{\delta \to 0} 0. \tag{3.150}$$

Specially, when $\delta < 1$, the convergence rate is of order $\left(\ln \frac{1}{\delta}\right)^{-\frac{1}{2}}$. Furthermore, the reconstruction formula of the approximation and the error estimate are given below:

1. ***Reconstruction formula***
   The approximate solution $\mathbb{g}_\delta$ is constructed by taking the minimizer of the following functional

$$\begin{aligned} \mathcal{J} : H_0^1(\Omega) &\to \mathbb{R} \\ \phi^0 &\mapsto \|\phi(\cdot,T) - \mathbb{f}_\delta\|^2_{L^2(\omega)} + \frac{\delta^2}{M^2}\|\phi^0\|^2_{H_0^1(\Omega)}. \end{aligned} \tag{3.151}$$

   where ϕ is solution of the following system

$$\begin{cases} \partial_t \phi - \Delta\phi = 0 & \text{in } \Omega \times (0,T), \\ \phi = 0 & \text{on } \partial\Omega \times (0,T), \\ \phi(\cdot,0) = \phi^0 & \text{in } \Omega. \end{cases} \tag{3.152}$$

2. ***Convergence rate***
   The error estimate between the approximate solution $\mathbb{g}_\delta$ and exact solution $u(\cdot,0)$ is given as

$$\|u(\cdot,0) - \mathbb{g}_\delta\|_{L^2(\Omega)} \leq \frac{(1+\sqrt{2})M\sqrt{T}}{\sqrt{\mathcal{Q}^{-1}\left(\frac{\sqrt{T}}{K_1 e^{\frac{K_2}{T}}}\left(\frac{M}{\delta}\right)^{\mu}\right)}}, \tag{3.153}$$

for some positive constants $K_1 > 0$, $K_2 > 0$, $\mu \in (0,1)$ and $\mathcal{Q}$ defined in (3.28) .

Remark 3.3. *By using the same method for Seidman problem, we get the following error estimate*

$$\|u(\cdot,t) - \mathfrak{g}_\delta\|_{L^2(\Omega)} \le (1+\sqrt{2}) M^{1-\frac{t}{T}} \delta^{\frac{t}{T}}. \tag{3.154}$$

By using the same method for backward problem, we get the following error estimate

$$\|u(\cdot,0) - \mathfrak{g}_\delta\|_{L^2(\Omega)} \le (1+\sqrt{2}) \frac{\sqrt{T}M}{\sqrt{\mathcal{Q}^{-1}\left(\frac{\sqrt{T}M}{\delta}\right)}}. \tag{3.155}$$

In both cases, under some assumption, the convergences in the Tikhonov method are order optimal in sense of Tautenhahn (see Subsection 3.6.1).

Proof of Theorem 3.6

The Tikhonov functional $\mathcal{J}$ has a unique minimizer $\mathfrak{g}_\delta$ on $H_0^1(\Omega)$ (see [Ho2, Th.2.1, p.14]). Now, we will estimate the error estimate$\|u(\cdot,0) - \mathfrak{g}_\delta\|_{L^2(\Omega)}$.

Step 1: Apply backward estimate.
Let $\mathfrak{g}$ be the solution of (3.152) corresponding to the initial data $\mathfrak{g}(\cdot,0) = \mathfrak{g}_\delta$. Let $w := \mathfrak{g} - u$, then w satisfies

$$\begin{cases} \partial_t w - \Delta w = 0 & \text{in } \Omega \times (0,T), \\ w = 0 & \text{on } \partial\Omega \times (0,T), \\ w(\cdot,0) = u(\cdot,0) - \mathfrak{g}_\delta & \text{in } \Omega, \\ w(\cdot,T) = \mathfrak{f} - \mathfrak{g}(\cdot,T) & \text{in } \omega. \end{cases} \tag{3.156}$$

Apply the local backward estimate (3.50) for the system (3.156), one has

$$\|w(\cdot,0)\|_{L^2(\Omega)} \le \frac{\sqrt{T}\|w(\cdot,0)\|_{H_0^1(\Omega)}}{\sqrt{\mathcal{Q}^{-1}\left(\frac{\sqrt{T}}{K_1 e^{\frac{K_2}{T}}}\left(\frac{\|w(\cdot,0)\|_{H_0^1(\Omega)}}{\|w(\cdot,T)\|_{L^2(\omega)}}\right)^{\mu}\right)}}. \tag{3.157}$$

Now, we will estimate $\|w(\cdot,0)\|_{H_0^1(\Omega)}$ and $\|w(\cdot,T)\|_{L^2(\omega)}$.

Step 2: Estimate $\|w(\cdot,0)\|_{H_0^1(\Omega)}$.
By triangle inequality, we have

$$\|w(\cdot,0)\|_{H_0^1(\Omega)} = \|u(\cdot,0) - \mathfrak{g}_\delta\|_{H_0^1(\Omega)} \le \|u(\cdot,0)\|_{H_0^1(\Omega)} + \|\mathfrak{g}_\delta\|_{H_0^1(\Omega)}. \tag{3.158}$$

Remind that $\mathfrak{g}_\delta$ is the minimizer of $\mathcal{J}$ on $H_0^1(\Omega)$, then one gets $\mathcal{J}(\mathfrak{g}_\delta) \le \mathcal{J}(u(\cdot,0))$. It implies that

$$\|\mathfrak{g}_\delta\|_{H_0^1(\Omega)} \le \frac{M}{\delta}\sqrt{\mathcal{J}(\mathfrak{g}_\delta)} \le \frac{M}{\delta}\sqrt{\mathcal{J}(u(\cdot,0))}. \tag{3.159}$$

Combining (3.158) and (3.159), one yields

$$\|w(\cdot,0)\|_{H_0^1(\Omega)} \le M + \frac{M}{\delta}\sqrt{\mathcal{J}(u(\cdot,0))}. \tag{3.160}$$

Step 3: Estimate $\|w(\cdot,T)\|_{L^2(\omega)}$.
By triangle inequality, we have

$$\|w(\cdot,T)\|_{L^2(\omega)} = \|\mathfrak{f} - \mathfrak{g}(\cdot,T)\|_{L^2(\omega)} \le \|\mathfrak{f} - \mathfrak{f}_\delta\|_{L^2(\omega)} + \|\mathfrak{g}(\cdot,T) - \mathfrak{f}_\delta\|_{L^2(\omega)}. \tag{3.161}$$

The fact $\mathcal{J}(\mathfrak{g}_\delta) \le \mathcal{J}(u(\cdot,0))$ also implies that

$$\|\mathfrak{g}(\cdot,T) - \mathfrak{f}_\delta\|_{L^2(\omega)} \le \sqrt{\mathcal{J}(\mathfrak{g}_\delta)} \le \sqrt{\mathcal{J}(u(\cdot,0))}. \tag{3.162}$$

Hence, one obtains from (3.161) and (3.162)

$$\|w(\cdot,T)\|_{L^2(\omega)} \le \delta + \sqrt{\mathcal{J}(u(\cdot,0))}. \tag{3.163}$$

Step 4: Estimate $\mathcal{J}(u(\cdot,0))$.
We have

$$\begin{aligned}\mathcal{J}(u(\cdot,0)) &= \|\mathfrak{f} - \mathfrak{f}_\delta\|^2_{L^2(\omega)} + \frac{\delta^2}{M^2}\|u(\cdot,0)\|^2_{H^1_0(\Omega)} \\ &\le \delta^2 + \frac{\delta^2}{M^2}M^2 = 2\delta^2. \end{aligned} \tag{3.164}$$

Step 5: Estimate $\|w(\cdot,0)\|_{L^2(\Omega)}$.
Combining (3.160) and (3.164) gives us

$$\|w(\cdot,0)\|_{H^1_0(\Omega)} \le (1+\sqrt{2})M. \tag{3.165}$$

Combining (3.163) and (3.164) gives us

$$\|w(\cdot,T)\|_{L^2(\omega)} \le (1+\sqrt{2})\delta. \tag{3.166}$$

Thus, it implies from (3.157) that

$$\|u(\cdot,0) - \mathfrak{g}_\delta\|_{L^2(\Omega)} \le \frac{(1+\sqrt{2})M\sqrt{T}}{\sqrt{\mathcal{Q}^{-1}\left(\frac{\sqrt{T}}{K_1 e^{\frac{K_2}{T}}}\left(\frac{M}{\delta}\right)^\mu\right)}}. \tag{3.167}$$

This completes the proof of Theorem 3.6.

Comments
It is well-known that Tikhonov method is a powerful tool to solve inverse problems. This method can even be used for dealing with nonlinear systems (see [ItJ], [EnKN] or [Ne]). However, for our heat backward problems, the filtering method has some following advantages:

1. According to Section 3.6.1, the convergence of the filtering method is optimal while the convergence of the Tikhonov method is order optimal. It means, the error between the exact solution and the approximate solution by the filtering method is better than the Tikhonov method.
2. The construction of the approximate solution by the filtering method is given explicitly by a formula while the construction of the approximate solution by the Tikhonov method is based on the minimizer of a functional. Hence, in some sense, the algorithm for constructing the approximate solution by the filtering method is more simple than Tikhonov method.
3. For the Tikhonov method, the backward estimate are required. However, for the filtering method, we can tackle our problem without using the backward estimate. Furthermore, from the error estimate, we can imply the backward estimates.

3.6.3 Time dependent thermal conductivity heat equation

In this section, we will apply our main results for solving the backward problem and local backward problem of the time dependent thermal conductivity heat equation, by using a changing variable technique. Precisely, let $T > 0$ and $p \in C^1([0,T])$ such that $p(t) > 0 \quad \forall t \in [0,T]$, we consider the following system:

$$\begin{cases} \partial_t w - p(t)\Delta w = 0 & \text{in } \Omega \times (0,T), \\ w = 0 & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{3.168}$$

Our target is recovering $w(\cdot,0)$ from $w(\cdot,T)|_\Omega$ (the backward problem) and from $w(\cdot,T)|_\omega$ (the local backward problem) for the system (3.168). The backward problem with the time dependent coefficients has been considered in [TrQTT], [TuKLT] or [TuQTT], however their results are only focused on one dimensional problems. Recently, Tuan et al. (see [TuKLT]) solve the backward heat equation in the multi-dimensional case by a new general filter regularization method. Here, based on the main results for solving backward (Theorem 3.1) and local backward problem (Theorem 3.2), we use a technique of changing variable to obtain the following results.

Firstly, let us denote that $\rho_T := \int_0^T p(s)ds$. Then we can state a result for the backward problem.

Theorem 3.7. *Let w be the solution of (3.168) such that $M := \|w(\cdot,0)\|_{H_0^1(\Omega)} < \infty$. Let $\mathbb{f}_\delta \in L^2(\Omega)$ and $\delta > 0$ such that:*

$$\|\mathbb{f} - \mathbb{f}_\delta\|_{L^2(\Omega)} \leq \delta. \tag{3.169}$$

Then there exists $\mathbb{g}_\delta \in L^2(\Omega)$ such that

$$\|w(\cdot,0) - \mathbb{g}_\delta\|_{L^2(\Omega)} \leq \epsilon(\delta) \quad \text{where} \quad \epsilon(\delta) \xrightarrow{\delta\to 0} 0. \tag{3.170}$$

Specially, when $\delta < 1$, the convergence rate is of order $\left(\ln\frac{1}{\delta}\right)^{-\frac{1}{2}}$. Furthermore, the reconstruction formula of the approximation and the error estimate are explicitly given below:

1. ***Reconstruction formula***
   The approximate solution $\mathbb{g}_\delta$ is constructed as below

$$\mathbb{g}_\delta := \begin{cases} 0 & \text{if } \delta \geq \frac{\sqrt{\rho_T}M}{b}, \\ \sum_{i\geq 1} \min\{e^{\lambda_i \rho_T}, \alpha\}\left(\int_\Omega \mathbb{f}_\delta(x)e_i(x)dx\right)e_i & \text{if } \delta < \frac{\sqrt{\rho_T}M}{b}, \end{cases} \tag{3.171}$$

   Here

$$\alpha = \mathcal{P}\mathcal{Q}^{-1}\left(\frac{\sqrt{\rho_T}M}{\delta}\right), \tag{3.172}$$

   where the functions $\mathcal{P}$ and $\mathcal{Q}$ are respectively defined in (3.27) and (3.28).

2. ***Convergence rate***
   The convergence of the approximate solution $\mathbb{g}_\delta$ in (3.34) is estimated as

$$\|u(\cdot,0) - \mathbb{g}_\delta\|_{L^2(\Omega)} \leq \begin{cases} \frac{b\delta}{\sqrt{\lambda_1 \rho_T}} & \text{if } \delta \geq \frac{\sqrt{\rho_T}M}{b}, \\ \frac{\sqrt{\rho_T}M}{\sqrt{\mathcal{Q}^{-1}\left(\frac{\sqrt{\rho_T}M}{\delta}\right)}} & \text{if } \delta < \frac{\sqrt{\rho_T}M}{b}. \end{cases} \tag{3.173}$$

Secondly, we state a result for the local backward problem.

Theorem 3.8. *Let w be the solution of (3.168) such that $M := \|w(\cdot,0)\|_{H_0^1(\Omega)} < \infty$. Let $\mathbb{f}_\delta \in L^2(\omega)$ and $\delta > 0$ such that*

$$\|\mathbb{f} - \mathbb{f}_\delta\|_{L^2(\omega)} \leq \delta. \tag{3.174}$$

Then there exists $\mathbb{g}_\delta \in L^2(\Omega)$ such that

$$\|w(\cdot,0) - \mathbb{g}_\delta\|_{L^2(\Omega)} \leq \epsilon(\delta) \quad \text{where} \quad \epsilon(\delta) \xrightarrow{\delta\to 0} 0. \tag{3.175}$$

Specially, when $\delta < 1$, the convergence rate is of order $\left(\ln\frac{1}{\delta}\right)^{-\frac{1}{2}}$. Furthermore, the reconstruction formula of the approximation and the error estimate are given below:

1. ***Reconstruction formula***
   The approximation solution $\mathbb{g}_\delta$ is constructed as below

$$\mathbb{g}_\delta := \begin{cases} 0 & \text{if } \delta \geq \left(\frac{\sqrt{\rho_T}}{bC_1 e^{\frac{C_2}{\rho}T}}\right)^{\frac{1}{\mu}} M, \\ \sum_{i\geq 1} \min\{e^{3\lambda_i \rho_T}, \beta\}e^{-\lambda_i \rho_T}\left(\int_\omega \mathbb{f}_\delta(x)h_i(x)dx\right)e_i & \text{if } \delta < \left(\frac{\sqrt{\rho_T}}{bC_1 e^{\frac{C_2}{\rho}T}}\right)^{\frac{1}{\mu}} M, \end{cases} \tag{3.176}$$

for some positive constants C_1, C_2 and $\mu \in (0,1)$ depending on Ω and ω. Here

$$\beta = \mathcal{P}\mathcal{Q}^{-1}\left(\frac{\sqrt{\rho_T}}{C_1 e^{\frac{C_2}{\rho_T}}}\left(\frac{M}{\delta}\right)^{\mu}\right)$$

where the functions $\mathcal{P}$ and $\mathcal{Q}$ are respectively defined in (3.27) and (3.28). And $h_i \in L^2(\omega)(i \geq 1)$ is the impulse control at time ρ_T (see Lemma 3.1)

2. ***Convergence rate***
The convergence of the approximate solution $\mathfrak{g}_\delta$ in (3.176) is estimated as

$$\|u(\cdot,0) - \mathfrak{g}_\delta\|_{L^2(\Omega)} \leq \begin{cases} \left(\frac{bC_1 e^{\frac{C_2}{\rho}T}}{\sqrt{\rho_T}}\right)^{\frac{1}{\mu}} \frac{\delta}{\sqrt{\lambda_1}} & \text{if } \delta \geq \left(\frac{\sqrt{\rho_T}}{bC_1 e^{\frac{C_2}{\rho}T}}\right)^{\frac{1}{\mu}} M, \\ \frac{\sqrt{3\rho_T}M}{\sqrt{\mathcal{Q}^{-1}\left(\frac{\sqrt{\rho_T}}{C_1 e^{\frac{C_2}{\rho_T}}}\left(\frac{M}{\delta}\right)^{\mu}\right)}} & \text{if } \delta < \left(\frac{\sqrt{\rho_T}}{bC_1 e^{\frac{C_2}{\rho}T}}\right)^{\frac{1}{\mu}} M. \end{cases} \quad (3.177)$$

Theorem 3.7 and Theorem 3.8 are respectively applications of Theorem 3.1 and Theorem 3.2 by using the following changing variable technique:
Define

$$\begin{aligned} f : [0,T] &\to [0,\rho_T] \\ t &\mapsto \int_0^t p(s)ds. \end{aligned} \quad (3.178)$$

Thanks to the fact that $f'(t) = p(t) > 0 \;\; \forall t \in [0,T]$, we get that f is a bijective function. Let us denote $f^{-1} : [0,\rho_T] \to [0,T]$ be the inverse function of f.
Now, put

$$\begin{aligned} u : \Omega \times [0,\rho_T] &\to \mathbb{R} \\ (x,t) &\mapsto w\left(x, f^{-1}(t)\right) \end{aligned} \quad (3.179)$$

then

$$\begin{aligned} \partial_t u(x,t) &= \partial_t w(x, f^{-1}(t))(f^{-1}(t))' \\ &= \partial_t w(x, f^{-1}(t))\frac{1}{f'(f^{-1}(t))} \\ &= \partial_t w(x, f^{-1}(t))\frac{1}{p(f^{-1}(t))}. \end{aligned} \quad (3.180)$$

Thus, thanks to the fact that

$$\partial_t w(x, f^{-1}(t)) - p(f^{-1}(t))\Delta w(x, f^{-1}(t)) = 0 \;\; \forall x \in \Omega \;\; \forall t \in (0,\rho_T), \quad (3.181)$$

we get

$$\partial_t u - \Delta u = \frac{1}{p(f^{-1}(t))}\left[\partial_t w(x, f^{-1}(t)) - p(f^{-1}(t))\Delta w(x, f^{-1}(t))\right] = 0. \quad (3.182)$$

Moreover, we also have $u(x,0) = w(x, f^{-1}(0)) = w(x,0)$ and $u(\cdot,\rho_T) = w(\cdot,T)$. Thus, u satisfies the following system

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega \times (0,\rho_T), \\ u = 0 & \text{on } \partial\Omega \times (0,\rho_T), \\ u(\cdot,\rho_T) = \mathfrak{f} & \text{in } \Omega. \end{cases} \quad (3.183)$$

Under the assumptions that $M := \|w(\cdot,0)\|_{H_0^1(\Omega)} < \infty$ and $\|\mathfrak{f} - \mathfrak{f}_\delta\|_{L^2(\Omega)} \leq \delta$ (backward problem) or $\|\mathfrak{f} - \mathfrak{f}_\delta\|_{L^2(\omega)} \leq \delta$ (local backward problem), one has

$$\|u(\cdot,0)\|_{H_0^1(\Omega)} = \|w(\cdot,0)\|_{H_0^1(\Omega)} = M \quad (3.184)$$

and

$$\|u(\cdot, \rho_T) - \mathfrak{f}_\delta\|_{L^2(\Omega)} \leq \delta, \tag{3.185}$$

or

$$\|u(\cdot, \rho_T) - \mathfrak{f}_\delta\|_{L^2(\omega)} \leq \delta. \tag{3.186}$$

Thus, we can apply Theorem 3.1 or Theorem 3.2 where the observation is available at time ρ_T.

3.7 Appendix

3.7.1 Proof of Lemma 3.2

Remind that $F_1(x) = e^{-xt} - \gamma e^{-xT}$. We have

$$F_1'(x) = -te^{-xt} + \gamma T e^{-xT}. \tag{3.187}$$

The equation $F_1'(\bar{x}) = 0$ is equivalent to

$$e^{-\bar{x}(T-t)} = \frac{t}{\gamma T}. \tag{3.188}$$

Thus, the equation $F_1'(\bar{x}) = 0$ has a unique solution

$$\bar{x} = \frac{1}{T-t} \ln \frac{\gamma T}{t}. \tag{3.189}$$

On the other hand, we also have

$$F_1''(x) = t^2 e^{-xt} - \gamma T^2 e^{-xT}. \tag{3.190}$$

It implies from (3.188) that

$$e^{-\bar{x}T} = \frac{t}{\gamma T} e^{-\bar{x}T}. \tag{3.191}$$

Hence $F''(\bar{x}) = \gamma T e^{-\bar{x}T}(t - T) < 0$. Thus, we get

$$\sup_{x\in\mathbb{R}} F_1(x) \quad = \quad F_1(\bar{x}) = e^{-\bar{x}t} - \gamma e^{-\bar{x}T}. \tag{3.192}$$

Thanks to (3.188), we get

$$\sup_{x\in\mathbb{R}} F_1(x) = e^{-\bar{x}t} - \frac{t}{T} e^{-\bar{x}t}. \tag{3.193}$$

Thanks to (3.189), one obtains

$$\begin{aligned} \sup_{x\in\mathbb{R}} F_1(x) &= \left(1 - \frac{t}{T}\right) e^{-\frac{t}{T-t}\ln\left(\frac{\gamma T}{t}\right)} \\ &= \left(1 - \frac{t}{T}\right)\left(\frac{t}{\gamma T}\right)^{\frac{t}{T-t}}. \end{aligned} \tag{3.194}$$

It completes the proof of Lemma 3.2.

3.7.2 Proof of Lemma 3.3

Remind that $F_2(x) = Ax^{-s} + Bx$. We have

$$F_2'(x) = -Asx^{-s-1} + B. \tag{3.195}$$

The equation $F_2'(\bar{x}) = 0$ has a unique solution $\bar{x} = \left(\frac{As}{B}\right)^{\frac{1}{1+s}}$. Moreover, we also have

$$F_2''(x) = As(s+1)x^{-s-2} > 0 \quad \forall x > 0. \tag{3.196}$$

It implies that

$$\begin{aligned}
\inf_{x\in(0,+\infty)} F_2(x) &= F_2(\bar{x}) \\
&= A\left(\frac{As}{B}\right)^{-\frac{s}{1+s}} + B\left(\frac{As}{B}\right)^{\frac{1}{1+s}} \\
&= A^{\frac{1}{1+s}}B^{\frac{s}{1+s}}s^{-\frac{s}{1+s}} + A^{\frac{1}{1+s}}B^{\frac{s}{1+s}}s^{\frac{1}{1+s}} \\
&= (As)^{\frac{1}{1+s}}B^{\frac{s}{1+s}}\left(1 + \frac{1}{s}\right).
\end{aligned} \tag{3.197}$$

This completes the proof of Lemma 3.3.

3.7.3 Proof of Lemma 3.4

Remnd that $F_\alpha(x) = \frac{1-\alpha e^{-x}}{\sqrt{x}}$. We have

$$\begin{aligned}
F_\alpha'(x) &= \frac{\alpha e^{-x}\sqrt{x} - \frac{1}{2\sqrt{x}}(1-\alpha e^{-x})}{x} \\
&= \frac{\alpha e^{-x}(1+2x) - 1}{2x\sqrt{x}}.
\end{aligned} \tag{3.198}$$

The equation $F_\alpha'(\bar{x}) = 0$ is equivalent to

$$\alpha = \frac{e^{\bar{x}}}{1+2\bar{x}}. \tag{3.199}$$

Thanks to the fact that $\alpha > 1$, we get: There exists a unique solution $\bar{x} = \mathcal{P}^{-1}(\alpha) \in [a, +\infty)$ satisfying (3.199). Here, $\mathcal{P}$ is the function defined in (3.27).

Furthermore, we also have

$$F_\alpha'(a) = \frac{\alpha - 1}{2a\sqrt{a}} > 0. \tag{3.200}$$

Thus, $\sup_{x\in(0,\infty)} F_\alpha(x) = F_\alpha(\bar{x}) = F_\alpha(\mathcal{P}^{-1}(\alpha))$. It completes the proof of Lemma 3.4.

3.7.4 Proof of Lemma 3.5

Step 1: Prove that $\lambda_i(\Omega) \geq Ci^{\frac{2}{n}} \quad \forall i = 1, 2, ..$ for some $C > 0$.
Let us recall the following monotonicity for inclusion of eigenvalues of Laplacian with Dirichlet boundary condition.

Lemma 3.7. *(see [He, p.13])*
Let U and V be open bounded sets in $\mathbb{R}^n$ such that $U \subset V$. Then

$$\lambda_k(V) \leq \lambda_k(U), \tag{3.201}$$

where $\lambda_k(U)$ (and $\lambda_k(V)$) is k^{th} eigenvalue of Laplacian with Dirichlet boundary condition on U (and V).

Thanks to the fact that Ω is bounded in $\mathbb{R}^n$, we get: there exists $R > 0$ such that $\Omega \subset [-R, R]^n$. Applying Lemma 3.7, one has

$$\lambda_i(\Omega) \geq \lambda_i([-R, R]^n) \quad \forall i = 1, 2, ... \tag{3.202}$$

Moreover, we also have a Lemma about boundedness of eigenvalues of Laplacian under the Dirichlet boundary condition, which is

Lemma 3.8. *(see [LiM])*
For any $R > 0$, there exists a positive constant C such that

$$\lambda_i([-R, R]^n) \geq C i^{\frac{2}{n}}. \tag{3.203}$$

Here $\lambda_i([-R, R]^n)$ denotes the i^{th} eigenvalue of Laplacian under the Dirichlet boundary condition on $[-R, R]^n$.

Combining (3.202) and (3.203), one obtains: There exists a positive constant C such that

$$\lambda_i(\Omega) \geq C i^{\frac{2}{n}} \quad \forall i = 1, 2, ... \tag{3.204}$$

Step 2: Prove $\sum_{i \geq 1} e^{-2\lambda_i T} < \infty$.
It deduces that

$$\sum_{i \geq 1} e^{-2\lambda_i T} \leq \sum_{i \geq 1} e^{-2C i^{\frac{2}{n}} T}. \tag{3.205}$$

Furthermore, using the property that $e^{-x} \leq \left(\frac{n}{x}\right)^n \quad \forall x > 0 \quad \forall n > 0$, yields

$$\sum_{i \geq 1} e^{-2\lambda_i T} \leq \left(\frac{n}{2CT}\right)^n \sum_{i \geq 1} \frac{1}{i^2} = \left(\frac{n}{2CT}\right)^n \frac{\pi^2}{6}. \tag{3.206}$$

This completes the proof of Lemma 3.5.

3.7.5 Proof of Lemma 3.6

Let

$$\begin{aligned} \mathcal{O} : \Omega &\rightarrow \mathbb{R} \\ x &\mapsto 0 \end{aligned} \tag{3.207}$$

be the zero function in $L^2(\Omega)$. Take $\mathbb{g} \in M$ such that $\|S\mathbb{g}\|_{L^2(\Omega)} \leq \delta$ then for any mapping $\mathcal{R} : L^2(\Omega) \rightarrow L^2(\Omega)$ solving $S\mathbb{g} = \mathbb{f}$ on the source set $\mathcal{M}$ with the noisy level δ, we get

$$W_{\mathcal{R}}(\delta, S, \mathcal{M}) \geq \|\mathbb{g} - \mathcal{R}(\mathcal{O})\|_{L^2(\Omega)}. \tag{3.208}$$

On the other hand, $-\mathbb{g} \in \mathcal{M}$ and $\|S(-\mathbb{g})\|_{L^2(\Omega)} = \|S(\mathbb{g})\|_{L^2(\Omega)} \leq \delta$. Hence, we also have

$$W_{\mathcal{R}}(\delta, S, \mathcal{M}) \geq \|\mathbb{g} + \mathcal{R}(\mathcal{O})\|_{L^2(\Omega)}. \tag{3.209}$$

Combining (3.208) and (3.209), we obtain

$$2W_{\mathcal{R}}(\delta, S, \mathcal{M}) \geq \|\mathbb{g} - \mathcal{R}(\mathcal{O})\|_{L^2(\Omega)} + \|\mathbb{g} + \mathcal{R}(\mathcal{O})\|_{L^2(\Omega)} \geq 2\|\mathbb{g}\|_{L^2(\Omega)}. \tag{3.210}$$

Thus

$$W_{\mathcal{R}}(\delta, S, \mathcal{M}) \geq \|\mathbb{g}\|_{L^2(\Omega)}. \tag{3.211}$$

On the other hand, the inequality (3.211) is true for any $\mathcal{R}$, hence one yields

$$W(\delta, S, \mathcal{M}) = \inf_{\mathcal{R}} W_{\mathcal{R}}(\delta, S, \mathcal{M}) \geq \|\mathbb{g}\|_{L^2(\Omega)}. \tag{3.212}$$

Furthermore, the inequality (3.212) is true for any $\mathbb{g} \in \mathcal{M}$ satisfying $\|S\mathbb{g}\|_{L^2(\Omega)} \leq \delta$. Hence, one gets

$$W(\delta, S, \mathcal{M}) \geq \sup\{\|\mathbb{g}\|_{L^2(\Omega)} : \mathbb{g} \in \mathcal{M} \text{ and } \|S\mathbb{g}\|_{L^2(\Omega)} \leq \delta\} = m(\delta, S, \mathcal{M}). \tag{3.213}$$

This completes the proof of Lemma 3.6.

Conclusion

In my thesis, we solve two main problems: The null controllability of a cubic semilinear heat equation and the local backward problem for a linear heat equation.

For the *null controllability of a cubic semilinear heat equation*, we use a new strategy to construct a control function which leads the solution of a cubic heat from a small initial data to null at any time later. The novelty of this method is the construction of the control function is explicitly given. Moreover, the size of the smallness of the initial data which ensures the null controllability of the cubic semilinear heat equation is quantitative computed. Our method can also be applied for studying the controllability of more general nonlinear parabolic systems.

For the *local backward problem*, we reconstruct a source of a linear heat equation from an observation acting on a subdomain at some time later. Our special method is using a connection between a control problem and an inverse problem. The achievement of this method is the explicit formula of the reconstruction, based on a family of impulse control functions. Furthermore, the convergence rate with the logarithmic type is also provided. In addition, we also tackle our local backward problem by Tikhonov regularization method and provide the comparison with our method. Another accomplishment in this section is a result on the local backward problem for the time dependent thermal conductivity heat equation.

Future works

In this thesis, we assure the local null controllability of a semilinear heat system, where the blow up occurs. Precisely, we construct a control function acting on $\omega \times (0, T)$ which steers the solution of a semilinear heat system from a small given data at the initial time to be null at the final time. On the other hand, we also build up a control function only acting on $\omega \times \{\tau\}$ $(0 < \tau < T)$ which leads the solution of a linear heat system from any given data at the initial data to a neighbourhood of zero at the final time T (called null approximate impulse controllability). As a consquence, a natural question is:

Question 1: *Does the null approximate impulse controllability property still true for the semilinear heat system?*

In this thesis, we also can recover the initial temperature of a linear heat system from the measurement on a subdomain at some time later (named the local backward problem). By the filtering method, it requires the impulse controllability of the adjoint system while by the Tikhonov method, a conditional stability estimate is commanded. Thus, if the answer for the Question 1 is yes, the second question is coming:

Question 2: *Can the local backward problem for a semilinear system be solved by the filtering method?*

Furthermore, the conditional stability estimate for the semilinear heat system is already established (see [PhWZ] or [PhW1]). Hence, another question appears:

Question 3: *How one can tackle the local backward problem for a semilinear system by the Tikhonov method?*

In addition, by using a technique of changing variable, we also deal with the local backward problem for the equation $\partial_t u - p(t)\Delta u = 0$. This makes appear another concern:

Question 4: *By using the same technique, can we get the null controllability for the semilinear system with time dependent coefficients?*

In both main problems, the construction of the null control and the reconstruction of the source are explicitly given. Hence, a natural question arises:

Question 5: *How one can illustrate our main results by numerical method?*

Our next target is finding the answers for above questions.

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Thi Minh Nhat VO

Construction d'un contrôle et reconstruction de source pour les équations linéaires et nonlinéaires de la chaleur

Résumé : Dans cette thèse, nous étudions un problème de contrôle et un problème inverse pour les équations de la chaleur.
Notre premier travail concerne la contrôlabilité à zéro pour une équation de la chaleur semi-linéaire. Il est à noter que sans contrôle, la solution est instable et il y aura en général explosion de la solution en un temps fini. Ici, nous proposons un résultat positif de contrôlabilité à zéro sous une hypothèse quantifiée de petitesse sur la donnée initiale. La nouveauté réside en la construction de ce contrôle pour amener la solution à l'état d'équilibre.
Notre second travail aborde l'équation de la chaleur rétrograde dans un domaine borné et sous la condition de Dirichlet. Nous nous intéressons à la question suivante: peut-on reconstruire la donnée initiale à partir d'une observation de la solution restreinte à un sous-domaine et à un temps donnée? Ce problème est connu pour être mal-posé. Ici, les deux principales méthodes proposées sont: une approche de filtrage des hautes fréquences et une minimisation à la Tikhonov. A chaque fois, nous reconstruisons de manière approchée la solution et quantifions l'erreur d'approximation.

Mots clés: équation de la chaleur, équation cubique de la chaleur, inégalité d'observation, contrôlabilité à zéro, problème inverse rétrograde.

Construction of a control and reconstruction of a source for linear and nonlinear heat equations

Abstract : My thesis focuses on two main problems in studying the heat equation: Control problem and Inverse problem.
Our first concern is *the null controllability of a semilinear heat equation* which, if not controlled, can blow up in finite time. Roughly speaking, it consists in analyzing whether the solution of a semilinear heat equation, under the Dirichlet boundary condition, can be driven to zero by means of a control applied on a subdomain in which the equation evolves. Under an assumption on the smallness of the initial data, such control function is built up. The novelty of our method is computing the control function in a constructive way. Furthermore, another achievement of our method is providing a quantitative estimate for the smallness of the size of the initial data with respect to the control time that ensures the null controllability property.
Our second issue is *the local backward problem for a linear heat equation*. We study here the following question: Can we recover the source of a linear heat equation, under the Dirichlet boundary condition, from the observation on a subdomain at some time later? This inverse problem is well-known to be an ill-posed problem, i.e their solution (if exists) is unstable with respect to data perturbations. Here, we tackle this problem by two different regularization methods: The filtering method and The Tikhonov method. In both methods, the reconstruction formula of the approximate solution is explicitly given. Moreover, we also provide the error estimate between the exact solution and the regularized one.

Keywords : linear heat equation, cubic semilinear heat equation, observation estimate, null controllability, inverse problem, local backward.

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