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Henshui Zhang

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Local analysis of Loewner equation

THÈSE DIRIGÉE PAR :

Michel ZINSMEISTER	Professeur, Université d'Orléans - Directeur de thèse

RAPPORTEURS :

Joan LIND	Assistant Professor, Université de Tennesse
Yuefei WANG	Professeur, AMSS

JURY :

Lu JIN	Professeur, Université de Fudan
Weiyuan QIU	Professeur, Université de Fudan
Weixiao SHEN	Professeur, SCMS
Dejun FENG	Professeur, The Chinese University of Hong Kong
Pierre NOLIN	Associate Professor, City University of Hong Kong

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Table des matières

Chapitre 1

Introduction

In 1923, the German mathematician Loewner found the Loewner equation when he managed to solve the Bieberbach conjecture in the univalent function theory, and solved the case of $n = 3$ in the conjecture(see [22]). In 1983, the French mathematician De Brange finally proved the Bieberbach conjecture [7], and used the Loewner equation as an important tool in the proof again.

Until the beginning of this century, the Israeli mathematician Oded Schranmm changed the driving function of the Loewner equation from a general continuous function to Brownian motion in order to find the scale limit of the Loop Erased Random Walk model, (see [31]). Thus, the Loewner equation was changed from a general complex differential equation to a stochastic differential equation, which combined complex analysis and stochastic analysis, and this combination drove a new theory called SLE (Stochastic Loewner Evolution). SLE has developed rapidly with the efforts of mathematicians in many directions, and solved a series of problems in stochastic theory and statistical physics. For example, Oded Schramm, Gregory Lawler and Wendelin Werner studied chordal SLE in a series of articles , such as [16], [17] and [14], and gave rigorous mathematical proof of a series of problems in random phenomenon, which included the famous Mandelbrot conjecture, that is the boundary dimension of the two-dimensional Brownian motion is $\frac{4}{3}$, and Werner won the Fields medal of 2006 for this. Later, Stanislav Smirnov used SLE to prove the cardy formula of the critical percolation model on the triangular lattice. He not only get a conformal invariant (see [36]) but also proved that the Ising model converges to SLE_3, and won the Fields medal of 2010.

With the rapid development of SLE theory, many mathematicians of complex analysis began to study the general Loewner equation and made a series of progress (refer to the introduction in 2.3.2 of this paper). Loewner equation became an important issue of the complex analysis. We first give a simple introduction of Loenwer equation, related results could be seen at [13].

In general, for a simple curve $\gamma : [0, +\infty) \to \bar{\mathbb{H}}$ in the upper half plane, satisfies $\gamma(0) \in \mathbb{R}, \gamma((0, +\infty)) \subset \mathbb{H}$. Define $K_t = \gamma((0, t])$, then there is a unique conformal mapping from $\mathbb{H} \setminus K_t$ to $\mathbb{H}$ that satisfies $\lim\limits_{z\to\infty} g_t(z)/z = 1$. Let $\dot{g}_t(z)$ denote the derivative with respect to t of function g_t, and g_t satisfies the following equation which is called the chordal Loewner equation :

$$\dot{g}_t(z) = \frac{b(t)}{g_t(z) - \lambda(t)}, \qquad g_0(z) = z,$$

where $\lambda : [0, +\infty) \to \mathbb{R}$ is a continuous function called the driving function of the Loewner equation, and $g_t(z)$ is called the Loewner process or Loewner chain driven by λ.

It is easy to prove that $b(t) > 0$ is a strictly increasing continuous function. We can re-parameterize γ to make $B(t) = 2t$, then the function becomes

$$\dot{g}_t(z) = \frac{2t}{g_t(z) - \lambda(t)}, \qquad g_0(z) = z, \tag{1.0.1}$$

which is the normalized Loewner equation. Under this condition, each simple curve γ corresponds to a unique driving function λ.

Conversely, given a continuous function λ, what the solution g_t of the equation (1.0.1) would be like? The domain $H_t = \mathbb{H} \setminus K_t$ of g_t is decreasing, and the corresponding K_t is increasing. One of the most important and fundamental problems in the study of the Loewner equation is to consider what is the relation between the solution of the Lowner equation $g_t(z)$ or K_t, and the driving function $\lambda(t)$. That is, find some sufficient or necessary condition of the driving function to make K_t to satisfy certain properties.

In the equation (1.0.1), if there is a continuous curve $\gamma : [0, T] \to \bar{\mathbb{H}}$, $\gamma(0) \in \mathbb{R}$ that for any $t \in [0, T]$, H_t is the unbounded component of $\mathbb{H} \setminus \gamma([0, t])$, $K_t = \mathbb{H} \setminus H_t$, then we say K_t is generated by the curve γ, also say the driving function $\lambda(t)$, the equation (1.0.1) and the process $g_t(z)$ are generated by the curve γ. For example, when γ is a simple curve in $\mathbb{H}$, then $K_t = \gamma((0, t])$ is generated by the curve γ. And if γ is the boundary of upper half unit circle, the starting point is $\gamma(0) = -1$ and the end point is $\gamma(1) = 1$. Then when $t < 1$, $K_t = \gamma((0, t])$, when $t = 1$, $K_1 = \bar{\mathbb{D}} \cap \mathbb{H}$, this K_t It is also generated by the curve γ.

Next comes to SLE. SLE is a stochastic differential equation in the form of :

$$\dot{g}_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}, \quad g_0(z) = z, \tag{1.0.2}$$

where $\kappa > 0$ is a constant, B_t is the standard Brownian motion, i.e. the driving function $\sqrt{\kappa} B_t$ is a stochastic function. Similar with the general Loewner equation, the domain $H_t = \mathbb{H} \setminus K_t$ of g_t is reduced over time but K_t is increasing. K_t is a random set. An important result in the SLE theory is the following theorem proved by Rohde and Schramm in [28] :

Theorem A. *In equation (1.0.2), if $\kappa \leq 4$, then w.p.1, the equation is generated by a simple curve. If $\kappa > 4$, then w.p.1, the equation is generated by a non-simple curve.*

According to the above SLE is always generated by curves w.p.1. And the curve is simple when $\kappa \leq 4$. Consider about the Loewner equation driven by the general continuous function $\lambda(t)$, whether the function is also generated by a curve ? It is easy to construct examples to show that the Loewner equation may not be generated by a curve. Hence, for the study of the Loewner equation, there are two important questions :

(1) For what kind of driving function, the Loewner equation is generated by a curve ?

(2) If the Loewner equation is generated by a curve, then for what kind of driving function, the curve is simple ?

The main work of this thesis is about the above two problems.

After mathematicians' efforts, many important properties about the Loewner equation have been obtained. Many of them show that the characterization of the driving function is about 1/2 order, such as 1/2-Hölder norm. Thus, the author finds that for a fixed time t, a local time transform of 1/2 order can be made on the left side, under this transformation, we can study the local properties of the original Loewner equation on the left of time t, by studying the properties of the equation after the transformation. And then we can get the the global properties by the local properties.

For example, for a certain time t, if we have a local condition of the driving function which makes the curve does not self-interest at time t. Then if the driving function satisfies this condition for all the time, then we shall obtain a sufficient condition that makes the entire curve simple.

The structure of this paper is as follow. First, in the second chapter, we introduce the basic theory of the Loewner equation, including the radial Loewner equation and the chordal Loewner equation, as well as some important results of the SLE and general Loewner equations, and some estimation of the chordal Loewner equation. Then in the following three chapters, we explain the three conclusions that the author obtained by introducing the local time transformation of the Loewner equation. We give a brief introduction below :

1.1 Simple curve problem

Theorem A tells us when the SLE is generated by a simple curve. For general Loewner equation (1.0.1), Marshall and Rohde gave a sufficient condition for the driving function λ generated by a simple curve in [23] firstly, and Lind improve this condition in [21] to make this condition optimal in some sense. Their conclusion is :

Theorem B. *In the Loewner equation* (1.0.1)*, if the 1/2-Hölder norm of the driving function λ satisfies*

$$\|\lambda\|_{1/2} = \max_{s,t} \frac{|\lambda(s) - \lambda(t)|}{\sqrt{|s-t|}} < 4,$$

then the Loewner equation is generated by a simple curve of the upper half plane. The above conclusions are optimal in the following sense : there is λ_0 *satisfies* $\|\lambda_0\|_{1/2} = 4$ *s.t. the Loewner equation not generated by a simple curve.*

Although the theorem B proves when $\|\lambda\|_{1/2} < 4$, Loewner equation is always generated by a simple curve, it may be not true when $\|\lambda\|_{1/2} \geq 4$. In this case, there are examples to show that the equation (1.0.1) can be generated by a simple curve, generated by a non-simple curve, or even not generated by a curve. In fact, when $\|\lambda\|_{1/2} \geq 4$, the problem of when the equation is generated by a cure, or even a simple curve has not been solved yet.

In this thesis, the author uses the left time transformation for the driving function to get the following main lemma :

Lemma 1. *If the Loewner equation* (1.0.1) *is generated by a curve* γ*, define*

$$a = \underline{\lim}_{s\to t^-} \frac{\lambda(t)-\lambda(s)}{\sqrt{t-s}}, \quad b = \overline{\lim}_{s\to t^-} \frac{\lambda(t)-\lambda(s)}{\sqrt{t-s}}.$$

If $b \geq a \geq 0$*, and* a, b *satisfies* $b < f(a)$*, where*

$$f(a) = \begin{cases} a, & a \geq 4, \\ 4, & 2 \leq a < 4, \\ a + \frac{4}{a}, & 0 < a < 2, \\ +\infty, & a = 0, \end{cases} \tag{1.1.1}$$

Then $\gamma(t) \notin \mathbb{R} \cup \gamma([0,t))$.

The above lemma is a kernel lemma of this paper. Using this lemma, we have the following three conclusions. The first one is

Theorem 2. *Assume* $\lambda : [0,T] \to \mathbb{R}$ *is a continuous function, and the corresponding Loewner equation is generated by the curve* γ*. For* $t > 0$*, define*

$$a(t) = \underline{\lim}_{s\to t^-} \frac{|\lambda(t)-\lambda(s)|}{\sqrt{t-s}}, \quad b(t) = \overline{\lim}_{s\to t^-} \frac{|\lambda(t)-\lambda(s)|}{\sqrt{t-s}}.$$

If

$$\forall t \in (0,T], \ a(t) < 2 \quad \textit{and } b(t) < a(t) + \frac{4}{a(t)}$$

or

$$\forall t \in (0,T], \ 2 < a(t) < 4 \quad \textit{and } b(t) < 4$$

Then γ *is a simple curve, and the above upper bound of this* $b(t)$ *is optimal.*

The theorem 2 improves the result of theorem B. For example, if the driving function satisfies the condition of the theorem 2, then $a(t), b(t)$ can take the value as $a(t) = 1, b(t) = 4.5$ at a certain time t, then the 1/2-Hölder norm of the driveing function λ is greater than 4 but it can be proved that the curve generated by the driving function is still a simple curve.

Our second conclusion is to apply the lemma 1 to the exception point set of Brownian motion. Brownian motion satisfies the following properties (see [12]) :

For any T,

$$\overline{\lim_{t\to 0^+}} \frac{B_{T+t} - B_T}{\sqrt{2t \log|\log t|}} = 1 \tag{1.1.2}$$

with probability 1. Hence the limitinf and limitsup $a(t)$ and $b(t)$ of the Brownian motion,, satisfy $a(t) = 0, b(t) = +\infty$ w.p.1. And when $s \to t^-$, $|B_t - B_s|/\sqrt{|t-s|}$ equals 0 infinite times for all t a.s.

On the other hand, for any interval $[T_1, T_2]$, the following equality holds w.p.1.(see [26])

$$\max_{T_1\le T\le T_2} \overline{\lim_{t\to 0^+}} \frac{|B_{T+t} - B_T|}{\sqrt{2t|\log t|}} = 1 \tag{1.1.3}$$

Combining with (1.1.2), we can get that the T which satisfies $\overline{\lim\limits_{t\to 0^+}} |B_{T+t} - B_T|/\sqrt{2t|\log t|} = 1$ exists , and it is dense on the positive real axis, but its measure is 0. We call such point set isexceptional set. The study of exceptional set is an important problem (see [25]).

Using the method of lemma 1, we have that if the speed of $a(t)$ tending to zero is known, and the speed of $b(t)$ tending to infinite is not fast enough, then the driving function is still generated by a simple curve. For the SLE driven by Bronwen motion, it is clear when SLE is generated by a simple curve (ie theorem A), and for all t, the speed of $b(t)$ tending to infinity can be controlled by above (1.1.3). Therefore, we have a estimation of the speed of $a(t)$ going to 0, that is, we get the following theorem :

Theorem 3. *In the equation* (1.0.2)*, if* $\kappa > 4$*, w.p.1. there is a exceptional set* $I_\kappa \subset [0,1]$ *s.t. for any* $T \in I_\kappa$*, we have :*

$$\underline{\lim_{t\to T^-}} \frac{|B_T - B_t|}{\sqrt{T-t}} \sqrt{\log \frac{1}{T-t}} \ge \frac{2\sqrt{2}}{\kappa}.$$

Where the exceptional set I_κ is a subset of the locally left side extreme point ssets of Brownian motion $\{t : \exists \delta > 0 \; s.t. \; B_s < B_t, \forall s \in (t-\delta, t)$ or $B_s > B_t, \forall s \in (t-\delta, t)\}$, and I_κ is increasing by κ. The measure of the locally extreme points set is 0, and the set is exceptional. This theorem gives a estimation of the speed in a exceptional set.

The third conclusion obtained by the lemma 1 is about the Loewner equation driven by Weierstrass function. Weierstrass function is defined as :

$$W_{a,b}(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x),$$

where $0 < a < 1,\ ab > 1$. When $a = 1/\sqrt{b}$, the Weierstrass function denotes as W_b, which is 1/2-Hölder continuous, also known as 1/2 order Weierstrass function. Since it shares similar properties with Brownian motion in many respects, we study the Loewner equation driven by cW_b.

When $b > 1$, the 1/2-Hölder norm of W_b satisfies (see [11])

$$\|W_b\|_{1/2} \leq \frac{b}{\sqrt{b}-1} + \frac{2}{1-\frac{1}{\sqrt{b}}} = C(b) \sim \sqrt{b}, \qquad b \to +\infty.$$

From the theorem B, we have that when $|c| < 4/C(b)$, cW_b is generated by a simple curve. But when b tends to infinity, $4/C(b)$ tends to 0, which means when b is very large, $|c|$ needs to be very small to ensure that cW_b is generated by a simple curve. And we use the proof of the theorem 2, combined with quasi-conformal mapping and conformal welding, get the following conclusion :

Theorem 4. *If $b > 9, c < \sqrt{8\pi}(\pi + 1)$, then cW_b is generated by a simple curve.*

This theorem improves the corollary of theorem B. It improves the coefficient c when b is large to a constant, which is an essential improvement. And in this theorem, we do not need to assume the Loewner equation is generated by a curve, we just prove that the cW_b with the condition above is generated by a curve. Actually, the curve generation problem which is the first problem at the beginning is much difficult than the second problem.

1.2 Imaginary Loewner equation and curve generation problem

In Chapter 4, we will further discuss the issue of when a drive function is generated by a curve. We still use the local time transform, and then consider the problem : if the driving function is generated by the curve γ in time $[0, T)$, then what condition should the driving function satisfy, so that the driving function is also generated by the curve in time $[0, T]$, i.e. the limit $\lim_{t\to T^-} \gamma(t)$ exists.

In [19], under the condition that the limit $\lim_{t\to T^-} \frac{\lambda(T)-\lambda(t)}{\sqrt{T-t}}$ exists , Lind, Marshall and Rohde proved the following results(we only state part of their theorem) :

Theorem C. *Let λ be the driving function of which the 1/2-Hölder norm is less than 4 on $[0, T)$*

$$\lim_{t\to T^-} \frac{\lambda(T)-\lambda(t)}{\sqrt{T-t}} = \kappa > 4,$$

The Loewner equation (1.0.1) driven by λ is generated by the curve $\gamma : [0, T] \to \bar{\mathbb{H}}$, and $\gamma(T) \in\ MathbbR$ or $\gamma(T) \in \gamma([0, T))$.

Where 1/2-Hölder norm of λ is less than 4 in $[0,T)$ is a very strong condition, and it is easy to prove that λ is generated by the curve in $[0,T]$ (see [19]). In this article, we have the following theorem :

Theorem 5. *Let $\lambda : [0,T] \to \mathbb{R}$ be a continuous driving function, assuming its corresponding Loewner procedure $g_t(z)$ is generated by the curve γ in $[0,T)$, and let*

$$a = \varliminf_{t\to T^-} \frac{\lambda(T)-\lambda(t)}{\sqrt{T-t}}, \qquad b = \varlimsup_{t\to T^-} \frac{\lambda(T)-\lambda(t)}{\sqrt{T-t}}.$$

If $a \geq 5$ and $b < a + \frac{4}{a}$, then the limit $\lim_{t\to T^-} \gamma(t) = \gamma(T)$ exists, i.e. the Loewner process $g_t(z)$ is generated by the curve γ in $[0,T]$, $\gamma(T) \in \mathbb{R}$ or $\gamma(T) \in \gamma([0,T))$.

The conclusions of the above theorem and the conclusion of theorem C are the same, but our condition does not need the limit : $\lim_{t\to T^-} \frac{\lambda(T)-\lambda(t)}{\sqrt{T-t}}$ exsits, so we generalize the theorem C. And in our theorem, we do not nee to assume that 1/2-Hölder norm is less than 4 in $[0,T)$, which makes it very difficult to prove the driving function is generated by a curve at time T.

In order to overcome the above difficulty, we introduce the analysis of the imaginary Loewner equation, which is one of the innovations in the thesis. For the Loewner equation (1.0.1), the real and imaginary parts of g_t are denoted as $X(t)$ and $Y(t)$ respectively, then we have

$$\dot{Y}(t) = -\frac{2Y(t)}{(X(t)-\lambda(t))^2 + Y(t)^2}.$$

In the above equation, although $X(t)$ is related to $Y(t)$, if we set $\theta(t) = X(t) - \lambda(t)$, and treat $\theta(t)$ as a general continuous function, then we will get a new equation

$$\dot{Y}(t) = -\frac{2Y(t)}{\theta(t)^2 + Y(t)^2}. \tag{1.2.1}$$

We call it the imaginary Loewner equation.

In equation (1.0.1), it can be seen that if the initial value of the equation (1.0.1) is real, then its solution is also a real function, which we call a real solution of (1.0.1). We define a dual transformation : take a real solution of the equation (1.0.1), $g_0(t)$, and let $\theta(t) = g_0(t) - \lambda(t)$, then for any other real solution $g_1(t)$, $W(t) = g_1(t) - g_0(t)$ satisfies the equation :

$$\dot{W}(t) = -\frac{2W(t)}{\theta^2(t) + \theta(t)W(t)}. \tag{1.2.2}$$

We call this equation the dual equation of the imaginary Loewner equation.

We can see that the above two equations are very similar. We can define the vanishing property of the above equation at time T, i.e. whether there is a positive initial value, s.t. the corresponding solution will just decrease to 0 at time T. We can analyse the left local properties of the above two equations at time T. After using the local time transformation, an important findings is that for most of the driving functions θ, the above equations either both vanishing at T, or not vanishing at the same time at T. The remaining few cases correspond the case that $\gamma(t)$ does not converge when t tends to T. So if we can find a condition that excludes the cases above, then we have the limit of the curve at T. The condition we give in the theorem 5 is such a condition, hence we get the theorem 5 by it.

1.3 Left self-similar driving function

In Chapter 5, we mainly study the properties of the left self-similar curve. The self-similarity we are talking about here means that for an graph, there are a point and a part of the image satisfy that if we magnified the part of the graph some times around the point, then the new graph we get is the same as the original one. Many typical fractals are self-similar, such as Von Koch curves, Sierpinski triangles, Hilbert curves.

Here, we can also define another self-similarity that depends on the Loewner equation. Suppose γ is a curve on the upper half plane with starting point on the real axis, then for time $s < t$, the solution g_s of Loewner equation (1.0.1) at time s maps $\gamma([0, s])$ to the real axis, and maps $\gamma([s, t])$ to the upper half plane as another curve with starting point on the real axis. If it is similar to $\gamma([0, t])$, then we will call this curve a left self-similar curve on the left of time t, as defined in Chapter 5. In the case of SLE, the SLE curves are left self-similar under the distribution.

Through our analysis, we have that the driving function corresponding to the left self-similar curve satisfies a condition that is exactly in the same form as the scale invariance that Brownian motion satisfies. If the Loewner equation is transformed into a left local time, the transformed Loewner equation becomes to

$$\dot{g}_t(z) = g_t(z) - \frac{4}{\xi(t) - g_t(z)},$$

where ξ is called the driving function of the equation after the transformation, then the condition that the driving function of the left self-similar curve satisfies is equivalent to the condition that ξ is a periodic function. For the driving function ξ with a period of T, if we define the mapping $f : z \mapsto g_T(z)$, then we have $g_{nT}(z) = f^{\circ n}(z)$, where $f^{\circ n}$ represents n iterations of f. So we got a complex power system. Using the complex dynamic system, we get :

Theorem 6. *If the driving function λ is self-similar on the left of* 1*, the self-similarity coefficient is $r > 1$, and λ is generated by a curve γ in* $[0, 1/r]$*, and* $\gamma((0, 1/r]) \subset \mathbb{H}$*, then the driving function must also be generated by a curve in* $[0, 1]$.

Chapitre 2

Theory of Loewner Equation

2.1 Background in complex cnalysis

2.1.1 capacity and half plane capacity

We introduce some basic results of complex analysis, the conclusions of the subsection refer to [2],[6],[13] et al.

Lemma 7 (Schwarz lemma). *f is a holomorphic function in $\mathbb{D}$, and $f(\mathbb{D}) \subseteq \mathbb{D}$, $f(0) = 0$, then$|f'(0)| \leq 1$, $|f'(0)| = 1$ if and only if f is a rotation of $\mathbb{D}$.*

Theorem 8 (Riemann mapping theorem). *Assume D is a simple connect region in the plane and D is not $\mathbb{C}$. $z \in D$, then there exists a unique conformal mapping $f : D \to \mathbb{D}$ s.t. $f(z) = 0, f'(z) > 0$.*

Lemma 9 (Koebe distortion theorem). *$f \in S \doteq \{g : g$ is a univalent function in $\mathbb{D}$ and $g(0) = 0, g'(0) = 1\}$. Then*

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2}$$

$$\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3}.$$

Definition 10. *If $\{D_n\}$ is a sequence of regions each of which contains 0, define kernel of $\{D_n\}$(with respect to 0) to be the component of the set*

$$\{z | \exists r > 0, s.t. \bar{B}(z,r) \subset D_n, \textit{for all but a finite number of integersn}\}$$

that contains 0. If the above set is empty, then $\{D_n\}$ does not have a kernel. When $\{D_n\}$ has a kernel, we say that $\{D_n\}$ converges to D if and only if D is the kernel of any subsequence $\{D_n\}$, denoted by $D_n \to D$.

Definition 11. *Suppose that $\{D_n\}$ is a sequence of regions each of which contains 0 and G is the kernel of $\{D_n\}$. $f_n : D_n \to \mathbb{C}$, if f_n convergfes uniformly on compacta to $f : D \to \mathbb{C}$ D, then we say f_n converges to f by Caratheodory kernel, denoted by $f_n \xrightarrow{Cara} f$.*

Theorem 12 ([6])**.** *$D_n \to D$, if f_n, f are the Riemann mapping from D_n, D to Ω_n, Ω respectively, then $f_n \xrightarrow{Cara} f$ if and only if $\Omega_n \to \Omega$.*

When the region is unbounded in $\mathbb{C}$, by the mapping $1/z$, we have the similar theorem :

Definition 13. *If a bounded compact set K in $\mathbb{C}$ and $\mathbb{C} \setminus K$ are both simple connected, and K contains at least two points, then we say K is a whole plane hull.*

Theorem 14 (Riemann mapping theorem of unbounded region)**.** *D is a region of $\mathbb{C}_\infty$,$\mathbb{C}_\infty \setminus D$ is a whole plane hull, then there exists an unique conformal mapping $f_D : \mathbb{C}_\infty \setminus \bar{\mathbb{D}} \to D$ satisfying $f_D(\infty) = \infty, f'_D(\infty) = \lim_{z\to\infty} f_D(z)/z > 0$.*

Definition 15. *The f above is called the Riemann mapping of D, and we say $f'_D(\infty)$ the whole plane capacity of the hull K, denoted by* $\mathrm{cap}(K)$.

For instance, the capacity of $\bar{\mathbb{D}}$ is 1,the capacity of the segment with length a is $a/4$.

By Schwarz lemma, we have the following basic property of capacity :

Proposition 16. *K_1, K_2 are two hulls,*
(1) Monotonicity : If $K_1 \subsetneqq K_2$, then $\mathrm{cap}(K_1) < \mathrm{cap}(K_2)$;
(2) Linear property :If $K_1 = rK_2, r > 0$, then $\mathrm{cap}(K_1) = r\mathrm{cap}(K_2)$;
(3) Rigid motion invariance : If T is a rigid motion ini the plane, then $\mathrm{cap}(K_1) = \mathrm{cap}(T(K_1))$.

By the Koebe distortion theorem and the monotonicity of the capacity, the capacity is controlled by the diameter of the hull. For the hull K which contains 0, define $\mathrm{rad}(K) = \sup\{|z| : z \in K\}$, then we have :

Lemma 17. *If $\mathrm{cap}(K) = 1$, then*

$$1 \leq \mathrm{rad}(K) \leq 4$$

the left equality holds when K is $\bar{\mathbb{D}}$, the left equality holds when K is a segment.

In $\mathbb{H}$, we can also discuss the half plane hull and its capacity.

Definition 18. *If $K \subset \mathbb{H}$ and K is connected, K satisfies that $\mathbb{H} \setminus K$ is simple connected and K is compact in $\mathbb{H}$, then we say K is a half plane hull .*

For a half plane hull K, set K^* to be the reflection of the real axis, then $\overline{K \cup K^*}$ is a whole plane hull, hence by theorem 14, we have :

Lemma 19. *For a half plane hull $K \subset \mathbb{H}$, there exists a unique conformal mapping $g_K : \mathbb{H} \setminus K \to \mathbb{H}$ satisfies $\lim_{z\to\infty}(g_K(z) - z) = 0$. And $g_K(z)$ can be extend to $\mathbb{C}_\infty \setminus \overline{K \cup K^*}$ by Schwarz reflection principle, and $g_K(z)$ can be extended at $z = \infty$ as :*

$$g_K(z) = z + \frac{a_1(K)}{z} + O(\frac{1}{|z|^2}).$$

we say $a_1(K)$ is the half plane capacity if K, which denotes as $\mathrm{hcap}(K)$.

We have two easy examples

Example 20. *The Half plane capacity of $\mathbb{D} \cap \mathbb{H}$ is 1.*

Example 21. *The Half plane capacity of the segment connected 0 and $e^{i\alpha\pi}, \alpha \in (0, 1)$ is $\alpha^{1-2\alpha}(1 - 2\alpha)^{2\alpha-1}/2$.*

Half plane capacity has similar properties as the whole plane capacity :

Proposition 22. *K_1, K_2 are two half plane hulls, then we have :*
(1) Monotonicity : if $K_1 \subsetneqq K_2$, then $\mathrm{hcap}(K_1) < \mathrm{hcap}(K_2)$ *;*
(2) Linear property : if$K_1 = rK_2, r > 0$, then $g_{K_1}(z) = rg_{K_2}(z/r)$, $\mathrm{hcap}(K_1) = r^2\mathrm{hcap}(K_2)$ *;*
(3) Invariance by translation and reflection : $T : x + y\mathrm{i} \mapsto \pm x + a, a \in \mathbb{R}$, then $\mathrm{hcap}(K_1) = \mathrm{hcap}(T(K_1))$.

Half plane capacity also has semi-gruoup property :

Proposition 23. *If $K_1 \subsetneqq K_2$, then* $\mathrm{hcap}(K_2) = \mathrm{hcap}(K_1) + \mathrm{hcap}(g_{K_1}(K_1 \setminus K_2))$.

By example 20,21 and monotonicity of the half plane capacity, we have

Proposition 24. *For a sequence of half plane hull $\{K_t\}, t \in [0, +\infty)$. If the half plane capacity* $\lim_{t\to+\infty} \mathrm{hcap}(K_t) = +\infty$, *then* $\lim_{t\to+\infty} \mathrm{rad}(K_t) = \infty$.

Definition 25. *A sequence half plane hull $\{K_t\}, t \in [0, +\infty)$ is called normalized half plane hull if $\forall s < t, K_s \subsetneqq K_t$ and* $\mathrm{hcap}(K_t) = 2t$,.

Like the hull of the plane,the capacity of the half plane hull can be controlled by its imaginary part :

Lemma 26. $\mathrm{hcap}(K) \geq \max_{z\in K} \mathrm{Im} z$, *the equality holds if and only if K is a segment perpendicular to* R.

Remark 27. The half plane capacity only can be control by one side. We will give a proof by chordal Loewner equation later.

2.1.2 the boundary behaviour of conformal mapping

The boundary behaviour of conformal mapping has important connection to the curve generation problem of Loewner equation, the subsection mainly refers to [27].

Suppose D is a simple connected region in the plane,$f : \mathbb{D} \to D$ is a conformal mapping. We consider an important question in complex analysis, whether f can be extended to the boundary of $\mathbb{D}$ continuously. We define local connectedness :

Definition 28. *The close set $A \subset \mathbb{C}$ is called locally connected if for every $\varepsilon > 0$ there is $\delta > 0$,s.t. for any two points $a, b \in A$ with $|a - b| < \delta$, we can find a continumm B with*

$$a, b \in B \subset A, \operatorname{diam} B < \varepsilon.$$

Whether a conformal mapping can be extended to boundary continuously is determined by the locally connectedness :

Theorem 29. *Suppose D is a bounded region in the plane, $f : \mathbb{D} \to D$ is the Riemann mapping, then the following are equivalent :*

(1) f has a continuous extension on $\mathbb{T} = \partial\mathbb{D}$;

(2) ∂D is locally connected ;

(3) ∂D is a curve, that is $\partial D = \varphi(\mathbb{T})$, where φ is continuous.

Remark 30. Since $\mathbb{D}$ and $\mathbb{H}$ are equivalent under conformal mapping, hence the above theorem also valid in $\mathbb{H}$.

Similarly, we can consider that whether f has a injective continuous extension to the boundary.

Definition 31. *E is a locally connected continumm, $a \in E$, if $E \setminus a$ is not connected, then we say a is a cut point of E.*

For instance, $\mathbb{T}$ has no cut points, but in $\mathbb{T} \cup [0, 1]$, the points of $(0, 1]$ are all cut points.

Theorem 32. *Suppose D is a bounded region in the plane, $f : \mathbb{D} \to D$ is the Riemann mapping, then the following are equivalent :*

(1) f has a continuous injective extension on $\mathbb{T} = \partial\mathbb{D}$;

(2) ∂D is locally connected and has no cut points ;

(3) ∂D is a Jordan curve.

Hence when we extend f to the boundary, we should fix a point in $\mathbb{T}$, then discuss whether the limit of f exists at this point.

Definition 33. *D is a bounded region, if a open simple curve C in D satisfies $\bar{C} = C \cup \{a, b\}, a, b \in \partial D$, then we say C is a crosscut of D.*

We say a family of crosscut $(C_n), n = 0, 1, \ldots$ is a null-chain of D, if (C_n) satisfies the following conditions :

(1) for all n, C_n is a crosscut of D ;

(2) $\bar{C}_n \cap \bar{C}_{n+1} = \varnothing$ *;*
(3) C_n *separates* C_0 *and* C_{n+1} *in* D *;*
(4) $\text{diam}C_n \to 0$ *as* $n \to \infty$*.*

For a null-chain (C_n), crosscut of (C_n) separates D to two components, we set V_n to be the component does not contain C_0. Then V_n is decreasing and converges to a boundary. But for one fixed boundary point, it may correspond to many null-chain, hence we can define a equivalent relation of null-chain :

Definition 34. *If two null-chain* (C_n) *and* (C'_n) *satisfy that for sufficient large* m*, there exists* n *s.t.* C'_m *separates* C_n *and* C_0 *in* D*,* C_m *separates* C'_n *and* C'_0 *in* D*, then we say* (C_n) *is equivalent to* (C'_n)*. It is easy to check that this is a equivalence relation. The equivalent class is called the prime end of* D*. Let* $P(D)$ *denote the set of prime ends of* D*.*

Whatever the conformal mapping has a continuous extension to boundary, the conformal mapping can induce a bijective between the prime ends of the two regions, that is the prime end theorem :

Theorem 35. f *is a conformal mapping of* $\mathbb{D}$ *to* D*. Then there is a bijective*

$$\hat{f} : \mathbb{T} \to P(D)$$

such that, if $z \in \mathbb{D}$ *and* (C_n) *is a representing prime end of* $\hat{f}(z)$*, then for sufficient large* n*,* $(f^{-1}(C_n))$ *is a null-chain in* $\mathbb{D}$ *separates* z *and 0.*

Remark 36. By prime end theorem and Riemann mapping theorem, it is easy to see that if f is a conformal mapping from D_1 to D_2, then f can also induced a bijective mapping from $P(D_1)$ to $P(D_2)$. But null-chain has no conformal invariant, that is the image of a null-chain under a conformal mapping may not be a null-chain.

The prime end describe how the null-chain converges, then we can define the following set :

Definition 37. $p \in P(D)$*,*$I(p) \doteq \cap_{n=1}^{\infty} \bar{V}_n$*, is the impression of* p*. Choose a representing null-chain* $(C_n)p$*,* $\Pi(p) \doteq \{w | \forall \varepsilon, \exists N(\varepsilon) \text{ s.t. } \forall n > N(\varepsilon), C_n \subset D(w, \varepsilon)\}$ *is called the principal points set of* p*, where* $D(w, r)$ *is the circle with center* w*,radius* r*.It is easy to see that* $\Pi(p) \subset I(p)$*.*

Now we consider for a $\zeta \in \mathbb{T}$, when z approach ζ in $\mathbb{D}$, or when z approach ζ by some curves, whether the limit of $f(z)$ exists. Suppose $E \subset \mathbb{D}$, define the cluster set $C_E(f, \zeta)$ is the limit points set of $f(z)$, when z approach ζ in E, the set of all w satisfy that there are sequence $z_n \in E$,$w \in \partial D, z_n \to \zeta$ s.t. $f(z_n) \to w$. And we denote $C_{\mathbb{D}}(f, \zeta) = C(f, \zeta)$ which is the unrestricted limit set of f. Let $[0, \zeta)$ be the radius from 0 to ζ, we say $C_{[0,\zeta)}(f, \zeta)$is the radial limit points set of f at ζ. We say the unrestricted limit or the radial limit exist if and only if the corresponding sets has only one point. Then we have the most important theorem in the subsection :

Theorem 38. *f is the conformal mapping from $\mathbb{D}$ to D, $\zeta \in \mathbb{T}$, then we have*

(1) $I(\hat{f}(\zeta)) = C(f, \zeta)$, that is the unrestricted limit exists if and only if the impression is a single point set ;

(2) $\Pi(\hat{f}(\zeta)) = C_{[0,\zeta)}(f, \zeta)$, that is the radial limit exists if and only if the principle point set has only one point.

Remark 39. Actually, the radial limit exist has more equivalent conditions. For example :

For $\zeta \in \mathbb{T}, \theta \in [0, \pi/2)$, we can define the Stolz angular $\Delta \doteq \{z \in \mathbb{D} | \arg((\zeta - z)/\zeta) \in (-\theta, \theta)\}$. Then we have $C_{[0,\zeta)}(f, \zeta) = C_\Delta(f, \zeta)$, the later set is called the angular limit set. Hence the radial limit exists if and only if the angular limit exists. Let Γ be a curve in $\mathbb{D}$ with endpoint ζ, denote $C_\Gamma(f, \zeta)$ is the limit along Γ, then we have $C_{[0,\zeta)}(f, \zeta) = \cap_\Gamma C_\Gamma(f, \zeta)$, where Γ runs through all curves in $\mathbb{D}$ ending at ζ, that is the radial limit exists if and only if the limit along some curves exists.

2.1.3 harmonic measure and Brownian motion

In this sub section, we introduce the relation between the harmonic measure and the Brownian motion, and the relation between Brownian motion and the half plane capacity. Related results refer to [1], [5], and [13]. We state these conclusions as simple as possible, hence we only consider the harmonic measure of the simple connect region.

Definition 40. *G is open set in $\mathbb{H}$, if function $u : G \to \mathrm{R}$ satisfies the Laplace equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

then we say u is a harmonic function on G.

The Laplace equation of function u in the plane is equivalent to

$$\frac{\partial^2 u}{\partial z \partial \bar{z}} = 0$$

hence we have the following property :

Proposition 41. *The harmonic functions on G form a linear space, that is if f_1, f_2 is harmonic and c is a constant, then $f_1 + f_2$ and cf_1 are harmonic function.*

Proposition 42. *$f : G_1 \to G_2$ is conformal mapping, u is a harmonic function on G_2, then $u \circ f$ is a harmonic function on G_1.*

By the Cauchy integral formula of the holomorphic function, we have the mean value theorem of harmonic function :

Theorem 43. *$u : G \to \mathbb{R}$ is harmonic function, $\bar{B}(z, r) \subset G$ is a close circle, then*

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{\theta \mathrm{i}}) \mathrm{d}\theta.$$

By the mean value theorem of harmonic function, we have the extreme theorem of harmonic function :

Theorem 44. *$u : G \to \mathbb{R}$ is a harmonic function, if there exists a inner point $z \in G$ s.t. $u(z)$ is maximum or minimum of u on G, then u is a constant function.*

In this thesis, we only consider the harmonic function on a simple connect region of the plane. By proposition 42 and Riemann mapping theorem 8, we should study the harmonic function on $\mathbb{D}$, the others regions can be studied by the Riemann mapping.

In the unit disc, we consider the boundary value problem, that is for a function f on $\partial\mathbb{D}$, is there a harmonic function u on $\mathbb{D}$ s.t. u has a continuous extension to boundary, and its value on the boundary is f. Apparently, f must be a continuous function either, we have the following theorem :

Theorem 45. *If f is a real continuous function on $\partial\mathbb{D}$, there there exists an unique continuous function u on $\bar{\mathbb{D}}$ satisfying :*

(1) $u(z) = f(z), \forall z \in \partial\mathbb{D}$;

(2) u is harmonic on $\mathbb{D}$.

And u satisfies :

$$u(re^{\theta\mathrm{i}}) = \frac{1}{2\pi}\int_0^{2\pi} P(z, re^{t\mathrm{i}})f(re^{t\mathrm{i}})\mathrm{d}t \tag{2.1.1}$$

where $P(z, re^{t\mathrm{i}}) = \dfrac{1-|z|^2}{|z-re^{t\mathrm{i}}|^2}$ is the Poisson kernel of the unit disc.

Now we define the harmonic measure. We take an arc I from $\mathbb{T}$, we define a piece-wise boundary function

$$f(z) = \begin{cases} 1 & z \in I \\ 0 & z \notin I \end{cases}$$

since f is not continuous, we can construct a sequence of continuous boundary function f_n s.t. f_n converge to f uniformly, set u_n to be the corresponding harmonic functions. Then we use the formula of the above theorem, we have that u_n converge to a harmonic function u on $\mathbb{D}$, then u and f also satisfy equation (2.1.1). But now, u has a continuous extension to $\mathbb{T}$ except for the two endpoints of I.

By (2.1.1), for any Borel set $I \subset \mathbb{T}$, we can define the harmonic measure of D of I as

$$\mathrm{hm}_{\mathbb{D}}(re^{\theta\mathrm{i}}, I) = \frac{1}{2\pi}\int_0^{2\pi} P(z, re^{t\mathrm{i}})\mathbb{1}_I(re^{\theta\mathrm{i}})\mathrm{d}t.$$

we $r = 0$, it is easy to compute that

$$\mathrm{hm}_{\mathbb{D}}(0, I) = \frac{1}{2\pi}\int_0^{2\pi} P(z, 0)\mathbb{1}_I(re^{\theta\mathrm{i}})\mathrm{d}t = \frac{|I|}{2\pi} \tag{2.1.2}$$

where $|I|$ is the measure of I.

Then $\mathrm{hm}_{\mathbb{D}}(\cdot, I)$ can extent to $\mathbb{T}$, and except a 0-measure set, $\mathrm{hm}_{\mathbb{D}}$ equals 1 in I and 0 in $\mathbb{T} \smallsetminus I$. Then harmonic measure satisfies :

$$\mathrm{hm}_{\mathbb{D}} : \mathbb{D} \times \mathcal{B}_{\mathbb{D}} \to [0, 1]$$

where $\mathcal{B}_{\mathbb{D}}$ is the set of Borel set of $\mathbb{T}$. The reason why the $\mathrm{hm}_{\mathbb{D}}(z, I)$ is called the harmonic measure on $\mathbb{D}$ is that when $z \in \mathbb{D}$ is fixed, $\mathrm{hm}_{\mathbb{D}}$ is a Borel measurable measure in $\partial\mathbb{D}$, when the Borel set I is fixed, $\mathrm{hm}_{\mathbb{D}}$is harmonic function on $\mathbb{D}$.

Since we have the harmonic measure of $\mathbb{D}$, we can define the harmonic measure on a general simple connect region by the Riemann mapping. For simple connect region D, denote $\mathcal{B}_D$ the set of Borel set of ∂D, then we can define the harmonic measure $\mathrm{hm}_D : (D \times \mathcal{B}_D) \to [0, 1]$ of D as follow :

Definition 46. *Suppose $f : \mathbb{D} \to D$ is a Riemann mapping, if f has continuous extension to boundary. Then for all $z \in \mathbb{D}, I \subset \mathcal{B}_D$, we have :*

$$\mathrm{hm}_D(z, I) = \mathrm{hm}_{\mathbb{D}}(f^{-1}(z), f^{-1}(I)).$$

For the general Riemann mapping $f : \mathbb{D} \to D$, by theorem 35, every point in $\mathbb{T}$ corresponds to a prime end of D. We can define :

Definition 47. *Assume that $\hat{f}$ is the bijective mapping from $\mathbb{T}$ to $P(D)$ which induced by f, $I \subset \mathcal{B}_D$, denote $\hat{I}$ the set of prime ends which the corresponding impressions intersects whit I. then we have*

$$\mathrm{hm}_D(z, I) = \mathrm{hm}_{\mathbb{D}}(f^{-1}(z), \hat{f}^{-1}(\hat{I})).$$

Now we introduce the connection between Brownian motion and the harmonic measure, we need to recall some knowledges of the stochastic theory.

First, the definition of the Brownian motion :

Definition 48. *one-dimentional standard Brownian motion $\{B_t : t \geq 0\}$ is a stochastic process with the following properties :*

(1) $B_0 = 0$;

(2) For any $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n, B_{t_2} - B_{t_1}, B_{t_3} - B_{t_2}, \ldots, B_{t_n} - B_{t_{n-1}}$ are independent ;

(3) For all $0 \leq s \leq t$, the distribution of $B_t - B_s$ is a normal distribution, with variance $t - s$ and centre 0 ;

(4) $t \mapsto B_t$ is continuous w.p.1.

Brownian motion has many properties, we only notice the scaling property here, because it has connection to the self-similarity of the Loewner equation :

Proposition 49. *For a one-dimentional standard Brownian motion B_t and positive constant $a > 0$, $\frac{1}{\sqrt{a}} B_{at}$ is still a one-dimentional standard Brownian motion .*

The higher dimension Brownian motion, for instance, the standard two-dimensional Brownian motion is $B_t = (B_t^1, B_t^2)$, where B_t^1 and B_t^2 are two independent one-dimentional standard Brownian motion. The two-dimensional Brownian motion has the following important property :

Theorem 50. *Suppose D_1, D_2 are two regions of the plane, f is the conformal mapping from D_1 to D_2. $z \in D_1, B_t$ is the standard two-dimensional Brownian motion starts at z. $\tau_{D_1} \doteq \inf\{t \geq 0, B_t \notin D_1\}, \sigma(t) \doteq \int_0^t |f'(B_s)|^2 \mathrm{d}s$. Then $f(B_{\sigma(t)}) : 0 \leq t < \tau_{D_1}$ is a standard two-dimensional Brownian starts at $f(z)$ and stops at the boundary of D_2 respect t as the time parameter.*

The proof of this theorem need the Ito formula, we omit it here. In short, the the image of a two-dimensional Brownian motion under a conformal mapping is still a two-dimensional Brownian motion .

Now we consider a two-dimensional Brownian motion starts at 0. We denote τ as the stopping when $|B_t| = 1$ happens firstly, since rotation is a conformal mapping, we have that B_τ is uniform distributed in $\mathbb{T}$. Hence for a Borel set I,

$$\mathbb{P}\{B_\tau \in I\} = |I|/2\pi.$$

This number is same as equation (2.1.2), that is $\mathrm{hm}_{\mathbb{D}}(0, I)$.Since harmonic measure and Brownian motion are all invariant under the conformal mapping, we have :

Theorem 51. *D is a simple connect region, $z \in D$, $I \subset \mathcal{B}_D$. B_t is a two-dimensional Brownian motion starts at z, denote B_t as the stopping time that B_t first reach the boundary of D, then*

$$\mathbb{P}^z(B_\tau \in I) = \mathrm{hm}_D(z, I).$$

This theorem describes the relation between Brownian motion and harmonic measure, which connected the complex analysis to stochastic analysis. For instance, we consider the region $D = \{z : 0 < \mathrm{Re}\,(z) < 1\}$, by the one dimensional random walk, we have that for $z \in D$, the probability of the two two-dimensional Brownian motion starts at z escape from D at the left side is $\mathrm{Re}\,(z)/(1 - \mathrm{Re}\,(z))$. Hence we obtain the harmonic measure of D.

Now we introduce the relation among half plane capacity, Brownian motion and harmonic measure. For the capacity of the whole plane, by Riemann mapping, we have :

Proposition 52. *K is a hull of the plane, $K \subset B(0, r)$, where $B(0, r)$ is disc with rarius r and center 0. Assume B_t is a two-dimensional Brownian motion, the starting point is a uniform distribution in the circle of $B(0, r)$. Denote τ as the stopping time that B_T reaches K firstly, we have*

$$\mathbb{E}[\log |B_\tau|] = \log \mathrm{cap}(K).$$

In upper half plane, we consider the harmonic measure at ∞. That is, for interval $I \subset \mathbb{R}$, we have :

$$|I|/\pi = \lim_{y\to+\infty} y\mathrm{hm}_{\mathbb{H}}(y\mathrm{i}, I)$$

This equality gives the probability of that the Brownian motion starts at ∞ escape $\mathbb{H}$ at I. For a half plane hull K, its Riemann mapping $f_K : \mathbb{H} \setminus K \to \mathbb{H}$ maps the boundary of K to a interval I, hence by the equality above, we denote τ as the stopping time that Brownian motion reach the boundary of $\mathbb{H} \setminus K$ firstly, then we have

$$|I|/\pi = \lim_{y\to+\infty} y\mathbb{P}^{y\mathrm{i}}(B_\tau \in K)$$

that is the the probability of that the Brownian motion starts at ∞ escape $\mathbb{H} \setminus K$ at K. For half plane capacity, we have the following property

Proposition 53. *K is a half plane hull, denote τ as the stopping time that Brownian motion reach the boundary of $\mathbb{H} \setminus K$ firstly, then*

$$\mathrm{hcap}(K) = \lim_{y\to+\infty} y\mathbb{E}^{y\mathrm{i}}(\mathrm{Im}\,(B_\tau)).$$

2.1.4 dynamic system

We introduce the related results of complex dynamic system, especially the hyperbolic system, refer to [24].

We only consider the discrete system which is the iteration of functions. For any set S, if we have a mapping $f : S \to S$,define $f^{\circ n}$ to be the $n$the iteration of f, then we get a dynamic system in S.

If $z \in G$ satisfies $f(z) = z$, then we say z is a fixed point of f. If $A \subset S$ s.t. $\forall z$, $z \in A$ if and only if $z \in f(A)$, then we say A is fully invariant under f. It is easy to see that the set of fixed point is fully invariant.

In this thesis, S is a subset of $\mathbb{C}$, and f is a analytic function. Then for a fixed z_0, if there exists a neighborhood $D \subset S$ s.t. $\forall z \in D$, $f^{\circ n}(z)$ converges to z_0, then we say z_0 is attracting. Conversely, if there exists a neighborhood $D \subset S$ s.t. $\forall z \in D$, $\{f^{\circ n}(z)\} \notin D$ for sufficient large n,then we say that z_0 is repelling. It is easy to see that if $|f'(z_0)| < 1$, then z_0 is attracting, if $|f'(z_0)| > 1$, then z_0 is repelling. When $|f'(z_0)| = 1$, it may have many cases.

For instance, $f : x \mapsto x^2 + x$ is a dynamic system in $\mathbb{R}$, 0 is a fixed point with $f'(0) = 1$. In the left side of 0, for $-1 < x < 0$, $x < f(x) < 0$, hence $f^{\circ n}(x)$ converges to 0 ;But in the right side, for $x > 0$, $f^{\circ n}(x)$ increasing to $+\infty$. In such case, we say the left side of 0 is stable, the right side is unstable.

We consider the case that S is a simple connect region of $\mathbb{C}_\infty$, f is a holomorphic mapping of S, such system is complex dynamic system. If $\mathbb{C}_\infty \setminus S$ contains at least two points, we say S is a hyperbolic Riemann surface. For instance, $\mathbb{H}$ is hyperbolic. The dynamics on the hyperbolic Riemann surface is called hyperbolic dynamics.

For a hyperbolic Riemann surface Riemann S, and $f : S \to S$, we define ρ is the hyperbolic metric of S, then the hyperbolic metric is decreasing under f, that is

Proposition 54. *$f : S \to S$ is a holomorphic function, then for all $z_1, z_2 \in S$:*

$$\rho(f(z_1, z_2)) \leq \rho(z_1, z_2)$$

the equality hold for some $z_1 \neq z_2$ if and only if f is a covering mapping.

We the inequality hold for all $z_1 \neq z_2$, we say the hyperbolic metric is decreasing strictly under f, by Schwarz lemma, if $f(S) \subsetneq S$, then the hyperbolic metric is decreasing strictly under f. We have the following property :

Proposition 55. *If the hyperbolic metric is decreasing strictly under $f : S \to S$, then one of the following hold :*
(1) For all $z \in S$,$f^{\circ n}(z)$ converges to ∞ in hyperbolic metric;
(2) For all $z \in S$,$f^{\circ n}(z)$ converges to the unique fixed point $z_0 \in S$ local uniformly.

When $G = \mathbb{D}$, we have the Wolff-Denjoy theorem :

Theorem 56. *$f : \mathbb{D} \to \mathbb{D}$ is a holomorphic mapping, then one of the following holds :*
(1) f is a rotation;
(2) $f^{\circ n}(z)$ converges to $z_0 \in \bar{\mathbb{D}}$ local uniformly in $\mathbb{D}$.

2.2 radial Loewner equation

In this section, we introduce the original Loewner theory briefly, include the radial Loewner equation on $\mathbb{D}$. Actually, in De Brange's proof of Bierberbach conjecture, he used the whole plane Loewner equation and the radial Loewner equation, refers to [7] or [6].

Suppose $\gamma : (-\infty, +\infty) \to \mathbb{C}$ is a simple curve, γ satisfies $\lim_{t\to-\infty} \gamma(t) = 0$, $\lim_{t\to+\infty} \gamma(t) = \infty$. Set $K_t = \gamma((0, t]), H_t = \mathbb{C} \setminus K_t$, by the Riemann mapping theorem and the Schwarz lemma, we have that there exists an unique $g_t : \mathbb{D} \to H_t$ s.t. $g_t(0) = 0$,$g_t'(0) = a(t) > 0$, and it is easy to see that $a(t)$ is increasing.

By the Koebe distortion theorem, $a(t)$ converges to 0 and $+\infty$ as t goes to 0 and $+\infty$ respectively. Hence we can reparameterize the γ in $(-\infty, +\infty)$, s.t. $a(t) = e^t$. Then we have the following lemma :

Lemma 57 (see [6])**.** *Using the symbols above, $\lambda(t) := \lim_{z\to\gamma(t), z\in\mathbb{C}\setminus K_t} g_t(z)$ exists and λ is continuous by t, denote $\lambda(t) = e^{iW_t}$, we can choose a proper angular function s.t. W_t is a continuous functionon $\mathbb{R}$. And $g_t(z)$ satisfies the following ODE*

$$\dot{g}_t(z) = z g_t'(z) \frac{e^{iW(t)} + z}{e^{iW(t)} - z}, \quad \lim_{t\to-\infty} e^{-t} g_t(z) = z, \forall z \in \mathbb{C} \setminus \{0\}.$$

this equation is called the whole plane Loewner equation, Loewner proved the case $n = 3$ of the Bieberbach conjecture by this equation. $W(t)$ is called the driving function of the Loewner equation, it is a real function on $\mathbb{R}$. $g_t(z)$ is called the Loewner process or Loewner chain.

It is easy to see that any curve connected 0 and ∞ corresponds to a continuous driving function on $\mathbb{R}$. Then for any continuous driving function, can we induce a simple curve? This the basic problem for Loewner equation. The disadantage of thewhole plane Loewener equation is that the initial value condition is $\lim_{t\to-\infty} e^{-t}g_t(z) = z$, hence when the driving function is fixed, it is hard to solve the solution. But the radial Loewner equation figure this problem out.

For $s < t$, define function $\phi_{s,t} \doteq g_t^{-1} \circ g_s$, then $\phi_{s,t}$ is a conformal mapping from $D_t \doteq \mathbb{D} \setminus g_t^{-1}(\gamma([s,t]))$ to $\mathbb{D}$, 0 is a fixed point, the derivative at 0 is e^{t-s}. For a fixed time T, denote $f_t = \phi_{T-t,T}$, we have :

Theorem 58 (see [6]). *$f_t(z)$ is defined as before, then limit $\lim_{z\in D_t, z\to g_T^{-1}(\gamma(T-t))} f(z)$ exists and is continuous by t, $f_t(z)$ satisfies the following ODE*

$$\begin{cases} \dot{f}_t(z) = f_t(z)\frac{\lambda(t)+f_t(z)}{\lambda(t)-f_t(z)} \\ f_0(z) = z. \end{cases} \tag{2.2.1}$$

equation (2.2.1) is the radial Loewner equation, as the whole plane, λ could be written as e^{iW}, and W is the driving function which is a real continuous function on $[0, +\infty)$. Notice that for all initial value z, the solution of the equation not always exist. Hence the domain of definition D_t of g_t will decreasing by t. Actually $D_t = \mathbb{D} \setminus \gamma([0,T])$, where γ is a simple curve on $[0, +\infty)$ start at $\mathbb{T}$ and converges to 0 as t goes to infinity. Hence we have $D_t \in D_s, \forall s < t$.

Actuall, the radial Loewner equation describe that the process of a set grows from a boundary point to a set contains 0 in a simple connect region. Of course this set may not be a simple curve. Now we introduce the general radial Loewner equation on $\mathbb{D}$.

Definition 59. *If $K \subset \mathbb{D}$ is compact in $\mathbb{D}$ satisfies $0 \notin K$, $K = \operatorname{var} K \cap \mathbb{D}$ and $\mathbb{D} \setminus K$ is simple connected, then we say K is a hull of $\mathbb{D}$. For the convenience,we say K is a hull simply only in this subsection.*

By the Riemann mapping theorem, there exists an unique $g_k : \mathbb{D} \setminus K \to \mathbb{D}$ s.t. $g_K(0) = 0, g_K'(0) > 0$, by the Schwarz lemma, $g_K'(0) > 1$. Similarly, we can define the capacity of K in $\mathbb{D}$:

$$\operatorname{dcap}(K) \doteq g_K'(0).$$

Hull of $\mathbb{D}$ has similar properties of the half plane hull, we omit them here. Now we consider a family of hulls $K_t, t \in [0, +\infty)$, if K_t satisfies $K_s \subset K_t, \forall s < t$ and $\operatorname{dacp}(K_t) = e^t$, then we say K_t is normalized hull. By the Koebe distortion theorem we have that when t goes to 0 and $+\infty$, the distance $d(0, K_t)$ from 0 to K_t tends to 1 and 0 respectively. Set $D_t = \mathbb{D} \setminus K_t$, g_t is the Riemann mapping from D_t to $\mathbb{D}$ which fix 0, then we have the following theorem :

Theorem 60. *Suppose $\mu_t, t \geq 0$ is a family of nonnegative Borel measure on $\partial\mathbb{D}$, the total measure of μ_t is 1, and μ_t is continuous by t under the weak topology. For any $z \in \mathbb{D}$, set $g_t(z)$ to be the solution the following equation :*

$$\dot{g}_t(z) = g_t(z) \int_0^{2\pi} \frac{e^{\mathrm{i}\theta} + g_t(z)}{e^{\mathrm{i}\theta} - g_t(z)} \mu_t(\mathrm{d}\theta), g_0(z) = z. \tag{2.2.2}$$

Denote T_z is the upper bound which $g_t(z)$ exists, $D_t = \{z | T_z > t\}, K_t = \mathbb{D} \setminus D_t$, then g_t is the Riemann mapping from D_t to $\mathbb{D}$, and K_t is normalized hull.

Remark 61. Actually, the solutions above can be extended by the Schwarz principle, and the the extension also satisfies equation (2.2.2) for all initial value $z \in \mathbb{C}$.

By the simple curve case, we have that $|g_t(z)|$ is increasing by t, and when $|g_t(z)|$ goes to 1, that is g_t maps z to a boundary point of $\mathbb{D}$, hence we have

Proposition 62. *$T_z = t$ if and only if* $\lim_{s \to t^-} |g_s(z)| = 1$.

In general, we only consider the case that μ_t is a single point measure, that is the equation (2.2.1) with a continuous driving function. Pommerenke gives a condition on D_t to show the connection between continuous driving function and the Loewner process (see [27]), we restate it as the following way :

Theorem 63. *g_t is the Riemann mapping from D_t to $\mathbb{D}$, $0 < a < b$, then the following are equivalent :*

(1) For all $t \in [a, b], g_t$ satisfes the Loewner equation

$$\dot{g}_t(z) = g_t(z) \frac{e^{\mathrm{i}W(t)} + g_t(z)}{e^{\mathrm{i}W(t)} - g_t(z)}, \quad g_0(z) = z$$

where $W(t)$ is a real continuous function on $[a, b]$;

(2) For all $\varepsilon > 0$, there exists $\delta > 0$ s.t. for all $s, t \in [a, b]$ with $0 < b - a < \delta$, there exists a crosscut C of D_s satisfying the diameter of C is less than ε, and C separates $D_s \setminus D_t$ and 0.

Remark 64. Tis theorem also valid in chordal Loewner equation.

Similar to the whole Loewner equation, for any simple curve connect $\mathbb{T}$ and 0, we can induce a continuous driving function by the radial Loewner equation. Conversely, for a continuous driving function, can be get a simple curve by the Loewner equation ? The answer is no. We give the definition of the curve generation at first.

Definition 65. *For a continuous driving function $\lambda : [0, T] \to \partial\mathbb{D}$, consider the hull K_t of* (2.2.1)*, if there exists a curve $\gamma : [0, T] \to \overline{\mathbb{D}}$ s.t. forall t, $D_t = \mathbb{D} \setminus K_t$ is the component of $\mathbb{D} \setminus \gamma[0, t]$ which contains 0, then we say equation* (2.2.1)*, the Loewner process, and the driving function are generated by γ.*

We will give an example (see example 76) later, in this example, the driving is continuous, but the Loewner equation is not generated by curve. As we mentioned in the introduction, the two important problems of Loewner equation are :

(1) For what kind of driving function, the Loewner equation is generated by the curve ?

(2) If the Loewner equation is generated by a curve, then for what kind of driving function, the curve is have a certain property ? For instance, the curve is simple.

In [31], Schramm change the driving function of equation (2.2.1) from a general continuous function to $e^{i\sqrt{2}B_t}$, then the Loewner equation becomes to a stochastic equation SLE_2. He proves :

Theorem 66. *SLE_2 is generated by curve a.e. And distribution of the curve equals the scaling limit of the loop-erased random walk in D.*

For $c < 1/2$, the Brownian motion is c-Hölder continuous. By discussing the 1/2-Hölder of the driving function, and using quasi conformal mapping, Marshall and Rhode give an important theorem in [23] :

Theorem 67. *λ is the driving function, $\forall K > 1$, there exists a constant $C > 0$ s.t. if the $1/2$-Hölder norm $\|\lambda\|_{1/2} \doteq \max_{s,t} \dfrac{|\lambda(s) - \lambda(t)|}{\sqrt{|s-t|}}$ satisfies $\|\lambda\|_{1/2} < C$, then the Loewner process is generated by a K quasislit in $\mathbb{D}$.*

where K quasislit is kind of simple curve, we give the definition here, and the knowledge of quasi conformal mapping refers to [18].

Definition 68. *We say a simple curve $\gamma : (0, 1] \to \mathrm{D}$ in $\mathbb{D}$ is a K quasislit if and only if there exists a K quasi conformal mapping $f : \mathbb{D} \to \mathbb{D}$, and $1 > r > 0$ s.t. $f(0) = 0$, and $f(\gamma((0, 1])) = [r, 1)$.*

By the results of quasi conformal mapping, the quasislit satisfies the following property :

Proposition 69. *The space of the K quasislit is compact.*

Remark 70. In the same paper, Marshall and Rhode also give an example to show that there is some driving functions with finite 1/2-Hölder norm, but the Loewner is not generated by curves. And there are also examples to show that the 1/2-Hölder norms of the driving functions generated simple curves can be any positive number.

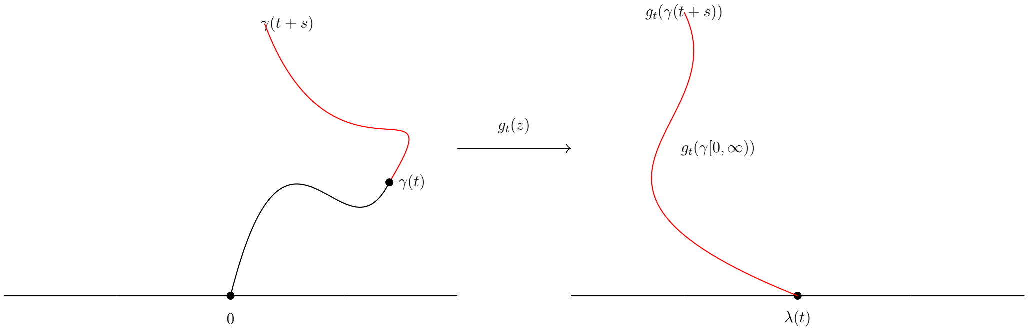
After that, the research of Loewner equation concentrate to the chordal Loewner equation, we will introduce in next section.

2.3 Chordal Loewner equation

In this section, we introduce the chordal Loewner equation, results refer to [13].

2.3.1 The definition of chordal Loewner equation

Assume $\gamma : [0, T]$ is a simple curve in $\overline{\mathbb{H}}$, with $\gamma(0) = 0$,$\gamma(0, T] \subset \mathbb{H}$. Hence for all $0 \le t \le T$, $K_t := \gamma((0, t])$ is a half plane hull, there is a unique conformal mapping $g_t : \mathbb{H} \setminus K_t \to \mathbb{H}$ satisfying $\lim\limits_{z\to\infty} g_t(z) - z = 0$.



Set $a(t) = \text{hcap}(K_t)$, it is easy to see that, $a(t)$ is a continuous function, can increasing strictly. We can normalize $a(t)$ as the radial Loewner equation s.t. $a(t) = 2t$. Under these assumption, we havethe following theorem

Theorem 71. *The $\lambda(t) = \lim\limits_{z\in\mathbb{H}\setminus K_t, z\to\gamma(t)} g_t(z) \in \mathbb{R}$ exists, and $\lambda(t)$ is a continuous function.*

and $g_t(z)$ satisfies ODE :

$$\dot{g}_t(z) = \frac{2}{g_t(z) - \lambda(t)}, \quad g_0(z) = z. \tag{2.3.1}$$

(2.3.1) is the chordal Loewner equation, λ is called the driving function. The driving function is induced by a simple curve of $\mathbb{H}$ right now. But actually, we have the general chordal Loewner equation :

Theorem 72. *Suppose $\mu_t, t \ge 0$ is a parameter family of nonnegative Borel measure on $\mathbb{R}$ with compact supp s.t. $t \mapsto \mu_t$ is continuous in the weak topology, and the total measure of μ_t is 1. Then for all $z \in \mathbb{H}$, set $g_t(z)$ to be the solution of the following equation :*

$$\dot{g}_t(z) = \int_{-\infty}^{+\infty} \frac{2}{g_t(z) - s} \mu_t(\mathrm{d}s), g_0(z) = z. \tag{2.3.2}$$

denote T_z as the suprem time which $g_t(z)$ exists, and $H_t = \{z \in \mathbb{H} | T_z > t\}, K_t = \mathbb{H} \setminus H_t$, then g_t is the conformal mapping from H_t to $\mathbb{H}$, with $\lim_{z\to\infty} g_t(z) - z = 0$, and K_t is normalized half plane hull.

For the chordal Loewner equation, we still concern about the relation between K_t and μ_t. We only consider the case that μ_t is a single point measure in $\mathbb{R}$, then the equation is (2.3.1). By the properties of the half plane capacity, we have

Proposition 73. *Assume λ_1, λ_2 are two driving functions, g_t^1, g_t^2 are the corresponding solutions of the Loewner equation, and K_t^1, K_t^2 are their half plane hull.*

(1) Translation : $\forall t, \lambda_1(t) = \lambda_2(t) + a, a \in \mathbb{R} \Longleftrightarrow \forall t, K_t^1 = K_t^2 + a, a \in \mathbb{R}$ *;*
(2) Symmetry : $\forall t, \lambda_1(t) = -\lambda_2(t) \Longleftrightarrow \forall t, K_t^1$ *is the reflection of* K_t^2 *about the imaginary axis.*
(3) Scaling propety :$\forall t, \lambda_1(t) = \lambda_2(r^2 t)/r, r > 0 \Longleftrightarrow \forall t, K_t^1 = K_{r^2 t}^2/r, r > 0.$
(4) Semi-group property :$\forall t, \lambda_1(t) = \lambda_2(t+s), s > 0 \Longleftrightarrow \forall t, K_t^1 = g_t^2(K_{t+s}^2), s > 0.$

By the first property, we assume $\lambda(0) = 0$ in (2.3.1) in general.

By proposition 24, for general chordal Loewner equation, K_t grows to ∞. When the equation is induced by the curve γ, γ goes to ∞. If we map the half plane to a simple connect region, $\mathbb{D}$ for example, by a conformal mapping, then the image of γ is a curve connect two points of $\partial\mathbb{D}$ like a chord, hence we call this equation the chordal Loewner equation .

$z \in \mathbb{H}$ is a fixed initial value, set $g_t(z) = X(t) + Y(t)\mathrm{i}$, then equation (2.3.1) is

$$\begin{cases} \dot{X}(t) = \dfrac{2(X(t) - \lambda(t))}{(X(t) - \lambda(t))^2 + Y(t)^2} \\ \dot{Y}(t) = -\dfrac{2Y(t)}{(X(t) - \lambda(t))^2 + Y(t)^2} \end{cases} \tag{2.3.3}$$

The imaginary part Y is always decreasing. By the existence theorem of ODE, solutions of (2.3.1) always exist until there is t s.t. $g_t(z) = \lambda(t)$, and we can obtain $Y(t) = 0$. This property is similar to the radial Loewner equation, we give a simple proof here :

Proposition 74. $\forall z \in \mathbb{H}, T_z = inf_t\{Y(t) = 0\}$.

Proof. Assume $g_t(z)$ is the solution of equation (2.3.1) with initial value z. If $Y(t) > 0$, then $|g_t(z) - \lambda(t)| > 0$, hence the solution $g_t(z)$ exists locally at time t, we have $T_z > t$ and $T_z \geq inf_t\{Y(t) = 0\} = T$. If $T_z > T$, then $Y(T) = 0, |X(t) - \lambda(t)| > 0, \forall t < T_z$. There exists a $a > 0$ s.t. $|X(t) - \lambda(t)| > a, \forall t < T$.Now we consider the equation

$$\dot{y}(t) = -\frac{2y(t)}{a^2 + y(t)^2}, y(0) = Y(0)$$

It is easy to see that $y(t) \leq Y(t)$, but $y(t) > 0$, this contradicts $Y(T) = 0$. □

Now we can give a proof of lemma 26 by this property :

proof of lemma 26. We assume $\mathrm{hcap}(K) = T$, and $K = K_T$, where K_t corresponds to a Loewner equation. For all $z \in K_T$, $T_z \leq T$. Hence the solution Y of (2.3.3) with initial value z decreasing to 0 at a time not lager than T. We consider equation

$$\dot{y}(t) = -\frac{2y(t)}{y(t)^2} = \frac{2}{y(t)}, \quad y(0) = \mathrm{Im}(z)$$

then $y \leq Y$ apparently, but we can solve $y(t) = \sqrt{\mathrm{Im}(z) - 4t}$, so we have $T \geq \mathrm{Im}(z)/4$. Since we can choose any $z \in K_T$, we have $\mathrm{hcap}(K) \geq \max_{z\in K} \mathrm{Im} z$. And the equality holds if and only if $X - \lambda$ is a constant, by the first equaion of (2.3.3), we have that λ is a constant, hence K is a segment perpendicular to the real axis. □

Chordal Loewner equation was invented by Schranmm after he proved theorem 66, then he change the driving function from a genera continuous function λ to $\sqrt{\kappa}B_t$ which is the standard Brownian motion times a constant, then the Loewner process become a stochastic process called SLE_κ. The SLE equation is :

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa}B_t}, \quad g_0(z) = z.$$

After this, many mathematicians begin to study the SLE with different κ, they found that many SLEs are the scaling limit of some models, such as SLE_3,SLE_4(see [33],[32]),SLE_6,SLE_8(see [15]), and they solved many related problems. Some of them will be introduced later.

2.3.2 problems of curve

In this section, we present the results of related study on the chordal Loewner equation. Similar to the radial Loewner equation, we can define the curve generation of the chordal Loewner process :

Definition 75. *Given a continuous driving function $\lambda : [0, T] \to \mathbb{R}$, consider the hull K_t corresponding to the solution of the differential equation* (2.3.1)*, if there is a curve $\gamma : [0, T] \to \overline{\mathbb{H}}$ such that for any $t \in [0, T]$, $H_t = \mathbb{H} \setminus K_t$ is the unbounded component of $\mathbb{H} \setminus \gamma[0, t]$, then we say that the equation* (2.3.1)*, the Loewner process and the driving function λ is generated by γ.*

Example 76. *Let us consider the logarithmic spiral, $\gamma^*(t) = (t-1)e^{\mathrm{i}\log|t-1|} - 1, t \in [0, 2]$. We define a piece-wise curve*

$$\gamma(t) = \begin{cases} F \circ \gamma^*(t) & t \in [0, 2] \setminus \{1\} \\ (t+2)\mathrm{i} & t \geq 2 \end{cases}$$

where F is a bijective mapping :

$$F(z) = \mathrm{i}[(|z| + 1)\frac{z}{|z|} + 2]$$

It is easy to see that γ spirals the circle centered on 2i *with radius* 1 *infinitely many times near time 1. So the Loewner process corresponding to this curve is not generated by the curve. But by theorem 63, the corresponding driving function is continuous.*

When we induce (2.3.1) by a simple curve, there is $\lambda(t) = \lim_{z\in\mathbb{H}\setminus K_t, z\to\gamma(t)} g_t^{-1}(z) \in \mathbb{R}$. In fact, judging whether a Loewner process is generated by a curve depends on whether the inverse of this limit exists :

Theorem 77. *The Lowner process is generated by the curve if and only if the limit* $\lim\limits_{z\in\mathbb{H},z\to\lambda(t)} g_t^{-1}(z)\doteq\beta(t)$ *exists, and* $\beta(t)$ *is continuous with respect to* t*.*

Remark 78. The limit in the theorem can be weakened to the angular limit $\lim\limits_{y\to 0} g_t^{-1}(\lambda(t)+yi)$.

Remark 79. In fact, if β is continuous, then β is γ.

Since K_t is growing, define $J_t\doteq K_t\setminus\cup_{s<t}K_s$ as the growth part of the hull at time t. Then if the process is generated by a simple curve, then J_t is always a single point, which is the value of the curve at t. When the process is not generated by a simple curve, J_t may not be a single point set. To describe whether it is generated by a curve, we define that z is t-accessible if and only if there is a curve $\eta:[0,1)\to\bar{\mathbb{H}}$ s.t. $\eta(0)=z,\eta((0,1))\subset H_t$ .The t-accessible points satisfy the following properties, and we give a simple proof :

Proposition 80. *For any* t*, the* t*-accessible point is either none or only one. If in the former case, the Loewner process is not generated by the curve. If in the latter, and the Loewner process is generated by a curve, then the value of the curve at* t *is that point.*

Proof. Firstly, we assume that there are at least two t-accessible points z_1,z_2, then there are two curves l_1,l_2 connected to z_1,z_2 and ∞ respectively, which are not intersected. For any $s<t$, we consider a crosscut C in H_s. If C separates $H_s\setminus H_t$ from ∞, then it must intersect $1_1,l_2$, so the diameter of C must be greater than a certain constant. By the theorem 63, the driving function is discontinuous, which makes a contradiction.

Secondly, we consider the remaining two cases. If the process is generated by a curve, then by the theorem 77, limit $\lim\limits_{y\to 0} g_t(\lambda(t)+yi)$ must exist, and this limit is a t-accessible point. Conversely, if there is no t-accessible point, then the process is not generated by a curve. At the same time, if there is only one t-accessible point, then it should correspond to the point of the curve according the former case. □

For the these cases of accessible points, refer to the figure 4.3.1 in Chapter 4, which gives examples of these cases. But it must be emphasized that when the Loewner process is not generated by curves, the situation may be very complicated.

It is difficult to prove that the Loewner process is generated by curves for a given class of driving functions. First, let's look at the situation of SLE. In the paper [28], they used martingale to calculate the expectation of the derivative of g_t^{-1} near $\lambda(t)$, which proved the existence and continuity of the above angular limit :

Theorem 81. *We* $\kappa\neq 8$*,* SLE_κ *is generated by a curve w.p.1.*

Remark 82. When $\kappa=8$, the SLE is also generated by curves. The proof was not directly obtained, but was derived from the corollary of the scale limit of the UST (see [15]). The direct proof is still very difficult now, and the specific difficulty is given by Lawler et al. in [3].

In a non-stochastic case, Lind transfers the theorem 67 from the radial Loewner equation to the chrodal Loewner equaion in [21]. And she uses the iterative method to give the specific upper bound :

Theorem 83. *For any $C < 4$, there is a constant $K > 1$, so that if the $1/2$-Hölder norm $\|\lambda\|_{1/2}$ of the driving function λ satisfies $\|\lambda\|_{1/2} < C$, then the Loewner process is generated by the K quasilit in the upper half plane.*

Similarly, the K quasislit on H is the image of a segment perpendicular to the real axis, under a K quasi conformal mapping from upper half plane to itself and fixes ∞.

Remark 84. The 4 in the theorem is a optimal upper bound. Because in [23], they find an example where the driving function satisfies the 1/2-Hölder norm equal to 4, but it is not generated by a curve. And this example looks like the case (b) in figure 4.3.1.

Since the function in the Cameron-Martin space is the skeleton of the Brownian motion, it is considered in [8] as the driving function, Friz and Shekhar prove the following conclusion :

Theorem 85. *If the driving function λ is absolutely continuous on $[0, T]$ and is satisfies*

$$\int_0^T |\dot{\lambda}(t)| \mathrm{d}t < +\infty$$

, then the Loewner process is generated by a simple rectifiable curve.

In addition, in [29], Rohde, Tran and Zinsmeister gives a sufficient condition for the Loewner process to be generated bycurve :

Theorem 86. *If there is constant $C_0 > 0$ satisfying that for the continuous driving function λ,*

$$\int_0^t \frac{N_{s,t}^{\lambda}}{(t-s)^{3/2}} \mathrm{d}s \leq C_0, \forall 0 < t < T$$

, where $N_{s,t}^{\lambda} = \sup\{|\lambda_s - \lambda_r| : s \leq r \leq t\}$, then the Loewner process is generated by a Lipschitz graph.

Next, we consider the types of the curve, and we assume that the Loewner process is generated by the curve. In fact, since the Bessel process is equivalent to the real SLE(see remark 97), the capture time of the SLE can be calculated by using the basic properties of the Bessel process. Schramm and Rohde prove the classification theorem of the SLE curve in [28] :

Theorem 87. *Let γ be the generation curve of SLE_{κ}, then*

i. $0 \leq \kappa \leq 4$, γ is almost surely injective, and $\gamma(0, \infty) \subset \mathbb{H}$ w.p.1.

ii. $4 < \kappa < 8$, *γ is almost surely a non-simple but nowhere-dense path, for any $z \in \overline{\mathbb{H}} \setminus \{0\}$, w.p.1., $z \notin \gamma[0,\infty)$, and $\cup_{t\geq 0} K_t = \overline{\mathbb{H}}$;*

iii. $8 \leq \kappa$, *$\gamma(t)$ is almost surely a space-filling curve, and $\gamma[0,\infty) = \overline{\mathbb{H}}$ w.p.1.*

There are many related conclusions about SLE in the paper [28], so we won't go into details here.

It can be seen from the theorem 87 that finding the relationship between the driving term and the curve of the Loewner equation is an important and interesting proposition. We analyze the three theorems mentioned before, the theorems 85, 83, and 87. The condition in 85 is whether $\int \dot{\lambda}^2(t)\mathrm{d}t$ converges. For the driving functions as t^α, the transition of whether the square of the derivative converges is $\alpha = 1/2$. And the characterization in 83 is 1/2-Hölder norm, in other words, for the α-Hölder continuous function, the transition for whether it is a simple curve is still $\alpha = 1/2$. As for the theorem 87, the driving function is Brownian motion, and for any $\alpha < 1/2$, the Brownian motion is α-Hölder continuous, so the transition is still 1/2. Therefore, when considering the types of curve problem, the characterization of the driving function should be the related to the 1/2 order.

For example, in [20] Lind and Rhode give a necessary condition for a curve to be a space-fill curve :

Theorem 88. *There is $C > 4$ satifies that if the Loewner process is generated by the curve γ, and there is an open set $D \subset \mathbb{H}$ s.t. $D \in \gamma([0,\infty))$, then the 1/2-Hölder norm of the driving function is greater than C.*

Remark 89. As described in Lemma 101, the space-fill curve is equivalent to that for any s and any x_1, x_2, capture time $T^s_{x_1} \neq T^s_{x_2}$ and the captured times are all bounded. This condition can be weakened as : any $x_1, x_2 > 0$, capture time $T_{x_1} \neq T_{x_2}$ and they are all bounded. And this weakening condition is sufficient to prove the theorem 88. In fact, the constant in this theorem is very closed to 4, and there should be better results for the space-fill curve.

In [37], the relationship between the smoothness of the drive function and the smoothness of the curve is discussed by Wong :

Theorem 90. *If $\beta > \frac{1}{2}, \beta \neq \frac{3}{2}$, then $\lambda \in C^\beta \Rightarrow \gamma \in C^{\beta+\frac{1}{2}}$, where λ and γ are the driving function and the generation curve, respectively.*

If the Loewner process is generated by the a curve γ, then by the semi-group property of the Loewner equation, the local properties of γ at a fixed time T should be determined by the local properties of the driving function at T. For example, when $\gamma(1) \in \mathbb{R}$ or γ is self-intersected at 1, following the symbols of proposition 73, for any $0 < s < 1$, we have $\gamma_s(1-s) \in \mathbb{R}$ or γ_s self-intersected at $1-s$. So it is the value of the driving function near 1 that determines whether the curve self-intersects or intersects with the real axis at this time. This and previous idea inspire us to consider the value of $lambda(T+t)/\sqrt{|t|}$ around $t = 0$ for a fixed time T.

First we consider two special cases of driving functions, $c\sqrt{t}$ and $c\sqrt{T-t}, t \in [0,T]$, where the above function $lambda(T+t)/\sqrt{|t|}$ is always equal to the constant on the left and right sides at time 0 and T respectively.

The solution of the former can be derived in many ways, which we will introduce later, and we give the conclusion directly here : the driving function $c\sqrt{t}$ is generated by the line which connect 0 and ∞ in the upper half plane, where the angle between this line and the positive real axis is α, then α satisfies $c = \dfrac{4\alpha - 2}{\sqrt{\alpha(1-\alpha)}}$. Under the assumption that the process is generated by the curve, the value of $\lambda(T+t)/\sqrt{t}$ near the right side of $t = 0$ determines the angle of the curve.

Remark 91. The limit $\lim\limits_{t\to 0+} \dfrac{\lambda(t)}{\sqrt{t}} = c$ is discussed in detail in [30], [10] and [38]. In fact, the existence of the limit corresponds to the special case above.

As for the latter one, [19] gives the solution of the special equation by using self-similarity, and gives following conclusions : When $|c| < 4$, the process is generated by a curve, then $\gamma((0,T]) \subset \mathbb{H}$. When $|c| \geq 4$, the process is generated by γ, $\gamma((0,T)) \subset \mathbb{H}, \gamma(T) \in \mathbb{R}$. If $c > 4$, the curve intersects γ the real axis at time T on the right of the starting point and the angle is $2\sqrt{1 - 2/(1 + \frac{c}{c^2-16})}$. If $c < -4$, then by symmetry, the curve is the reflection about the imaginary axis of the curve corresponding to the case $c > 4$. In this paper, we will use the local time transformation to give another method to solve these solution.

Later, in the same paper, the authors discuss the case where the limit $\lim\limits_{t\to T^-} \dfrac{\lambda(T-t)}{\sqrt{T-t}}$ exists and find that the curves are very similar with those in the special cases above. We introduce one conclusion of it as follow :

Theorem 92. *$C < 4$ is a constant, λ is a driving function that satisfies the local Lipschitz-$\frac{1}{2}$ norm less than C on $[0,T]$, then :*

(1) if $\lim\limits_{t\to T^-} \dfrac{|\lambda(T) - \lambda(t)|}{\sqrt{T-t}} = \kappa < 4$, then the process is genenrated by simple curve $\gamma : [0,T] \to \bar{\mathbb{H}}$, and $\gamma(T) \in \mathbb{H}$.

(2) if $\lim\limits_{t\to T^-} \dfrac{|\lambda(T) - \lambda(t)|}{\sqrt{T-t}} = \kappa > 4$, then the process is generated by the curve $\gamma : [0,T] \to \bar{\mathbb{H}}$, and $\gamma(T) \in \mathbb{R}$ or $\gamma(T) \in \gamma([0,T))$.

Remark 93. The condition that the norm of the local Lipschitz-$\frac{1}{2}$ is less than a constant less than 4 is very strong, which implies that the process is generated by the curve before the time T, and at here we won't go into details.

Another important conclusion of this paper is about the case of the limit equal to 4, using the example in the remark 70, they prove that :

Theorem 94. *If the Loewner process is generated by a sufficiently smooth simple curve $\gamma : [0,T) \to \bar{\mathbb{H}}$, and γ wraps around a closed set of half planes infinitely times* *in one direction near time T, then the driver function λ satisfies :*

$$\lim_{t\to T^-}\frac{\lambda(T)-\lambda(t)}{\sqrt{T-t}}=\pm 4$$

In many cases, the types of curve problem of Loewner equation is solved in this way : Fist, find a special solution of the equation and we get a special curve. Then, we consider the driving functions which close to or have similar properties with the driving function of the special solution. At last, we prove that the corresponding curves share similar properties with the special curves. But actually, finding a special solution is eqpiuvalent to fingding a Riemann mapping, but the construction of the Riemann mapping is very hard, hence the construction of the driving function is also very difficult. For example, except the Brownian motion, there is no certain driving function which is generated by a fractal curve.

At last, we introduce the converges theorem of the Loewner equation :

Theorem 95 (see [13])**.** *If a sequence of driving function λ_n converges to another driving function λ in $[0,T]$ uniformly, then the corresponding Loewner processes g_n, g satisfy $g_n \xrightarrow{Cara} g$.*

Since g_n has a extension in a region of $\mathbb{C}_\infty$, and ∞ is a inner point, so the kernel is based on ∞.

Chapitre 3

Local Analysis of Real Loewner Equation

In this chapter, we introduce the local time transformation of the real Loewner equation at first, then we use this method to obtain some results, which include the related theorems in the first section of the introduction.

3.1 Real Loewner equation and its basis properties

At first, we give the definition of real Loewner equation. Refer to [13] for the relevant conclusions in this section. For general chordal Loewner equation, let $\hat{K}_t$ be the symmetry region of the K_t to the real axis . $g_t$is a conformal mapping from H_t to $\mathbb{H}$, and $g_t(\infty) = \infty$, so we can use the symmetry extend theorem, then g_t can be extend to a conformal mapping from $\mathbb{C}_\infty \setminus (K_t \cup \hat{K}_t)$ to $\mathbb{C}_\infty \setminus I_t$, and $I_t$is a close inteval in real axis. g_t fixes ∞ and the derivative at ∞ is 1, and g_t maps the point of $\mathbb{R}$ to $\mathbb{R}$. Then we can proof :

Theorem 96. *Using the symbols before, $\forall z \in \mathbb{C} \setminus (K_t \cup \hat{K}_t), g_t(z)$ satisfies equation* (2.3.1).

Proof. The points of upper half plane have already satisfy the equation, for the points of lower half plane, because we extend g by conjugation, which is $g_t(\bar{z}) = \overline{g_t(z)}$. Put it in (2.3.1), then we see it is right. Then for the points of real axis, because those points are the inner point of $\mathbb{C} \setminus (K_t \cup \hat{K}_t), g_t(z)$, and the right side of (2.3.1)is greater than 0, hence the solution of the equation is continuous to the initial value, so the equation are also valid for the points of the real axis. □

When the initial value is a real number, we write the Loewner equation as :

$$\dot{X}(t) = \frac{2}{X(t) - \lambda(t)}, X(0) = X_0 \neq 0 \in \mathbb{R} \tag{3.1.1}$$

we call it the real Loewner equation, it is a real ODE, driving function λ satisfies $\lambda(0) = 0$. For $s \geq 0$, set $\lambda_s(t) = \lambda(t + s)$.

Remark 97. In the case of SLE, let $Z_t = X(t)/\sqrt{\kappa} - B_t$, then the equation become :

$$\mathrm{d}Z_t = \frac{2}{\kappa Z_t}\mathrm{d}t + \mathrm{d}B_t$$

the solution of this stochastic differential equation is called Bessel process, its properties can be seen in [15].

From equation 3.1.1, we can see the solutions of real equation have monotonicity :

Proposition 98. *The solution of equation* (3.1.1) *with positive initial value is increasing strictly, and the solution of equation*(3.1.1) *with neagtive initial value is decreasing strictly.*

Definition 99. *For $x \in \mathbb{R}$, let T_x be the maximal time that the solution with initial value x exists, we call it to be the capture time of x. For $s \geq 0$, define T_t^s to be the capture time of x with driving function λ_s .*

The easiest example of capture time is $T_0 = 0$. For $x \neq 0$, T_x satisfies $\lim_{t \to T_x^-} X(t) = \lambda(T_x)$. From the graph of K_t, we have $T_x = \max_t\{x \notin K_t\}$. Actually, if the Loewner process is generated by a curve, whelther T_x is infinite determines the properties of the curve. One of the most important theorem is :

Lemma 100. *If Loewner process is generated by curve γ, then $\gamma((0, +\infty))$ is a simple curve if and only if $\forall s \geq 0, x \neq \gamma_s(0)$, capture time $T_x^s = \infty$.*

Proof. Let γ_s, K^s to be the curve and hull respectively which are generated by λ_s. If γ satisfies the condition of the lemma, it is easy to see that γ_s also satisfies the condition. For all $s, t > 0$, $\gamma_s((0, t]) \subset \mathbb{H}$, hence $K_t^s \cap \mathbb{R} = \gamma_s(0)$, so we have $\forall x \neq \gamma_s(0)$, and $x \neq K_t^s$, then $T_x^s > t$. Because t can be any positive time, we have $T_x^s = +\infty$.

Conversely, if all the capture times are infinity and γ does not satisfy the condition, then there exists $t > 0$ s.t. $\gamma(t) \in \mathbb{R}$ or $\gamma(t) \in \gamma((0, t))$. If the first case hold, then $T_{\gamma(t)} < \infty$; If the second case hold, we assume that $\gamma(t_1) = \gamma(t)$ and choose a time s between t_1 and t, then we have $g_s(\gamma(t_1)) \in \mathbb{R}$, therefor γ_s will intersect the real axis, which is the same as the first case, this gives a contradiction to the condition. □

By the similar method :

Lemma 101. *If Loewner process is generated by a curveγ, then γ is a space-filling curve of $\mathbb{H}$, that is $\gamma([0, \infty)) = \bar{\mathbb{H}}$, if and only if $\forall s \geq 0, x_1, x_2$, capture time $T_{x_1}^s \neq T_{x_2}^s$.*

Assume initial value $x > 0, T_x < \infty$, let X to be the corresponding solution, the X is increasing and $\lim_{t \to T_x} X(T) = \lambda(T)$, hence $\lambda(T_x) = X(T_x) > x(t) > \lambda(t), \forall t < T_x$, which means T_x is a left extreme point of the driving function. Combine withe lemma 100, we have :

Lemma 102. *If the Loewner process is generated by curve γ, and $\gamma(t) \in \mathbb{R}$ or $\gamma(t) \in \gamma((0,t))$, then t is a left local extremal point of driving function λ, which means there exists $\delta > 0$ s.t. $\forall s \in (t-\delta, t), \lambda(s) < \lambda(t)$ or $\forall s \in (t-\delta, t), \lambda(s) > \lambda(t)$ hold.*

Now we consider the interval I_t, we can see that I_t is the image of ∂K_t under g_t, use the symbol of boundary behaviour, that is :

$$I_t = C_{H_t}(g_t, \partial K_t)$$

so we have $\lambda(t) \in I_t$.

Since I_t is an inteval of real axis, we only need consider the two tips of the inteval. Form all $X_1 \in I_t$, there are no initial values s.t. the corresponding solution X of equation (3.1.1) satisfies $X(t) = X_1$. It is easy to see the converse is also true. Hence we can consider the reverse equation of (3.1.1) :

$$\dot{X}(t) = -\frac{2}{X(t) - \lambda(T-t)}, X(0) = x \in \mathbb{R} \tag{3.1.2}$$

Fix$T > 0$, since $g_t(\infty) = \infty$, for sufficient large or sufficient small initial value, the solutions of (3.1.2) are always exist in$[0,T)$. We define $I_T = [X_T^-, X_T^+]$. Then for the initial value in I_t, the solutions of (3.1.2) does not exist in $[0,T)$. And since $\lambda(t) \in I_t$, we define $I_t^+ = [\lambda(t), X_t^+], I_t^- = [X_t^-, \lambda(t)]$.

Proposition 103. *We assume the Loewner process is generated by a curveγ, then I_t has the following properties :*

(1) X_t^+ and X_t^- are increasing and decreasing by t respectively ;

(2) $X_t^+ \geq \max_{s\in[0,t]} \lambda(s), X_t^- \leq \min_{s\in[0,t]} \lambda(s)$;

(3) $\gamma(t) \in \mathbb{R}$ if and only if $\lambda(t) = X_t^+$ or X_t^-.

Proof. First, beacause the solutions of the real Loewner equation are always increasing or decreasing, we can obtain (1) by the definition of X_t^+ and X_t^-.

As for (2), we prove the first part. If $\max_{s\in[0,t]} \lambda(s) = \lambda(t) \in I_t$, then it is true. Otherwise we consider the solution$X$ of (3.1.1), with initial value $X(t) = \max_{s\in[0,t]} \lambda(s) > \lambda(t)$, since this solution is increasing, but the initial value equals the maximum of the driving function, so the solution can not exist in $[0,t]$, we have $\max_{s\in[0,t]} \lambda(s) \in I_t$, which finishes the proof.

Now we consider (3), since $\gamma(t) = \lim_{z\in\lambda(t)} g_t^{-1}(z)$ and g_t is a conformal mapping from $\mathbb{C}_\infty \setminus \overline{K_t \cup \hat{K}_t}$ to $\mathbb{C}_\infty \setminus I_t$, these imply the conclusion. □

According to the properties above, we can prove that the length of I_t, which defined as $|I_t|$, can be controlled by the 1/2-Hölder norm of the driving function :

Theorem 104. *λ is the driving function, $\lambda(0) = 0$. If $\|\lambda\|_{\frac{1}{2}} = C < \infty$, then there exists two positive constant C_1, C_2 which only depend on C, s.t.*

$$C_1 < \frac{|I_t|}{\sqrt{t}} < C_2.$$

We prove a lemma at first :

Lemma 105. *$T > 0$ is a fixed time, λ is a driving function, $\lambda(0) = 0$. λ satisfies that $\forall t \in [0, T), |\lambda(T) - \lambda(t)|/\sqrt{T-t} < C$, C is a positive constant. Then there exists a solution x in $[0, T]$ of the real Loewner equation (3.1.1) with positive initial value, s.t.*

$$x(T) - \lambda(T) < (C + 2)\sqrt{T}.$$

Proof. We consider the solution x with initial value $x(0) = \lambda(T) + C\sqrt{T}$: since x is increasing, and $\lambda(t) < \lambda(T) + C\sqrt{T-t} < x(0)$, so the solution exists in$[0, T]$. Set $\hat{x}$ to be the solution of equation (3.1.1) with driving function $\hat{\lambda} \equiv x(0)$ and initial value $\hat{x}(0) = x(0)^+$, then we by the special solution of Loewner equation, we have $\hat{x}(t) = x(0) + 2\sqrt{t}$. Because $\hat{\lambda} > \lambda$, then $x < \hat{x}$, hence we get $x(T) < \hat{x}(T) = 2\sqrt{T} + x(0) = (2 + C)\sqrt{T}$. □

Now we prove theorem 104.

Proof. We prove the left side at first. We choose two solutions x_1, x_2 which are exist in $[0, t]$, the initial value of x_1 is positive and x_2 is negative. Subtract them in real Loewner equation(3.1.1), we have

$$\dot{x}_1 - \dot{x}_2 = \frac{2}{x_1 - \lambda} - \frac{2}{x_2 - \lambda} = \frac{2(x_1 - x_2)}{(x_1 - \lambda)(\lambda - x_2)}$$

it can be written as

$$\begin{aligned} \frac{\mathrm{d}(x_1 - x_2)}{\mathrm{d}x} &= \frac{2(x_1 - x_2)}{(x_1 - \lambda)(\lambda - x_2)} \\ &\geq \frac{8(x_1 - x_2)}{(x_1 - x_2)^2} = \frac{8}{x_1 - x_2} \end{aligned}$$

solve this ODE, then we have

$$x_1(t) - x_2(t) \geq \sqrt{16t + (x_1(0) - x_2(0))} > 4\sqrt{t}.$$

By the definition of I_t,

$$|I_t| = \inf_{x_1, x_2 \text{exist in}[0,T]} x_1 - x_2$$

Hence we let $C_1 = 4$ which finishes the proof of the left side.

Lemma 105 implies the right side.It is obvious that the condition of the theorem satisfies the condition of lemma 105, so we can get two solutions x_1, x_2, the initial value are positive and negative respectively, and satisfy :

$$x_1(t) - \lambda(t) < (C + 2)\sqrt{t}, \lambda(t) - x_2(t) < (C + 2)\sqrt{t}$$

add them together, we have $x_1(t) - x_2(t) < (2C+4)\sqrt{t}$. Notice that $I_t < x_1(t) - x_2(t)$, hence let $C_2 = 2C + 4$, we obtain the right side. □

Now we consider the Loewner process which is generated by a simple curve γ. In this case, I_t^+ and T_t^- are all closed inteval. By theorem 29, g_t^{-1} can be continuous extended to the boundary. And for all $s < t$, $\gamma(s)$ corresponds to two points of in the boundary of H_t, that is $x_+^s \in I_t^+$ and $x_-^s \in I_t^-$, $g_t^{-1}(x_+^s) = g_t^{-1}(x_-^s) = \gamma(s)$. Conversely, for any point of I_t, g_t^{-1} maps it to one point of γ.

Hence we can define a continuous homeomorphism$\phi_t : I_t^+ \to I_t^-$ s.t. $g_t^{-1}(x) = g_t^{-1}(\phi(x))$. And we can shift λ s.t. $\lambda(t) = 0$. This homeomorphism is called conformal welding. For example, when γ is a straight line perpendicular to the real axis at 0, $\phi_t(x) = -x$. It's easy to see the following connection between conformal welding and the curve :

Proposition 106. *Any simple curve $\gamma : [0, t] \to \bar{H}$ with $\gamma(0) \in \mathbb{R}$ corresponds a conformal welding uniquely.*

conformal welding is similar to the "sewing" in quasi conformal theory, and they also have some connections. In stochastic case, it is also very useful, see [34]. Some related conclusions will be introduced later.

3.2 Local time transformation

3.2.1 local time transformation of real equation

In last chapter, we introduced our idea for analysing the driving function of Loewner process, that is fix a time$T$, and discuss the local properties which are related to 1/2 order at time T. For instance, if the Loewner process is generated by a curve , then the left local properties of the driving function determines whether the curve is simple. Like the capture time, we can give a definition :

Definition 107. *Let $\lambda(t)$ be a driving function,and T a positive real. We say that λ is captured at time T if $\exists s > 0, x \in \mathbb{R}$ s.t. $T_x^s = T - s$. And in that case, we say x is a captured solution at time T. We say that λ is uncaptured if it is not captured at any time T.*

We can see that if the Loewner process is generated by a curve, then the curve is simple if and only if λ is not capture. Hence we only need to discuss if λ is captured at each time T. By the scaling property of chordal Loewner equation, we assume $T = 1$ wlog. In the rest of this thesis, unless state specially, we assume $t = 1$. A natural idea is making a transformation on the driving function, for real Loewner equation (3.1.1), if it is captured at time 1, we assume that $s = 0$. And we can also assume the X is a captured solution, with the initial value $X_0 > 0$. Then X satisfies $\lim_{t \to 1^-} X(t) = \lambda(1)$, set

$$\lambda^-(t) = (\lambda(1) - \lambda(t))/\sqrt{1-t}, X^-(t) = (\lambda(1) - X(t))/\sqrt{1-t}.$$

The equation (3.1.1) becomes

$$\dot{X}^{-}(t) = \frac{1}{2(1-t)}\left(-\frac{4}{\lambda^{-}(t) - X^{-}(t)} + X^{-}(t)\right).$$

We now use a time change to get rid of the time term, namely $\sigma : [0, +\infty) \to [0, 1)$, $t \mapsto 1 - e^{-2t}$. Setting $x(t) = X^{-}(\sigma(t)), \xi(t) = \lambda^{-}(\sigma(t))$, we have

$$\dot{x}(t) = x(t) - \frac{4}{\xi(t) - x(t)},\ x(0) < \xi(0). \tag{3.2.1}$$

This is the real Loewner equation in the Hölder-$\frac{1}{2}$ point of view : the functions ξ and x are continuous function in$[0, +\infty)$, and ξ is the new driving function. It is easy to see that once the solution of equation 3.2.1 equals 0, it will keep decreasing. ξ and x are all continuous function on $[0, +\infty)$, we say ξ is the driving function of (3.2.1). ξ and x satisfy the followling property :

Proposition 108. *$\xi > x$, ξ and x satisfies :*

$$\lim_{t\to+\infty} e^{-t}\xi(t) = \lim_{t\to+\infty} e^{-t}x(t) = 0 \tag{3.2.2}$$

Proof. Since the initial value of X is positive, X is increasing on $[0, 1)$ with $\lambda(1) = X(1) > X(t) > \lambda(t)$, and λ satisfies $\lambda(1) > \lambda(t)$, $t \in [0, 1)$, after the transformation, we have $\forall t > 0, \xi(t) > x(t) > 0$.

Because λ is continuous, we have :

$$0 = \lim_{t\to 1^{-}} \lambda(1) - \lambda(t) = \lim_{t\to 1^{-}} \sqrt{1-t}\lambda^{-}(t) = \lim_{t\to+\infty} e^{-t}\xi(t)$$

Combine with $\xi(t) > x(t) > 0$, we finish the proposition. □

Although equation(3.2.1) is induced by a capture solution, but for a driving function ξ, if it is positive, and satisfies $\lim_{t\to+\infty} e^{-t}\xi(t) = 0$, then we can study the capture property of equation (3.2.1). For the convenience, we give the definition at first :

Definition 109. *Assume the driving function of equation* (3.2.1) *is a positive continuous function on* $[0, +\infty)$*, and it satisfies equation* (3.2.2)*, if there is a initial value*x_0 *s.t. the corresponding solution is also a positive continuous function on* $[0, +\infty)$*, then we say the equation or the driving function are captured, and say solution is a capture solution. If equation* (3.2.1) *has no capture solution, then we say it is not captured.*

Remark 110. For a captured driving function, the capture solution may not be unique. If x_1, x_2 are two captured solutions, then for any initial values between $x_1(0)$ and $x_2(0)$, the corresponding solutions are also between x_1 and x_2, hence they are also captured solution. So if the driving function is captured, then there are infinitely many captured solutions, or there is only one.

Example 111. *For $c \geq 4$,$\xi \equiv c$ is captured, and $x \equiv (c \pm \sqrt{c^2 - 16})$ is a captured solution.*

Example 112. *For $c < 4$, $\xi \equiv c$ is not captured.*

About the capture property, a basic result is

Lemma 113. *Assume ξ_1, ξ_2 are two driving functions of equation (3.2.1) , $\forall t,\ \xi_1(t) \geq \xi_2(t)$. If ξ_2 is captured, then ξ_1 is also captured.*

Proof. Assume ξ_2 is captured, $x_2 : [0, +\infty) \to [0, +\infty)$ is a capture solution of ξ_2. x_1 is the solution of equation(3.2.1), with initial value $x_2(0)$and driving function ξ_1. If x_1 exists in$[0, +\infty)$, then it is easy to see $x_1 \geq x_2 > 0$, hence x_1 is a captured solution which implies ξ_1 is captured.

Now we consider the case x_1 does not exist in $[0, +\infty)$, x_1 will equal ξ in a finite time. We consider a set A of the initial values, the elements of A satisfy that their corresponding solutions of ξ_1 do not exist in $[0, +\infty)$. By the assumption, A is not empty. And for these negative initial values, solutions are exist in $[0, +\infty)$ obviously, hence we can set $x_0 = \inf A$. Let $\tilde{x}$ be the solution with initial value x_0, and ξ_1 is the driving function. If $\tilde{x}$ is a captured solution, then the proof is finished. Otherwise we have two cases :

If $\tilde{x}$ does not exist in$[0, +\infty)$, then we can assume T is the capture time, that is $\tilde{x}(T) = \xi_1(T)$. And consider another solution $\hat{x}$, with driving function ξ_1, and the initial value condition it $\hat{x}(T) = x_2(T) < \xi_2(T) \leq \xi_1(T) = x(T)$. It is easy to see $\hat{x}$ at least exist in $[0, T]$ and we have$\hat{x} < \tilde{x}$, since $\hat{x}(0) < \tilde{x}(0) = x_0 = \inf A$, so we know that $\hat{x}$ exist in $[0, +\infty)$. After time T, we have $\hat{x} \geq x_2 > 0$, hence $\hat{x}$ is a captured solution of ξ_1.

If $\tilde{x}$ are not always positive, then we assume $\tilde{x}(T) < 0$. Since the solutions of ODE is continuous by its initial value, we know that for the initial values which are sufficient close to a , the corresponding solutions must be negative at time T, so these solutions are exist in $[0, +\infty)$. This is contradict to that a is a boundary point of A. Hence the lemma is true. □

By this lemma and the examples before, we have

Proposition 114. *If for all $t, \xi(t) \geq 4$, then ξ is captured. If for all $t, \xi(t) < c < 4$, then ξ is not captured.*

For the driving function before and after the time transformation, whether it is captured is the same. To study the capture property and the connection between ξ and x, we give some definitions at first :

$$\mathcal{L} = \{\varphi \in L^1([0, +\infty),\ \varphi > 0,\ \lim_{t \to \infty} e^{2t}\varphi(t) = +\infty\}$$
$$\mathcal{I} = T(\mathcal{L}),$$

T is a linear integration operator :

$$T(\varphi)(t) = e^t \int_t^{\infty} \varphi(s)\mathrm{d}s.$$

We can write (3.1.1) as :

$$\xi(t) = x(t) + \frac{4}{x(t) - \dot{x}(t)},\ x(0) < \xi(0) \tag{3.2.3}$$

Define a nonlinear operator $F(\varphi) \doteq \varphi + 4/(\varphi - \dot{\varphi})$. Then we have a basic theorem :

Theorem 115. *ξ is the driving function of equation* (3.2.1)*, then ξ is captured if and only if there exists a $\Phi \in \mathcal{I}$ s.t. $\xi = F(\Phi)$.*

Proof. Let us first assume that x is a captured solution. Since X and λ are both continuous at 1, it is easy to check that

$$\lim_{t\to+\infty} e^{-t}x(t) = \lim_{t\to1} \sqrt{1-t}X^-(t) = 0,$$

$$\lim_{t\to+\infty} e^{-t}\xi(t) = \lim_{t\to1} \sqrt{1-\lambda}\lambda^-(t) = 0.$$

Multiplying (3.2.3) by e^{-t}, and noticing that $\dot{x}(t) - x(t) < 0$ since $x < \xi$, we have

$$\lim_{t\to+\infty} \frac{4}{e^t(x(t) - \dot{x}(t))} = 0,$$

and

$$\lim_{t\to+\infty} e^{2t}\frac{\mathrm{d}}{\mathrm{d}t}(e^{-t}x(t)) = \lim_{t\to+\infty} e^t(\dot{x}(t) - x(t)) = -\infty.$$

Setting $\varphi(t) = -\mathrm{d}(e^{-t}x(t))/\mathrm{d}t$, we get

$$\int_0^{+\infty} \varphi(s)\mathrm{d}s < +\infty,\ \varphi(t) > 0,\ \lim_{t\to+\infty} e^{2t}\varphi(t) = +\infty,$$

and φ is continuous, so that the function $\Phi :\ t \mapsto e^t \int_t^{+\infty} \varphi(s)\mathrm{d}s$ is an element of $\mathcal{I}$.

Conversely, if $\Phi \in \mathcal{I}$ is such that $\xi = F(\Phi)$, then, by the time change, we have

$$\lambda(t) = \lambda(1) - \int_{-\frac{1}{2}\ln(1-t)}^{+\infty} \varphi(s)\mathrm{d}s - \frac{4(1-t)}{\varphi(-\frac{1}{2}\ln(1-t)),}$$

$$X(t) = \lambda(1) - \int_{-\frac{1}{2}\ln(1-t)}^{+\infty} \varphi(s)\mathrm{d}s,$$

and we easily check that λ and X satisfy the original Loewner equation (1.0.1) and that they are continuous and equal at time $t = 1$. □

The set $\mathcal{I}$ is a cone, meaning that if $\Phi_1, \Phi_2 \in \mathcal{I}$ and $c > 0$, then $\Phi_1 + \Phi_2 \in \mathcal{I}$ and $c\Phi_1 \in \mathcal{I}$. The set $\mathcal{I}$ is also a subspace of $C_1(\mathbb{R}_+)$. Lind has shown in [21] that if $\forall t \geq 0, \xi(t) < 4$, then $\xi \notin F(\mathcal{I})$.

3.2.2 special solutions of chordal Loewner equation

The first application of local transformation is we can get some special solutions. For instance, in last chapter we have mentioned the driving function $c\sqrt{T-t}$, we can use local transformation to solve the corresponding chordal Loewner equation :

It is easy to check that if $\lambda(t) = c - c\sqrt{1-t}, c > 0$, then after the transformation, we have $\xi(t) = c, \forall t \in [0, +\infty)$. We can write (3.2.1) as :

$$\mathrm{d}t = \frac{c-x}{cx - x^2 - 4}\mathrm{d}x,$$

this is a autonomous ODE, we discuss it in three cases :

(1) $c < 4$ equation is :

$$\mathrm{d}t = \frac{x-c}{(x-c/2)^2 + 4 - c^2/4}\mathrm{d}x$$

integrating it, we have

$$t + C = \frac{1}{2}\ln((x-c/2)^2 + 4 - c^2/4) + \frac{1}{\sqrt{4-4-c^2/4}} \arctan \frac{x - c/2}{\sqrt{4-4-c^2/4}}$$

C is the integral constant ;

(2) $c = 4$, equation is :

$$\mathrm{d}t = \frac{x-4}{(x-2)^2}\mathrm{d}x$$

and after integration, we have

$$t + C = \ln|x-2| + \frac{1}{x-2}$$

C is the integral constant ; And $x(t) = 2, \forall t \in [0, +\infty)$ is a special solution ;

(3) $c > 4$, equation is :

$$\mathrm{d}t = \frac{x - c_1 - c_2}{(x-c_1)(x-c_2)}\mathrm{d}x$$

$c_1 = (c + \sqrt{c^2-16})/2, c_2 = (c - \sqrt{c^2-16})/2$, integrating it :

$$t + C = (\frac{1}{2} - \frac{c}{2\sqrt{c^2-16}}) \ln|x-c_1| + (\frac{c}{2\sqrt{c^2-16}} + \frac{1}{2}) \ln|x-c_2|$$

C is the integral constant ; And$x(t) = c_1, c_2, t \in [0, +\infty)$ are two special solution.

We can see that $c = 4$is a transition of equation, this is the same as the theorem 83. Since we have the solution of equation (3.2.1), do the inverse local transformation, we can get the solution of equation (3.1.1).

Remark 116. Since the analytic function can be extent to the plane, the above method is also valid to the the general Loewner equation, we only need to solve the ODE :

$$\mathrm{d}t = \frac{c - z}{cz - z^2 - 4}\mathrm{d}z.$$

Actually, the solution is same as the real equation, but it need to define the specific logaritheorem function and the inverse trigonometric function in the plane. The result can been seen in [19].

as we mentioned before, for the general Loewner equation, find some special solutions is very hard. But in equation (3.2.1), if we assume that ξ is differentiable, and set $\hat{x}(t) = \xi(t) - x(t)$, $F(t) = \xi'(t) - \xi(t)$, then solve equation (3.2.1) is equivalent to solve equation :

$$\frac{\mathrm{d}\hat{x}}{\mathrm{d}t} = \hat{x} + \frac{4}{\hat{x}} + F(t) \tag{3.2.4}$$

If we can find some F, and some special solutions for (3.2.4), then we can find some special solutions for Loewner equation. For example, when F is a constant, equation (3.2.4) is solvable, the correspond ξ is the type of $C_1 e^t + C_2$, when $C_1 = 0$, the driving function λ is $c\sqrt{T - t}$, when $C_1 \neq 0$, the driving function λ is linear function.

When λ is linear function, when can assume that $\lambda(t) = ct$, in this case, real Loewner equation (2.3.1)can be write as :

$$\frac{\mathrm{d}t}{\mathrm{d}X} = \frac{X - ct}{2}$$

this is a linear ODE of variable t. When $c = 0$, Loewner process is generated by the upper half imaginary axis. When $c \neq 0$, we can solve the equation as :

$$t = \frac{X(t)}{c} + Ce^{-cX(t)/2} - \frac{2}{c^2}$$

C is a constant. Similar as before, this solution can also be extend to the upper half plane, the specific solution is :

$$t = \frac{g_t(z)}{c} + (\frac{2}{c^2} - \frac{z}{c})e^{\frac{c}{2}(z - g_t(z))} - \frac{2}{c^2}$$

$g_0(z) = z$. By this solution, we can compute the curve γ which generate this Loewner process . Notice that γ satisfies :

$$\gamma(t) = \lim_{\zeta \in \mathbb{H}, \zeta \to \lambda(t)} g_t^{-1}(\zeta)$$

hence we only need to put $g_t(z) = \lambda(t) = ct$ into the solution above, then we have

$$e^{\frac{c^2 t}{2}} = (1 - \frac{cz}{2})e^{\frac{cz}{2}}$$

set $\gamma(t) = z$, we can obtain the curve γ. It is not difficult to check that γ is intersect with real axis at 0 perpendicularly.

3.2.3 local time transformation at the right side

The time transformation we introduced is worked at the left side of the driving function, similarly, we can do the same time transformation at the right side. We use the same symbol for convenience, but in other part of this thesis, these symbols are used for the transformation of the left side.

We assume the fixed time $T = 0$ wlog. If a positive solution Xsatisfies $\lim\limits_{t\to 0+} X(t) = \lambda(0) = 0$, we set

$$\lambda^-(t) = (\lambda(t) - \lambda(0))/\sqrt{t} = \lambda(t)/\sqrt{t}, X^-(t) = (X(t) - X(0))/\sqrt{t} = X(t)/\sqrt{t}$$

then the equation (3.1.1) become

$$\dot{X}^-(t) = \frac{1}{2t}\left(\frac{4}{X^-(t) - \lambda^-(t)} - X^-(t)\right).$$

Let $\sigma : (-\infty, 0] \to (0, 1]$, $t \mapsto e^{-2t}$, set $x(t) = X^-(\sigma(t)), \xi(t) = \lambda^-(\sigma(t))$. Then we have

$$\dot{x}(t) = \frac{4}{x(t) - \xi(t)} - x(t),\ x(1) = x_1 > \xi(1). \tag{3.2.5}$$

When ξ is a constant c, the equation is autonomous :

$$\frac{x - c}{x^2 - cx - 4}\mathrm{d}x + \mathrm{d}t = 0$$

integrating it, we have

$$t + C = (\frac{1}{2} - \frac{c}{2\sqrt{c^2 + 16}}) \ln|x - c_1| + (\frac{c}{2\sqrt{c^2 + 16}} + \frac{1}{2}) \ln|x - c_2|$$

$c_1 = (c+\sqrt{c^2 + 16})/2, c_2 = (c+\sqrt{c^2 - 16})/2$. like the left side case, we can change x to z, and put the solution back to the local transformation, then we get the solution of equation (1.0.1) with driving function $c\sqrt{t}$.

3.3 local condition of simple curve

3.3.1 limitinf and limitsup condition

In this section, we will prove theorem 2 to show how the local time transformation works.

As we mentioned before, if we assume the Loewner process is generated by a curve, then in the case that limit $\lim_{t\to T^-}(\lambda(T) - \lambda(t))/\sqrt{T - t}$ exists, the left local property at time T has been almost made clear in [19]. Hence we discuss the case that limit does not exist. We define

$$a = \varliminf_{t\to 1^-} \frac{\lambda(1) - \lambda(t)}{\sqrt{1 - t}},\ b = \varlimsup_{t\to 1^-} \frac{\lambda(1) - \lambda(t)}{\sqrt{1 - t}}.$$

We discuss when the driving function λ is captured at time 1. A sufficient condition is that 1 is a left extremal point of λ. It implies that a and b are both positive or negative. We prove the follow lemma, as we mentioned in the introduction, it is the kernel lemma in this chapter :

Lemma 117. *a, b as defined above, assumeb $b \geq a \geq 0$, if driving function is captured at time* 1*, then a, b satisfies $b \geq f(a)$, where f is a piece-wise function as follow :*

$$f(a) = \begin{cases} a, & a \geq 4 \\ 4, & 2 \leq a < 4 \\ a + \frac{4}{a}, & 0 < a < 2 \\ +\infty, & a = 0 \end{cases} \tag{3.3.1}$$

and $f(a)$ is the best lower bound.

Proof. Assume that x is a captured solution with driving function ξ. From the equation [3.1.1] we see that X must be strictly monotone, implying that a and b are both either negative or positive. Changing λ to $-\lambda$ if necessary, we assume from now on that $a \geq 0$. Using the time change, we get $a = \underline{\lim}_{t\to\infty} \xi(t)$ and $b = \overline{\lim}_{t\to\infty} \xi(t)$.

If $a \geq 4$, then $b \geq a = f(a)$, and the result is sharp as seen by taking $\xi(t) \equiv a$ for which $x(t) \equiv (a + \sqrt{a^2 - 16})/2$ is a captured solution for ξ making $b = f(a)$.

Suppose now that $a < 4$: let $\varepsilon_0 > 0$ such that $a + \varepsilon_0 < 4$. Because $\underline{\lim}_{t\to\infty} \xi(t) = a$ there exist two sequences $t_n \uparrow +\infty$ and $\varepsilon_n \downarrow 0$ such that $\xi(t_n) < a + \varepsilon_n \leq a + \varepsilon_0 < 4, n \geq 1$. Since $x(t_n) + \frac{4}{x(t_n)} \geq 4$, (3.2.3) together with the above imply that $\dot{x}(t_n) < 0, n \geq 1$.

There are two cases :

i. $\exists n_0 > 0; \xi(t_{n_0}) < a + \varepsilon_0 < 4$ and $\dot{x}(t) < 0, \forall t \geq t_{n_0}$. Then the function x, being positive and decreasing on $(t_0, +\infty)$, must converge to a limit $c \geq 0$ as $t \to \infty$, which implies that $\overline{\lim}_{t\to\infty} \dot{x}(t) = 0$. Thus there exists a sequence $s_n \uparrow +\infty$ such that $c \leq x(s_n) < c + \frac{1}{n}, 0 > \dot{x}(s_n) > -\frac{1}{n}$. Putting this information in (3.2.3) we get

$$\xi(s_n) = x(s_n) + \frac{4}{x(s_n) - \dot{x}(s_n)} \geq c + \frac{4}{c + \frac{2}{n}}$$

which implies that $b = +\infty$ if $c = 0$ and $b \geq c + \frac{4}{c}$ otherwise. On the other hand :

$$c \leq x(t_n) \leq \xi(t_n) < a + \varepsilon, n \leq 1 \Rightarrow c \leq a$$

Finally, we get $b \geq \min_{c \leq a}\{c + \frac{4}{c}\} = f(a)$, as seen from the fact that f decreases on $[0, 2]$ and is non decreasing after time 2.

ii. If the first case does not occur we define for all $n \geq 1$ $T_n \geq t_n$ as the first time when $\dot{x}(T_n) = 0$. Putting again this information in (3.1.1), we get that

$$\xi(T_n) = x(T_n) + \frac{4}{x(T_n)}, \ x(T_n) \leq x(t_n) \leq \xi(T_n) \leq a + \varepsilon_n.$$

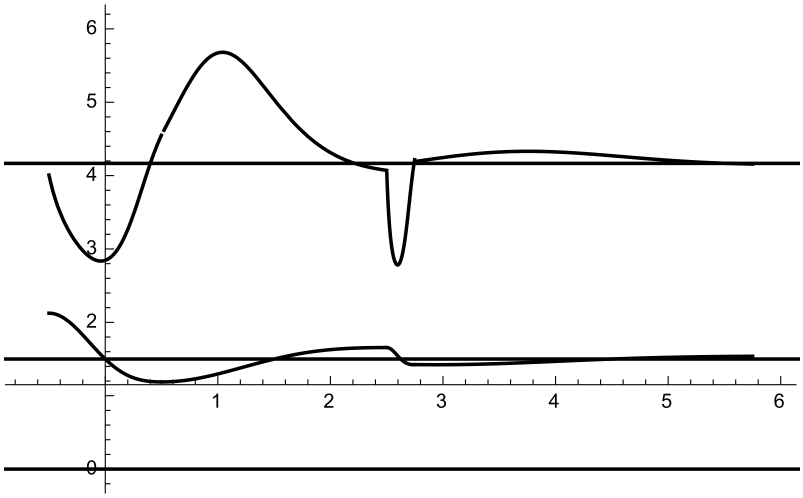


FIGURE 3.3.1 – Example of lemma 2.4

We conclude that $\xi(T_n) \geq \min_{d<a+\varepsilon_n}\{d+4/d\} = f(a+\varepsilon_n)$. As above, $b = \overline{\lim_{t\to\infty}}\,\xi(t) \geq \lim_{n\to\infty} f(a+\varepsilon_n) = f(a)$.

To make sure that the bound is optimal, we give an example first in the case $a \leq 2$:

Set $\alpha_k = \frac{k(k+1)\pi}{2} + \pi - \frac{\pi}{k+1}, \beta_k = \frac{k(k+1)\pi}{2} + \pi - \frac{\pi}{k+2}, k = 0, 1, \ldots$

$$x(t) = \begin{cases} a + \frac{1}{\ln t}\cos((k+1)(k+2)(t-\alpha_k)) & t \in [\alpha_k, \beta_k) \\ a - \frac{1}{\ln t}\cos(\frac{t-\beta_k}{k+1}) & t \in [\beta_k, \alpha_{k+1})). \end{cases} \tag{3.3.2}$$

It is not difficult to check that

$$\underline{\lim_{t\to+\infty}}\,\xi(t) = \lim_{n\to+\infty}\xi(\frac{\alpha_n+\beta_n}{2}) = a, \overline{\lim_{t\to+\infty}}\,\xi(t) = \lim_{n\to+\infty}\tilde{\xi}(\beta_n) = a + \frac{4}{a}.$$

Figure 3.3.1 shows the case $c = 3/2$: the upper graph is that of ξ, the lower one of x, and the three lines are $y = 0, c, c+4/c$. We can see that x increasing slowly,and then decreasing in a very short time, and x converges to c. Hence ξ will approach x when x decreasing,then go back to $c+4/c$ rapidly, and ξ stay closed to $c+4/c$ when x increasing slowly.

If $4 > a > 2$, put $\alpha_k = \frac{k(k+1)\pi}{2} + \frac{a-2}{4-a}\frac{k}{k+1}\pi, \beta_k = \frac{k(k+1)\pi}{2} + \frac{a-2}{4-a}\frac{k+1}{k+2}\pi, k = 0, 1, \ldots$

$$x(t) = \begin{cases} 2 + \frac{1}{t}\cos\frac{4-a}{a-2}(k+1)(k+2)(t-\alpha_k) & t \in [\alpha_k, \beta_k) \\ 2 - \frac{1}{t}\cos\frac{t-\beta_k}{k+1} & t \in [\beta_k, \alpha_{k+1}) \end{cases} \tag{3.3.3}$$

As in the example above, at $(\alpha_n+\beta_n)/2$ and β_n, this function will attain the two limits. This implies that 4 is the best bound of b when $4 > a > 2$.

Finally, from the proof before, we can see when $a = 0$, $b = +\infty$. □

now we proof theorem 2 by the lemma above :

proof of theorem 2. Assume λ is a driving function which satisfies the condition of theorem 2, by lemma 100, we only need to prove that for all time$t, s > 0$, driving function λ_s are not captured at time t.

If t is not a left extremal point of λ_s, then λ_s is not captured. Hence we only need to consider the rest case, now we do local transformation for λ_s at time t, then a, b are both non-positive or non-genative. If they are non-positive, we consider a driving function $-\lambda_s$, by the symmetry property of chordal Loewner equation, $-\lambda_s$ and λ_s are both captured or not captured at time t, the a, b of both non-negative. Now it is easy to check a, b do not satisfy the condition of lemma 100, hence λ_s is not captured. □

3.3.2 other conditions for driving function

In this sub section, we discuss other conditions. We use the symbols before, and define the local 1/2-Hölder norm at time t as follow :

$$\|\lambda\|_{\frac{1}{2}}^{t} \doteq \lim_{\delta \to 0^+} \max_{s,r \in [t-\delta, t]} \frac{|\lambda(s) - \lambda(r)|}{\sqrt{|s-r|}}$$

In the examples of lemma 117, the local 1/2-Hölder norm at time 1 satisfies

$$\lim_{n \to \infty} \frac{\xi(\beta_n) - \xi(\frac{\alpha_n + \beta_n}{2})}{\beta_n - \frac{\alpha_n + \beta_n}{2}} = +\infty$$

And it is obvious that $\|\lambda\|_{\frac{1}{2}}^{1} \geq b(t)$, so if we know $a(t)$, then the lower bound of local 1/2-Hölder norm must be larger than $a(t)$. We have the following lemma for captured driving function, the proof is similar to lemma 117, so we only state the result here :

Lemma 118. *a is the limit inferior as before. There exists a constant $C > 1$, s.t. if $0 < a < 2$, and λ is captured at time 1,then*

$$\|\lambda\|_{\frac{1}{2}}^{1} \geq Cf(a)$$

.

If we want to estimate the 1/2-Hölder norm of driving function by the limit inferior, then this lemma is stronger than lemma 117.

Now we consider the case $a = 0$, in this case, if Loewner process is captured, then $b = +\infty$. But we can discuss the speed of the limit approach 0 and $+\infty$.

Lemma 119. *If ξ is a captured driving function, $a = 0$, $b > 0$, and ξ satisfies*

$$\varliminf_{t\to+\infty} \xi(t)h(t) < C,$$

then we have

$$\varlimsup_{t\to+\infty} \xi(t)\frac{1}{h(t)} > \frac{4}{C}$$

where C is a positive constant, h is a continuous positive function in $\mathbb{R}^+$ with

$$\lim_{t\to+\infty} h(t) = \infty.$$

Proof. By the idea of lemma reflimit1, we can choose $t_n \uparrow +\infty, \varepsilon_n \downarrow 0$, s.t. $x(t_n) < \xi(t_n) < (C + \varepsilon_n)/h(t_n)$. Then we use the same method, have two cases. Whatever which case hold, we always have time $T_n > t_n$ s.t. $\dot{x}(T_n)$ is sufficient small, then

$$\xi(T_n) > \frac{4}{x(T_n)} > \frac{4h(T_n)}{C + \varepsilon_n}$$

let n goes to infinity, we get the lemma. □

By propositon 108, the increasing speed of ξ are control by e^t, hence by the lemma above, set $h(t) = e^t$, we obtain the following lemma :

Proposition 120. *If ξ is captured, then*

$$\varliminf_{t\to+\infty} \xi(t)e^t = \lim_{t\to+\infty} \xi(t)e^t = +\infty.$$

Actually, from (3.2.1) we know that, if $\xi(t) < 4$, then x is decreasing at t, hence if x is a captured solution, then ξ can be less than 4 for too long. By this idea, we can construct another condition.

We define the function G as :

$$G(x) = \begin{cases} 4 - x & x \geq 2 \\ 4/x & 0 \leq x \leq 2 \end{cases} \tag{3.3.4}$$

then we have :

Proposition 121. *If $\exists\, t_2 > t_1 \geq 0$ such that*

$$\int_{t_1}^{t_2} G(\xi(s))\mathrm{d}s \geq \xi(t_1) \tag{3.3.5}$$

then no solution x with initial time $t_0 \leq t_1$ is captured.

Proof. If x is a captured solution, consider the function $h(t) = t - \frac{4}{c-t}$, c is a positive constant, whose derivative is $h'(t) = 1 - \frac{4}{(c-t)^2}$, For $t \in (0, c)$,

$$h(t) \leq \begin{cases} h(c-2) = c-4 & c \geq 2 \\ h(0) = -\frac{4}{c} & 2 > c \geq 0 \end{cases},$$

or, in other words, $h(t) \leq -G(c)$. Since $x(t) < \xi(t)$, we have

$$\begin{aligned} x(t_2) = \int_{t_1}^{t_2} \dot{x}(s)\mathrm{d}s + x(t_1) &\leq \int_{t_1}^{t_2} -G(x(s))\mathrm{d}s + x(t_1) \\ &\leq \int_{t_1}^{t_2} -G(\xi(s))\mathrm{d}s + \xi(t_1) \leq 0, \end{aligned}$$

which is impossible since $x(t) > 0$, being captured. Hence there is no captured solution starting at a time $s < t_1$. □

Remark 122. Actually, this estimation can be finer, because the in the inequation of h of the proof, equality can not hold in any inteval. And it is enough to set $G(x) = 4 - x$ for the following proposition.

If Loewner process is generated by a curve, then we use this lemma and lemma 100, combine with that capture means for all initial values, the captured time are infinity, we get a sufficient condition to insure the curve is simple :

Proposition 123. *If Loewner process is generated by a curve , and for all time $s, T > 0$, after the local time transformation of λ_s at time T, the driving function the condition of satisfies lemma 121, then the curve is simple curve.*

By this proposition, we can proof half of theorem 83 easily :

Theorem 124. *If the Loewner chain (g_t) is generated by the curve γ, and the Hölder-$\frac{1}{2}$ norm of the driving function is less than 4, then γ is a simple curve.*

Proof. Assume that ξ is the driving function after time change at time T. Since $\xi(t) < c < 4$, for $t_1 > 0$ we can choose $t_2 = t_1 + c/(4-c)$, so that the condition of proposition 121 holds. This means there is no captured solution at time T since t_1 is arbitrary. Hence we get that ξ is not captured at any time, implying that γ is a simple curve. □

3.3.3 the exceptional point of brownian motion

In this subsection, we introduce the exceptional set of brownian motion, then we give a estimation of a exceptional set by lemma 119. The related results are refered to [25].

Since lemma 117 gives a condition on driving function, by theorem 87 the types of the SLE curves are clear, hence we can get the local behaviour of the brownian motion, we can discuss the $a(t)$ and $b(t)$ of brownian motion. We have :

Theorem 125. *For all t, a.s. we have*

$$\overline{\lim_{h\to 0^+}} \frac{B_{t+h}-B_t}{\sqrt{2h\log\log(1/h)}} = 1$$

and

$$\underline{\lim_{h\to 0^+}} \frac{B_{t+h}-B_t}{\sqrt{2h\log\log(1/h)}} = -1.$$

Then for all t, a.s. we have $a(t)=0, b(t)=+\infty$. And for a fixed path $B(t)$ of the brownian motion, the measure of set of one-side(left side or right side) local extremal point is 0. This set is a exceptional set, because for all t, it satisfies the condition of theorem 125 a.s. except for this set.

Another important conclusion of brownian motion is the Lévy's modulus of continuity theorem :

Theorem 126. *With probability* 1*,*

$$\max_{t\in[0,1]} \overline{\lim_{h\to 0^+}} \frac{|B_{t+h}-B_t|}{\sqrt{2h\log(1/h)}} = 1.$$

$a>0$, $t\in[0,1]$,we say t is a a-fast time of the brownian motion if and only if t satisfies

$$\overline{\lim_{h\to 0^+}} \frac{|B_{t+h}-B_t|}{\sqrt{2h\log(1/h)}} \geq a$$

From theorem 125 and 126, we know that as a random set, a-fast time set is a exceptional set, the measure is always 0. And one result of this set is :

Theorem 127. *For $a\in[0,1]$, the Hausdorff dimension of a-fast time set is $1-a^2$.*

An intuition is that there should be some connection between a-fast time set local extremal set. In theorem 87, we know that if $\kappa>4$, then SLE curves are not simple curves, that is $\sqrt{\kappa}B_t$ is a captured driving function. We define

$$I_\kappa \doteq \{t|\gamma_\kappa(t)\in\mathbb{R} \text{ or } \exists s<t,\ \text{s.t. } \gamma_\kappa(s)=\gamma_\kappa(t)\}.$$

It is easy to see that $I_{\kappa_1}\subset I_{\kappa_2}$ if $\kappa_1<\kappa_2$(a rigorous proof is similar as lemma 137). Since the captured time must be a left local extreme point of the Brownian motion, the Lebesgue measure of I_κ is 0, which means that I_κ is an exceptional set for Brownian motion. By theorem 126, for all $T\in I_\kappa$, we have :

$$\overline{\lim_{t\to T^-}} \frac{|\sqrt{\kappa}B_T-\sqrt{\kappa}B_t|}{\sqrt{(T-t)\log(1/(T-t))}} \leq \sqrt{2\kappa} \tag{3.3.6}$$

Using lemma 119, we induce the following proposition about the local behaviour of the points in I_κ, which is theorem 3 in the introduction :

Proposition 128. *If $\kappa > 4$, then almost surely, for all $T \in I_\kappa$, we have :*

$$\varliminf_{t \to T^-} \frac{|B_T - B_t|}{\sqrt{T-t}} \sqrt{\log \frac{1}{T-t}} \geq 2\frac{\sqrt{2}}{\kappa}.$$

Proof. If this is not true, in lemma 119, we set $\lambda = \sqrt{\kappa}B$ and $h(T-t) = \log(1/(T-t))$. Then we get the inequality which is contradiction to 3.3.6. □

I_κ is similar to the fast time set, but if we want study the connection, maybe we need to estimate the dimension of I_κ, up to now, there is are results about $\gamma_\kappa(I_\kappa)$, where γ_κ is the SLE curves. But we still don't know the dimension of I_κ.

3.3.4 Weierstrass function as driving function

In this subsection, we consider the chordal Loewner equation with driving function cW_b, as we mentioned in the introduction, c is a constant, and W_b is the Weierstrass function with $a = 1/\sqrt{2}$:

$$W_b(t) = \sum_{n=1}^{\infty} \frac{\cos(b^n t)}{\sqrt{b^n}} \tag{3.3.7}$$

The local behavior of the Weierstrass function has been studied since long ago, see [9]. This function shares some properties with Brownian motion. Such as self-similar and the dimension, refer to [35],[4]. The Hölder-$\frac{1}{2}$ norm is finite when $b > 1$:

$$\|W_b\|_{1/2} \leq \frac{b}{\sqrt{b}-1} + \frac{2}{1-\frac{1}{\sqrt{b}}} = C(b) \sim \sqrt{b}, b \to +\infty \tag{3.3.8}$$

Combine with theorem 83 (see [11]) :

Theorem 129. *For $c < 4/C(b) \sim 4/\sqrt{b}$, the Loewner process driven by cW_b is generated by a K quasislit, in particular a simple curve.*

Since theorem 2 partly improves the result of [21], we may apply it to improve this result too. That is theorem 4 in the introduction, we change another way to express :

Theorem 130. *$\forall l_0 > 1, \exists C > 0$ s.t. if $c < C$, then the Loewner equation with driving function cW_b is generated by a quasislit curve when $b > l_0$.*

If b is small, the proof of theorem 130 follows easily from the methods of [23] and [21], so we may assume b is large. Let W_b^N be the N^{th} partial sum of the right side of (3.3.7). It is obvious that $\{W_b^N\}$ converge to W_b uniformly on the real line when N goes to infinity, and the estimate of the Hölder -1/2 norm above applies to W_b^N as well.

We first prove a lemma showing that cW_b^N satisfies the hypothesis of theorem 2 for large N.

Lemma 131. *For all N and T we have*

$$\varlimsup_{t\to T^-}\frac{|W_b^N(T)-W_b^N(t)|}{\sqrt{T-t}}<(\sqrt{\pi}+\frac{1}{\sqrt{\pi}})\frac{\sqrt{2}}{\sqrt{b}-1}\sim\frac{c_1}{\sqrt{b}}$$

Proof. Set $t_m=2\pi/b^{m-1}$,$m\in\mathbb{N}$, when $N>m-1$, we have

$$\begin{aligned}
\frac{|W_b^N(T)-W_b^N(T-t_m)|}{\sqrt{t_m}} &= \left|2\sum_{n=0}^{N}\frac{1}{\sqrt{b^nt_m}}\sin(b^nT-\frac{b^nt_m}{2})\sin(\frac{b^nt_m}{2})\right|\\
=&2\left|\sum_{n=0}^{m-2}\frac{\sin(\frac{b^nt_m}{2})}{\sqrt{b^nt_m}}\sin(b^nT+\frac{b^nt_m}{2})+\sum_{n=m}^{N}\frac{\sin(\frac{b^nt_m}{2})}{\sqrt{b^nt_m}}\sin(b^nT+\frac{b^nt_m}{2})\right|\\
<&\sum_{n=0}^{m-2}\frac{b^nt_m}{\sqrt{b^nt_m}}+2\sum_{n=m}^{N}\frac{1}{\sqrt{b^nt_N}}=\frac{\sqrt{2\pi}}{\sqrt{b}}\frac{1-\frac{1}{\sqrt{b^m}}}{1-\frac{1}{\sqrt{b}}}+\frac{\sqrt{2}}{\sqrt{\pi b}}\frac{1-\frac{1}{\sqrt{b^{N-m}}}}{1-\frac{1}{\sqrt{b}}}\\
<&(\sqrt{\pi}+\frac{1}{\sqrt{\pi}})\frac{\sqrt{2}}{\sqrt{b}-1}
\end{aligned}$$

when $N\le m$, similar estimate shows that

$$\frac{|W_b^N(T)-W_b^N(T-t_m)|}{\sqrt{t_m}}<\frac{\sqrt{2\pi}}{\sqrt{b}-1}<(\sqrt{\pi}+\frac{1}{\sqrt{\pi}})\frac{\sqrt{2}}{\sqrt{b}-1}$$

which finishes the proof. □

Using the above lemma , we compute $a(t)$ and $b(t)$ in theorem 2. For the driving function cW_b^N,

$$a(t)<c(\sqrt{\pi}+\frac{1}{\sqrt{\pi}})\frac{\sqrt{2}}{\sqrt{b}-1},b(t)<c\frac{b}{\sqrt{b}-1}+\frac{2}{1-\frac{1}{\sqrt{b}}}$$

when b is sufficient large(actually, we only need $b>9$), it is easy to see $a(t)<2$, now if we have

$$c<\sqrt{8\pi}/(\pi+1)\Rightarrow\forall t,b(t)<f(a(t)),$$

then $a(t)$ and $b(t)$ satisfies the hypothesis of theorem 2. Actually we have already proved a half of theorem 131,that is if cW_b satisfies the hypothesis of theorem 131,and cW_b is generated by a curve, then the curve is simple.

In order to prove the complete theorem 131, we need some results. In the proof of theorem 83, there are three steps : Firstly, the K quasislit in the upper half plane and its conformal welding satisfy the following lemma(see [21] and [23] for the proof) :

Lemma 132. *if a Loewner equation is generated by a curve γ, ϕ is the conformal welding corresponds to$\gamma([0,T])$, then γ is a quasislit if and only if $\exists M>0$ only depend on K s.t.*

$$\frac{1}{M}<\frac{x-\lambda(T)}{\lambda(T)-\phi(x)}<M$$

for all x and

$$\frac{1}{M} < \frac{\phi(x) - \phi(y)}{\phi(x) - \phi(z)} < M$$

for all $\lambda(T) \leq x < y < z$ with $z - y = y - x$.

Secondly, we choose some driving functions λ_n which are uniformly converge to λ on $[0, T]$, and use lemma 132 prove that there exists K s.t. for all n, λ_n is generated by a K quasislit.

At last, using proposition 69 and theorem 95, we get that λ is also generated by a K quasislit .

In our theorem, for driving function W_b, we only need to finish the second step. It is natural that we can choose W_b^N, then W_b^N are converge uniformly to W_b. So what we only need to do is for all N, find a constant M s.t. the conformal welding of W_b^N satisfy the two conditions of lemma 132.

Proof. We prove the first inequality, it need the following lemma, in this lemma and its proof, we use the same symbol of lemma 117.

Lemma 133. *If c and b satisfy the hypothesis of theorem 130, then for all N, there exists a constant $C_2 > 0$ s.t. for all $t < T$ and $x(0) > W_b^N(t)$, the solution x of the real Loewner equation which is driven by cW_b^N with initial value $x(t) = x_0$ satisfies*

$$x(T) - W_b^N(t) > C_2\sqrt{T - t}.$$

Proof. We claim that $x(t+(T-t)/b) > W_b^N(T)$ on the whole line. If $x(t) > W_b^N(T)$, then because x is increasing and exists in all $\mathbb{R}$, the claim is true. If $x(t) \leq W_b^N(T)$ we can use the transformation and time change. It is easy to check that W_b^N satisfy the condition of 2.The claim then follows from the proof of theorem 2 and lemma 131.

Since $(\|W_b^N\|_{1/2})_{N>0}$ are uniformly bounded by $C(b)$, we have

$$W_b^N(T) - W_b^N(s) < C(b)\sqrt{T - s}, \forall s \in [t + (T - t)/b, T].$$

Let $l = x(t + (T - t)/b) - W_b^N(T)$: considering the solution $\hat{x}_l$ of (3.1.1) driven by $-C(b)\sqrt{T - s}$ with initial value $\hat{x}_l(t + (T - t)/b) = l$, we have $x(T) - W_b^N(t) > C\sqrt{T - t} > \hat{x}_l(T) \geq \min_{l>0}\{\hat{x}_l(T)\}$.

So we only need to prove that the last term is larger than $C(T - t)$. There are two methods, the first one is using the solution of remark 91. The second one is using the self-similar property of this special solution. We omit the details. □

Now we consider the conformal welding, by lemma 105, we have :

$$|I_T^+| < (C(b) + 2)\sqrt{T - t}$$

the estimation above is also valid to I_t^- obviously, hence we have :

$$\frac{x-\lambda(T)}{\lambda(T)-\phi(x)} < (C(b)+2)\sqrt{T-t}\Big/\left(\frac{4}{\sqrt{4+C(b)^2}+C(b)}\sqrt{\frac{T-t}{C}}\right) = M_1$$

this finishes the proof of the first inequality of lemma 132.

Now we prove the second one. We choose two times, $t_1 < t_2 < T$. Then by the first inequality, we have

$$\frac{1}{M_1} < \frac{|I_{t_1}^+|}{|I_{t_1}^-|} < M_1$$

Now we consider the chordal Loewner equation with the driving function $\lambda_{t_1}(t) = cW_b(t_1+t)$, set $g_t^{t_1}$ to be the solution, then we claim that for all $t < T$, there exists a constant M_2 only depend on M_1 s.t.

$$\frac{1}{M_2} < \frac{|g_t^{t_1}(I_{t_1}^+)|}{|g_t^{t_1}(I_{t_1}^-)|} < M_2$$

We take the two tips of $I_{t_1}^+$, and write the solutions $X_1 > X_2$of real Loewner equation driven by λ_{t_1} and the initial values are the two tips(one of the tips is λ_{t_1}, so when we solve the Loewner equation, let the initial value be $\lambda_{t_1}^+$). Then we subtract the two solutions, we have

$$\frac{\mathrm{d}|g_t^{t_1}(I_{t_1}^+)|}{\mathrm{d}t} = -\frac{2|g_t^{t_1}(I_{t_1}^+)|}{\theta(t)(\theta(t)+|g_t^{t_1}(I_{t_1}^+)|)}$$

where $\theta(t) = X_2(t) - \lambda_{t_1}$. By the proof of the first inequaluty, we have already proved :

$$C_2 = \frac{4}{\sqrt{4+C(b)^2}+C(b)}\sqrt{\frac{1}{C}} < \frac{\theta(t)}{\sqrt{t}} < (C(b)+2) = C_1$$

this conclusion is also valid for $I_{t_1}^-$.

Hence for proving the claim, we only need to prove that in equation :

$$\frac{\dot{w}}{w} = -\frac{2}{\theta(\theta+w)}$$

two solutions w_1,w_2, with driving functions $\theta_1(t) = C_1\sqrt{t}, \theta_2(t) = C_2\sqrt{t}$ respectively, and their initial values $w_1(0), w_2(0)$ are positive and satisfy $0 < w_1(0)/w_2(0) < M_1$, then there exists M_2, s.t. for all $t < T$, we have $w_1(t)/w_2(t) < M_2$.

We have two methods to show this, the first is using the local time transformation on the right side to this equation, after the transformation, the equation will become a autonomous equation, hence we can solve the solution directly. The second one is a very rough estimation. Notice that the left side of the equation equals the derivative of $\ln w$, hence we integrate the equation from 0 to t, since $t < T$, the right side of equation has a lower bound depend only on T, C_2, so $w_2(t)/w_2(0)$ also has a lower bound. Combine with that w_1 is also decreasing, we can get M_2 depends on T, C_1, C_2, M_1. This proved the claim.

Now we go back to the conformal welding. For x, y, z and $\phi(x), \phi(y), \phi(z)$ in lemma 132. We solve the real Loewner equation with initial value condition $X(T) = x, y, z, \phi(x), \phi(y), \phi(z)$. By the definition of conformal welding, the solutions correspond to x and $\phi(x)$ are exist in a same interval, we assume this interval is $[0, T]$ wlog. And we assume the solutions of y and z exist in $[t_1, T]$ and $[t_2, T]$ respectively, then $0 < t_1 < t_2$. And we have the following equalities

$$y - x = |g_t^{t_1}(I_{t_1}^+)|, \phi(x) - \phi(y) = |g_t^{t_1}(I_{t_1}^-)|$$

$z - y$ and $\phi(y) - \phi(z)$ also has the similar equalities. By the claim, we have

$$\frac{1}{M_2} < \frac{y - x}{\phi(x) - \phi(y)} < M_2, \frac{1}{M_2} < \frac{z - t}{\phi(t) - \phi(z)} < M_2,$$

Combine with $y - x = z - y$, we obtain the second inequality of lemma 132, and let M equals the square of M_2 is enough. □

Remark 134. Theorem 130 proves that the Loewner process is generated by a quasislit, the coefficient K is depend on not only b, but also time T.

Chapitre 4

Imaginary Loewner Equation

In this chapter, we disscuss the imaginary Loewner equation and it's dual equation, include the definition, basic properties, and the local transformation. Then we discuss the relation between these two equations and the curve generation problem, include theorem 5 and other results.

4.1 Imaginary Loewner equation

4.1.1 the definition of imaginary Loewner equation

We fist recall the general chordal Loewner equation, as we mentioned before, if we set $g_t = X(t) = Y(t)\mathrm{i}$, then the Loewner equation can be written as (2.3.3).Setting $\theta(t) = X(t) - \lambda(t)$, the second equation of (2.3.3) becomes

$$\dot{Y}(t) = -\frac{2Y(t)}{\theta(t)^2 + Y(t)^2}, Y(0) = Y_0 > 0 \tag{4.1.1}$$

This equation is called imaginary Loewner equation. In the rest of this section, we consider a continuous function $\theta : [0, +\infty) \to \mathbb{R}_+$, and we say θ is the driving function of (4.1.1).

It is easy to see that for positive initial value, the solution Y is always decreasing, until it equals 0. And by the Cauchy theorem of ODE, the solutions exist uniquely and locally before they decreasing to 0. And the solution will be 0 forever after it decreasing to 0. And by the symmetry of equation, we only need to consider the positive initial value. And we can also assume the driving function θ is non-negative.

Then like the capture problem of the real Loewner equation, we study whether there is a solution decreasing to 0, we first give a definition :

Definition 135. *If there exists an initial value $y_0 > 0$ and $T > 0$ such that $Y(T) = 0$ while $Y(t) > 0$ if $t < T$,where Y is the solution of the equation with initial value y_0, we say that θ is a vanishing driving function at T and that Y a vanishing solution at T. Conversely, for all $T > 0$ if there is no solution satisfying the condition above,then we say the equation or driving function are not vanishing at time T.*

This problem will be shown below to be connected to the first problem of the introduction. Like in the real Loewner equation, the vanishing property undergoes a phase-transition :

Theorem 136. *If $\theta(t) \geq 2\sqrt{T-t}$, $t \in [0,T]$, then θ is not time vanishing at time . If there exists a constant $C < 2$ s.t. $\theta(t) < C\sqrt{T-t}$, $t \in [0,T]$, then θ is vanishing at time T*

The proof of this theorem requires two lemmas, the first is similar to lemma 113 :

Lemma 137. *Let η_1 and η_2 be two driving functions for the equation (4.1.3). If $\forall t \geq 0, \eta_1(t) \geq \eta_2(t)$ and if η_1 is vanishing at time T, then η_2 is also vanishing at time T.*

Proof. We assume that Y_1 is a vanishing solution at time T driven by η_1 with initial value $Y_1(0) = y_1$. If the solution with initial value y_1 and driving function θ_2 is also vanishing at time T, then we are done.

Otherwise, we have at least one solution driven by θ_2 which is vanishing at a time $< T$. We consider the set A of initial values which make the solutions driven by η_2 vanish at a time $\leq T$. Let a be the least upper-bound of A, Y_2 the solution with driving function η_2 and initial value a, and T' be the vanishing time of Y_2, if $T' = T$ we are done.

Assume now $T' < T$. We consider the solution Y_2' which is driven by η_2 with initial condition $Y_2'(T') = Y_1(T')$. Since $Y_2'(T') = Y_1(T') > 0 = Y_2(T')$, it is easy to see that $Y_2'(0) > Y_2(0) = a$, hence $Y_2'(0) \notin A$. On the other hand, since $Y_2'(T') = Y_1(T')$, we have $Y_2'(t) \leq Y_1(t), \forall t \geq T'$, and $Y_2'(t)$ will vanish at T or before T, so $Y_2'(0) \in A$, contrarily to the assumption. □

Lemma 138. *Let c be a nonnegative number, $\varepsilon, T > 0$ and let y_ε be the solution of following equation :*

$$\dot{y} = \frac{2y}{y^2 + ct}, \quad y(0) = \varepsilon. \tag{4.1.2}$$

Then we have

$$\lim_{\varepsilon \to 0} y_\varepsilon(T) = \begin{cases} \sqrt{T(4-c)} & c < 4 \\ 0 & c \geq 4. \end{cases}$$

Proof. We observe that (4.1.2) becomes linear if we consider y as the variable and t as the function :

$$\frac{\mathrm{d}t}{\mathrm{d}y} = \frac{c}{2y}t + \frac{y}{2}$$

If $c \neq 4$, then we have $t = \frac{1}{4-c}y^2 - \frac{1}{4-c}\varepsilon^{2-\frac{c}{2}}y^{\frac{c}{2}}$. Putting $t = T$, we have : $\varepsilon^{\frac{c}{2}-2} = \frac{y(T)^{\frac{c}{2}}}{(c-4)T + y(T)^2}$, and it is easy to check that $\lim_{\varepsilon \to 0} y_\varepsilon(T) = 0$ for $c > 4$.

When $c < 4$, $\lim_{\varepsilon\to 0} y_\varepsilon(T) = \sqrt{T(4-c)}$ because the left-hand side of the equality goes to $+\infty$.

If $c = 4$, the solution is $t = \frac{1}{2}y^2 \ln y - \frac{1}{2}y^2 \ln \varepsilon$. Let $t = T$ as before : $\ln \varepsilon = \ln y(T) - \frac{2T}{y(T)^2}$. Since the left side tend to $-\infty$, then $\lim_{\varepsilon\to 0} y_\varepsilon(T) = 0$. This finishes the proof. □

Combine the twos lemmas above, we can proof theorem 136 :

Proof of theorem 136. By lemma 137, we only need to prove that the driving function $t \mapsto 2\sqrt{T-t}$ is not vanishing at time T while $t \mapsto c\sqrt{T-t}$ is if $c < 2$. From lemma 138, we know that if $c = 2$, then $\varepsilon \to y_\varepsilon(T)$ is a self homeomorphism of $(0, +\infty)$. Conversely, when the driving function is $t \mapsto 2\sqrt{T-t}$, any initial value will lead to a solution which is not 0 at time T.

When $c < 2$, we let the initial value equals $\sqrt{T(4-c^2)}$, then the solution $Y_2(t)$ will be smaller than all the solutions $t \mapsto y_\varepsilon(T-t)$, hence we have $Y_2(T) < y_\varepsilon(0) = \varepsilon$ for arbitrary positive ε, implying that Y_2 is vanishing at T, and the proof is finished. □

4.1.2 local transformation of imaginary Loewner equation

For a fixed time T,we can use local transformation to study whether the equation(4.1.1)is vanishing at time T. Like the real Loewner equation, we may assume $T = 1$ wlog, then we set

$$Y_-(t) = Y(t)/\sqrt{1-t}, \theta_-(t) = \theta(t)/\sqrt{1-t}$$

The equation becomes

$$\dot{Y}_-(t) = \frac{1}{2(1-t)}\left(Y_-(t) - \frac{4Y_-(t)}{\theta_-(t)^2 + Y_-(t)^2}\right)$$

Using the same time change $\sigma(t) = 1 - e^{-2t}$, setting $y(t) = Y_-(\sigma(t))$ and $\eta(t) = \theta_-(\sigma(t))$, we have

$$\dot{y}(t) = y(t) - \frac{4y(t)}{\eta(t)^2 + y(t)^2}, y(0) = y_0 > 0 \tag{4.1.3}$$

the equation can be written as

$$\frac{\mathrm{d}\ln y}{\mathrm{d}t} = \frac{\dot{y}}{y} = 1 - \frac{4}{\eta^2 + y^2}, y(0) = y_0 > 0. \tag{4.1.4}$$

In this equation, η is called driving function, in rest of this thesis, we only consider the case that the driving function is positive continuous function on $[0, +\infty)$.

The vanishing solutions satisfy the following property :

Proposition 139. *Under the local transformation, Y is a vanishing solution at time 1 if and only if y is a positive continuous function on $[0, +\infty)$, with $\lim_{t \to +\infty} e^{-t} y(t) = 0$.*

Proof. If Y is a vanishing solution at time 1, then we have $Y(1) = 0, \forall t < 1, Y(t) > 0$. Put them in to the local transformation, we get the condition in the proposition. □

The condition on y that implies the vanishing property may be considerably relaxed :

Lemma 140. *The solution y is vanishing if and only if $0 < y(t) < 2, \forall t > 0$.*

Proof. If $y(t) < 2$, then it is obvious that $\lim_{t \to +\infty} e^{-t} y(t) = 0$. Conversely, assume that there exists t_0 s.t. $y(t_0) \geq 2$. From the equation (4.1.4), $\dot{y}(t_0) > 0$. Hence after a small time, $y(t)$ is larger than 2. If $y(T) > 2$, we have

$$\frac{\mathrm{d}(y(s))}{\mathrm{d}s} > y(s) - \frac{4}{y(s)}, \forall s > T$$

Solving this differential inequation we get

$$y(s) > \sqrt{ce^{2t} + 4} > \sqrt{c}e^t,$$

which means that y is not a vanishing solution. □

Remark 141. It is easy to see that the condition of the lemma is also equivalent to y is bounded.

Keeps the vanishing property consistent before and after the transformation, we can define the vanishing property of the equation after the local transformation as follow :

Definition 142. *If there exists a solution y of (4.1.3) s.t. $t \in [0, +\infty), 0 < y(t) < 2$, then we say that the driving function η, the equation (4.1.3) and the solution are vanishing. Otherwise, we say that the equation, the solution and the driving function are not vanishing.*

Using the form of the equation (4.1.4) we can provide an alternative proof of theorem 136. After the local transformation, a equivalent expression of theorem 136 is

Theorem 143. *If $\eta(t) \geq 2, \forall t \in [0, +\infty)$, then η is not vanishing. If there exists a positive constant $C < 2$ s.t. $\eta(t) < C < 2, \forall t \in [0, +\infty)$, then η is vanishing.*

Proof. If $\eta(t) \geq 2, t \in [0, +\infty)$, the right side of (4.1.4) is positive, which means that y is a increasing function. For all positive initial value ε, we have

$$\frac{\mathrm{d}(\ln y(t))}{\mathrm{d}t} \geq \frac{\varepsilon}{4 + \varepsilon} = C_\varepsilon$$

so that

$$y(\frac{1}{C_\varepsilon}\ln\frac{2}{\varepsilon}) \geq y(0)\exp(\frac{1}{C_\varepsilon}\ln\frac{2}{\varepsilon}\cdot C_\varepsilon) = 2.$$

Hence there will be no vanishing solution, and this gives the proof of the first part.

If $\eta(t) < c < 2, t \in [0, +\infty)$, we choose $y(0) = 2 - c$. Then y will decrease on $[0, +\infty)$, so $y(t) < 2$ holds. Applying lemma 137 and 140 finishes the second part. □

Remark 144. $\eta(t) < 2, \forall t \in [0, +\infty)$ does not imply that *eta* is vanishing, we will give a counterexample later.

4.1.3 vanishing property of the driving function

In this subsection, we study the vanishing property of the driving function .Since 2 is the transition that determines whether the driving function disappears. We define the "lower bound function" as

$$L(t) = \int_0^t (1 - \frac{4}{\eta(s)^2})\mathrm{d}s. \tag{4.1.5}$$

This function plays a important role in the rest of this paper, because for any solution y of equation (4.1.4) and time interval (a, b), we have

$$\int_a^b \frac{\dot{y}(t)}{y(t)}\mathrm{d}t = \int_a^b (1 - \frac{4}{\eta(t)^2 + y(t)^2})\mathrm{d}t > L(b) - L(a).$$

hence we call the function the "lower bound function".The next lemma gives a necessary condition on the lower bound function to make a driving function η vanish.

Lemma 145. *If η is a vanishing driving function, then* $\lim_{t\to+\infty} L(t) = -\infty$.

Proof. We first prove that $\lim_{t\to+\infty} L(t)$ exists. If $L(t)$ does not converge, then we can find two constants $C_1 > C_2$ s.t. $L(t)$ takes these two value infinitely many times as $t \to +\infty$. Hence there exists an increasing sequence $\{a_n\}$ such that $L(a_n) = C_1$ and such that between time a_n and a_{n+1}, there is a time b_n s.t. $L(b_n) = C_2$. And without losing generality, we may assume that b_n is the first time when $L(t)$ equal C_2 after a_n.

From the definition of $L(t)$, we see that

$$\int_{a_n}^{b_n} (1 - \frac{4}{\eta(t)^2})\mathrm{d}t = L(b_n) - L(a_n) = C_2 - C_1$$

Notice that $1-\dfrac{4}{\eta(t)^2}>0$ when $\eta(t)>2$, so we can split this integral in two parts and drop the second one

$$\begin{aligned} C_2-C_1 &= (\int_{(a_n,b_n)\cap\{\eta\le 2\}}+\int_{(a_n,b_n)\cap\{\eta>2\}})(\left(1-\frac{4}{\eta(t)^2}\right))\mathrm{d}t \\ &\geq \int_{(a_n,b_n)\cap\{\eta\le 2\}}(1-\frac{4}{\eta(t)^2})\mathrm{d}t > -\int_{(a_n,b_n)\cap\{\eta\le 2\}}\frac{4}{\eta(t)^2}\mathrm{d}t \end{aligned}$$

Integrating equation (4.1.3) from a_n to b_n, we get

$$\begin{aligned} \ln\left(\frac{y(b_n)}{y(a_n)}\right) &= \int_{a_n}^{b_n}\mathrm{d}(\ln(y(t))) = \int_{a_n}^{b_n}(1-\frac{4}{\eta(t)^2+y(t)^2})\mathrm{d}t \\ &= \int_{a_n}^{b_n}\left[1-\frac{4}{\eta(t)^2}+\frac{4}{\eta(t)^2}(1-\frac{1}{1+\frac{y(t)^2}{\eta(t)^2}})\right]\mathrm{d}t \end{aligned}$$

Assuming $y(0)=\varepsilon$, since L is the lower bound function, we see that for $t\in[a_n,b_n]$, we have

$$\ln\left(\frac{y(t)}{y(0)}\right) = \int_0^t 1-\frac{4}{\eta(t)^2+y(t)^2}\mathrm{d}t \geq L(t) \geq L(b_n) = C_2,$$

which gives for $y(t)$ the lower bound $\varepsilon e^{C_2}=C_3$. We can then estimate the variation of $y(t)$ in $[a_n,b_n]$ as follows :

$$\begin{aligned} \ln\left(\frac{y(b_n)}{y(a_n)}\right) &> C_2-C_1+\int_{(a_n,b_n)\cap\{\eta\le 2\}}\frac{4}{\eta(t)^2}(1-\frac{1}{1+\frac{y(t)^2}{\eta(t)^2}})\mathrm{d}t \\ &\geq C_2-C_1+\int_{(b_n,a_n)\cap\{\eta\le 2\}}\frac{4}{\eta(t)^2}(1-\frac{1}{1+\frac{C_3^2}{2^2}})\mathrm{d}t \\ &\geq C_2-C_1+C_4(C_1-C_2) \end{aligned}$$

where $0<C_4=1-\dfrac{1}{1+\frac{C_3^2}{4}}<1$ depends only on C_1,C_2 and ε. But

$$\ln\left(\frac{y(a_{n+1})}{y(a_n)}\right) = \ln\left(\frac{y(a_{n+1})}{y(b_n)}\right)+\ln\left(\frac{y(b_n)}{y(a_n)}\right)$$

the first term being greater than $L(a_{n+1})-L(b_n)=C_1-C_2$, we get that

$$y(a_{n+1}) > y(a_n)e^{C_4(C_1-C_2)} = C_5 y(a_n).$$

Since $C_5>1$, for arbitrary ε, when n is sufficiently large, $y(a_n)>2$. That means that $\eta(t)$ is not a vanishing driving function. This contradiction leads to the conclusion that the limit of $L(t)$ must exist.

We now prove that $\lim_{t\to+\infty} L(t) = -\infty$. Otherwise this limit is a constant C_6, and $\exists C_7$ s.t. $L(t) > C_7, \forall t > 0$. Just like before, assuming $y(0) = \varepsilon$, we see that $y(t) > y(0)e^{L(t)} > \varepsilon e^{C_7} = C_8$. Recalling that

$$\ln\left(\frac{y(t)}{y(0)}\right) = L(t) + \int_0^t \frac{4}{\eta(s)^2}(1 - \frac{1}{1+\frac{y(s)^2}{\eta(s)^2}})\mathrm{d}s,$$

we may write

$$\begin{aligned}\ln\left(\frac{y(t)}{y(0)}\right) &\geq L(t) + \int_{(0,t)\cap\{\eta<3\}} \frac{4}{\eta(s)^2}(1 - \frac{1}{1+\frac{y(s)^2}{\eta(s)^2}})\mathrm{d}s \\ &\geq L(t) + \int_{(0,t)\cap\{\eta<3\}} \frac{4}{3^2}(1 - \frac{1}{1+\frac{C_8^2}{3^2}})\mathrm{d}s \\ &= L(t) + C_9|(0,t)\cap\{\eta<3\}|\end{aligned}$$

where $|A|$ denotes the Lebesgue measure of the Borel set A. If $|(0,+\infty)\cap\{\eta<3\}| = +\infty$, then $y(t) > 2$ for t large, which leads to a contradiction. Hence we may assume that $|(0,+\infty)\cap\{\eta<3\}| = C_{10} < +\infty$ and consequently that $|(0,+\infty)\cap\{\eta\geq 3\}| = \infty$. But from the definition of $L(t)$

$$L(t) = t - \left(\int_{(0,t)\cap\{\eta<3\}} + \int_{(0,t)\cap\{\eta\geq 3\}}\right)\frac{4}{\eta(s)^2}\mathrm{d}s$$

$$\int_{(0,t)\cap\{\eta<3\}} \frac{4}{\eta(s)^2}\mathrm{d}s > t - L(t) - \frac{4}{9}l((0,+t)\cap\{\eta\geq 3\}) > \frac{5}{9}t - L(t).$$

Going back to the inequality before, we have

$$\int_{(0,t)\cap\{\eta<3\}} \frac{4}{\eta(s)^2}\left(1 - \frac{1}{1+\frac{y(s)^2}{\eta(s)^2}}\right)\mathrm{d}s > \left(1 - \frac{1}{1+\frac{C_8^2}{9}}\right)\int_{(0,t)\cap\{\eta<3\}} \frac{4}{\eta(s)^2}\mathrm{d}s,$$

from which it follows that $y(t) \to +\infty$ since

$$\ln\left(\frac{y(t)}{y(0)}\right) \geq L(t) + C_{11}\frac{5}{9}t - L(t) = C_{12}t.$$

It follows that y is not a vanishing solution, contradicting the assumption : the lemma is proven. □

Remark 146. $\lim_{t\to+\infty} L(t) = -\infty$ is not a sufficient condition to make the driving function vanishing. We can construct a counterexample which is also the counterexample in 144.

Example 147. *We choose a driving function η is increasing and converges to 2, and makes $L(t)$ converges to $-\infty$. Then for any initial value y_0, we get he upper bound $y_0 e^{L(t)}$ of the corresponding solution $y(t)$. Notice that if $y(t)$ is a vanishing solution, then y is decreasing and satisfies $y(t) < \sqrt{4-\eta^2(t)}$. Hence we have*

$$y_0 e^{L(t)} < \sqrt{4-\eta^2(t)}$$

But we can we proper η, make the inequality does not hold.

Although the lower bound function of a vanishing driving function tends to $-\infty$, there still exist vanishing solutions that do not tends to 0, a simple example being $\eta(t) = y(t) = \sqrt{2}$. Hence we would like to investigate the behaviour of vanishing solutions when time goes to $+\infty$.

Lemma 148. *There is at most one vanishing solution satisfying $\lim_{t\to+\infty} y(t) \neq 0$.*

Proof. Assume that $y_1 > y_2$ are two vanishing solutions. Substracting the two equations (4.1.3) we have

$$\frac{\mathrm{d}(y_1(t)-y_2(t))}{\mathrm{d}t} = (y_1(t)-y_2(t)) - \frac{4(y_1(t)-y_2(t))(\eta(t)^2 - y_1(t)y_2(t))}{(\eta(t)^2+y_1(t)^2)(\eta(t)^2+y_2(t)^2)},$$

putting $u(t) = y_1(t) - y_2(t)$, we get

$$\frac{\dot{u}(t)}{u(t)} = 1 - \frac{1}{1+\frac{y_1(t)^2}{\eta(t)^2}} \frac{4}{\eta(t)^2+y_2(t)^2} + \frac{4y_1(t)y_2(t)}{(\eta(t)^2+y_1(t)^2)(\eta(t)^2+y_2(t)^2)}. \tag{4.1.6}$$

There are three cases :

(1) $\lim_{t\to+\infty} y_2(t)$ does not exist, (2) $\lim_{t\to+\infty} y_2(t) = c > 0$, (3) $\lim_{t\to+\infty} y_2(t) = 0$.

In the first case, we find two constant $C_1 > C_2$, and two sequences (a_n) and (b_n) converging to $+\infty$ s.t. $y_2(a_n) = C_1, y_2(b_n) = C_2, a_n < b_n < a_{n+1} < b_{n+1}$, and b_n is the first time that y_2 is equal to C_2 after a_n, $\forall n > 0$. Integrating the last term of (4.1.6) from a_n to b_n, we have

$$\begin{aligned}\int_{a_n}^{b_n} \frac{4y_1(t)y_2(t)\mathrm{d}t}{(\eta^2+y_1(t)^2)(\eta^2+y_2(t)^2)} &\geq \int_{(a_n,b_n)\cap\{\eta<2\}} \frac{4C_2^2\mathrm{d}t}{(2^2+2^2)(2^2+C_2^2)} \\ &= C_3|(a_n,b_n)\cap\{\eta<2\}|\end{aligned}$$

From equation (4.1.3) for y_2

$$\begin{aligned}\ln\frac{C_2}{C_1} &= \int_{a_n}^{b_n}\left(1-\frac{4}{\eta(t)^2+y(t)^2}\right)\mathrm{d}t > \int_{(a_n,b_n)\cap\{\eta<2\}}\left(1-\frac{4}{\eta(t)^2+y(t)^2}\right)\mathrm{d}t \\ &> \int_{(a_n,b_n)\cap\{\eta<2\}}\left(1-\frac{4}{C_2^2}\right)\mathrm{d}t = |(a_n,b_n)\cap\{\eta<2\}|(1-\frac{4}{C_2^2})\end{aligned}$$

Since $1 - 4/C_2^2$ is negative, we have

$$|(a_n, b_n) \cap \{\eta < 2\}| > \frac{C_2^2}{C_2^2 - 4} \ln \frac{C_2}{C_1} = C_4 > 0.$$

Going back to (4.1.6), integrating from a_1 to b_n,

$$\begin{aligned} \ln \frac{u(b_n)}{u(a_1)} &> \int_{a_1}^{b_n} 1 - \frac{4}{\eta(t)^2 + y_2(t)^2} \mathrm{d}t + \sum_{i=1}^{n} \int_{a_i}^{b_i} \frac{4y_1(t)y_2(t)\mathrm{d}t}{(2^2 + y_1(t)^2)(2^2 + y_2(t)^2)} \\ &> \ln \frac{y_2(b_n)}{y_2(a_1)} + nC_3 \frac{C_2^2}{C_2^2 - 4} \ln \frac{C_2}{C_1} = \ln \frac{C_2}{C_1} + nC_3C_4 \end{aligned}$$

It follows that $\lim_{n\to+\infty} y_1(b_n) = +\infty$, and we have a contradiction since y_1 is a vanishing solution.

If now $\lim\limits_{t\to+\infty} y_2(t) = c > 0$, we use the same method, and first prove that $|(0, +\infty) \cap \{\eta < 2\}| = +\infty$ as above. Integrating (4.1.3) with y_2,

$$\begin{aligned} \ln \frac{y_2(t)}{y_2(0)} &= t - \int_0^t \frac{4\mathrm{d}s}{\eta(s)^2 + y_2(s)^2} \\ &> t - \int_{(0,t)\cap\{\eta\geq 2\}} \frac{4\mathrm{d}s}{4 + y_2(s)^2} - \int_{(0,t)\cap\{\eta<2\}} \frac{4\mathrm{d}s}{y_2(s)^2} \end{aligned}$$

Let t tend to infinity : if $|(0, +\infty) \cap \{\eta < 2\}| < +\infty$, then the left side of the inequality is bounded while the right one goes to infinity as $\frac{c^2}{4+c^2}t$, thus leading to a contradiction. Now take A large enough so that $y(t) \geq c/2$, $t \geq A$. Then, as above,

$$\int_A^t \frac{4y_1(s)y_2(s)\mathrm{d}s}{(\eta^2(s) + y_1(s)^2)(\eta^2(s) + y_2(s)^2)} \geq \frac{c^2}{64}|[A, t] \cap \{\eta < 2\}|,$$

and

$$\ln \left(\frac{u(t)}{u(A)}\right) \geq \ln \left(\frac{y_2(t)}{y_2(A)}\right) + \frac{c^2}{64}|[A, t] \cap \{\eta < 2\}|,$$

which is impossible. The only left possibility is that $y_2(t) \to 0$ as $t \to +\infty$, and the claim is proven. □

Remark 149. In general, for a vanishing driving function, there exists a solution y satisfying $\lim\limits_{t\to+\infty} y(t) \neq 0$, but this is not always truth. For instance, if $\eta < 2$ but η converges to 2, all the vanishing solutions are converged to 0.

4.2 Dual imaginary Loewner equation

In this section, we discuss the properties of the dual imaginary Loewner equation.

4.2.1 local transformation

We induce the dual imaginary equation equation from the real Loewner equation 3.1.1. X is a solution of equation (3.1.1) ; If (3.1.1) is captured at time 1, then we choose a captured solution X_0 at time 1, set $W(t) = X(t) - X_0(t), \theta(t) = X(t) - \lambda(t)$. then the equation (3.1.1) becomes

$$\dot{W}(t) = -\frac{2W(t)}{\theta(t)(W(t) + \theta(t))}, W(0) = w_0$$

We can see that this equation looks similar to the imaginary Loewner equation, we call it the dual imaginary Loewner equation, and θ is the driving function. The driving function can be negative, this is different to the imaginary Loewner equation, but in this thesis, we still only consider the case that θ is a positive continuous function.

We can do the same local transformation at time 1,

$$W_-(t) = W(t)/\sqrt{1-t}, \theta(t) = \theta_1(t)/\sqrt{1-t}$$

With time change $\sigma(t) = 1 - e^{-2t}, w(t) = W_-(\sigma(t)), \eta(t) = \theta_-(\sigma(t))$, then w satisfies the following equation :

$$\dot{w}(t) = w(t) - \frac{4w(t)}{\eta^2(t) + \eta(t)w(t)}, w(0) = w_0 \tag{4.2.1}$$

like equation (4.1.4), this equation can be written as :

$$\frac{\mathrm{d}\ln w}{\mathrm{d}t} = \frac{\dot{w}}{w} = 1 - \frac{4}{\eta^2 + \eta w}, w(0) = w_0 > 0. \tag{4.2.2}$$

where the driving function η is a positive continuous function on $[0, +\infty)$. Although we don't consider the relationship between η and X and λ now, we still derive the equality of η and λ which is the driving function of the original Loewner equation.

Notice that $\theta = X - \lambda$, recall the local transformation of the Loewner equation, we have $\eta = \xi - x$, where ξ and x are the driving function and solution of equation (3.2.1). Hence we have

$$\dot{x} - x = -\frac{4}{\xi - x}$$

times e^{-x} on the two sides

$$\frac{\mathrm{d}(e^{-t}x(t))}{\mathrm{d}t} = -\frac{4e^{-t}}{\eta(t)}$$

integrate from t to $+\infty$

$$\lim_{s\to+\infty} e^{-s}x(s) - e^{-t}x(t) = -\int_t^{+\infty} \frac{4e^{-s}}{\eta(s)}\mathrm{d}s$$

by 3.2.2, we have

$$\lim_{t\to+\infty} e^{-t}x(t) < \lim_{t\to+\infty} e^{-t}\xi(t) = 0$$

put it in the equality before

$$x(t) = \int_0^{+\infty} \frac{4e^{-s}}{\eta(t+s)}\mathrm{d}s$$

put this to $\xi = \eta + x$, we have the equality between η and ξ :

$$\xi(t) = \eta(t) + \int_0^{+\infty} \frac{4e^{-s}}{\eta(t+s)} \tag{4.2.3}$$

For convenience, we define a operator H on $C_+[0, +\infty)$:

$$H(\eta)(t) = \eta(t) + \int_0^{+\infty} \frac{4e^{-s}}{\eta(t+s)}\mathrm{d}s.$$

then (4.2.3) can be simplified as $\xi = H(\eta)$.

It is easy to see, η and ξ satisfy :

Proposition 150. *η and ξ as above, then*

(1) If η is unbounded, then ξ is also unbounded;

(2) If η is bounded, and $\varliminf_{t\to+\infty} \eta(t) > 0$,then ξ is bounded.

Now we go back to equation (4.2.1). We can define the vanishing of this equation, we only discuss the equation after the local transformation.

Definition 151. *We say the driving function $\eta(t)$ and equation (4.2.1) are vanishing if there exists a solution $w(t)$ s.t. $\lim_{t\to+\infty} e^{-t}w(t) = 0$, and w vanishing solution.*

This is different with the imaginary Loewner equation, the equation is vanishing not equivalent to the equation has a bounded solution. But it is easy to prove the following property :

Proposition 152. *If there exits $C > 0$ s.t. the driving function satisfying $\eta > C$, then equation* (4.2.1) *is vanishing if and only if the equation has a bounded positive solution.*

Proof. If the equation has a bounded positive solution, then the equation is vanishing.Conversely, if all the solutions of equation (4.2.1) are bounded, set $c = (\sqrt{C^2 + 16} - C)/2$. Then c satisfies $C^2 + Cc = 4$. We take a solution w_1, assume $w_1(T) > c$ wlog, put this into (4.2.2), we have that w is increasing after time T. Solve equation

$$\frac{\mathrm{d}w}{\mathrm{d}t} = w - \frac{4w}{C^2 + Cw}, w(T) = w_1(T),$$

the solution is

$$C_1 \ln w + C_2 \ln(w + C - 4) = t + C_3,$$

where C_1, C_2, C_3 are constant, and $C_1 + C_2 = 1$, notice that w is increasing to $+\infty$, hence

$$\lim_{t\to+\infty} e^{-t} w(t) = \lim_{t\to+\infty} \left(\frac{w(t)}{w(t) + C - 4} \right)^{C^2} e^{C_3} = e^{C_3} > 0$$

compare with equations of w_1 and w, it is easy to see that $w_1 > w$ after time T, so we have $\lim_{t\to+\infty} e^{-t} w_1(t) > 0$. Since w_1 can be any solution of equation (4.2.1), the equation is not vanishing. □

The transition for equation (4.2.1) is the same as for (4.1.3). We can use the same method to prove it :

Theorem 153. *If $\eta(t) \geq 2, \forall t \geq 0$, then the equation (4.2.1) is not vanishing. If $\eta(t) < C < 2, \forall t \geq 0$, then the equation (4.2.1) is vanishing.*

Proof. Change $\sqrt{16 - C^2}$ of the proof of theorem 143 into $(\sqrt{C^2 + 16} - C)/2$, then we finish the proof.

The conclusion of remark 144 is also right here, and the counterexample is similar.

□

4.2.2 vanishing property of the dual equation

We continue to study the vanishing property of the dual equation. We can also define the "lower bound function" L here, but in dual equation, L does not work in many cases. Similar to lemma 148, most of the solutions of (4.2.1) converge to 0. But the statement is different since (4.2.1) may have unbounded vanishing solution, and the proof also needs to be modified.

Lemma 154. *Equation (4.2.1) has at most one vanishing solution w such that $\overline{\lim}_{t\to+\infty} w(t) > 0$.*

Proof. The proof is similar to that of lemma 148. Assume $w_1 > w_2$ are two solutions of (4.2.1), and that w_1 satisfies $\overline{\lim}_{t\to+\infty} w_1(t) = C > 0$: we want to prove that $\lim_{t\to+\infty} w_2(t) = 0$.
Subtracting these solutions as before, we get

$$\dot{w}_1(t) - \dot{w}_2(t) = w_1(t) - w_2(t) - \frac{4(w_1(t) - w_2(t))}{(\eta(t) + w_1(t))(\eta(t) + w_2(t))}.$$

Set $v(t) = w_1(t) - w_2(t)$: we have

$$\begin{aligned} \frac{\dot{v}(t)}{v(t)} &= 1 - \frac{4}{(\eta(t) + w_1(t))(\eta(t) + w_2(t))} \\ &= 1 - \frac{4}{\eta(t)(\eta(t) + w_1(t))} + \frac{4w_2(t)}{\eta(t)(\eta(t) + w_1(t))(\eta(t) + w_2(t))} \end{aligned}$$

Integrating this equality from 0 to t we get

$$\begin{aligned}\ln\frac{v(t)}{v(0)} &= \ln\frac{w_1(t)}{w_1(0)} + \int_0^t \frac{4\mathrm{d}s}{\eta(s)(\eta(s)+w_1(s))(1+\frac{\eta(s)}{w_2(s)})} \\ &= \ln\frac{w_1(t)}{w_1(0)} + \int_0^t M(s)\mathrm{d}s\end{aligned}$$

then we have

$$w_1(t) - w_2(t) = v(t) = w_1(t)\frac{v(0)}{w_1(0)}\exp\int_0^t M(s)\mathrm{d}s$$

which is equivalent to

$$\frac{w_2}{w_1} = 1 - \frac{v(0)}{w_1(0)}\exp\int_0^t M(s)\mathrm{d}s \tag{4.2.4}$$

It is easy to see w_2/w_1 is decreasing, we write the above equality to

$$w_2(t)/w_2(0) = c(t)w_1(t)/w_1(0)$$

where c is a decreasing function converging to a constant $C_1 < 1$. If $C_1 = 0$, then we finish this lemma. So we assume $0 < C_1 < 1$. Just like in lemma 148, there are two cases : the limit of w_2 as $t \to +\infty$ does not exist or it is a positive number.

In the first case, we choose two sequences $\{a_n\}$ and $\{b_n\}$ increasing to infinity as before : $w_2(a_n) = C_2, w_2(b_n) = C_3, C_2 > C_3 > 0, a_n < b_n < a_{n+1}$ and b_n is the first time after a_n such that $w(b_n) = C_3$. For sufficiently large n, we have

$$\frac{w_1(a_n)}{w_1(b_n)} = \frac{w_1(b_n)}{w_2(b_n)}\frac{c(b_n)}{c(a_n)} = \frac{C_2}{C_3}\frac{c(b_n)}{c(a_n)} > \sqrt{C_2/C_3} = C_4 > 1$$

so we have

$$\begin{aligned}\int_{a_n}^{b_n} M(t)\mathrm{d}t &> \int_{(a_n,b_n)\cap\{\eta<2\}} \frac{4\mathrm{d}t}{\eta(t)(\eta(t)+w_1(t))(1+\frac{2}{C_3})} \\ &> C_5\int_{(a_n,b_n)\cap\{\eta<2\}} \left(\frac{4}{\eta(t)(\eta(t)+w_1(t))} - 1\right)\mathrm{d}t \\ &> C_5\int_{a_n}^{b_n} \left(\frac{4}{\eta(t)(\eta(t)+w_1(t))} - 1\right)\mathrm{d}t \\ &= -C_5\ln\frac{w_1(b_n)}{w_1(a_n)} > -C_5\ln\frac{1}{C_4} = C_6 > 0\end{aligned}$$

implying that the integral of M is unbounded.

In the second case, since $c(t)$ converges, w_1 also converges to a constant C_7, and for all t, we have $w_2 > C_8 > 0$. Integrate equation 4.2.2 which corresponds to w_1

$$\begin{aligned}\ln\frac{w_1(t)}{w_1(0)} &= t - \int_0^t \frac{4\mathrm{d}s}{\eta(s)^2+\eta(s)w_1(s)}\\ &= t - \int_{(0,t)\cap\{\eta\geq 2\}} \frac{4\mathrm{d}s}{\eta(s)^2+\eta(s)w_1(s)} - \int_{(0,t)\cap\{\eta< 2\}} \frac{4\mathrm{d}s}{\eta(s)^2+\eta(s)w_1(s)}.\end{aligned}$$

Since when t goes to $+\infty$, the left side converges to the constant $\ln\dfrac{C_7}{w_1(0)}$, hence in the last two terms of the right side, there are at least one term converge to $-\infty$. If the third term converges to $-\infty$, we have

$$\begin{aligned}\int_0^t M(s)(d)\mathrm{d}s &> \int_{(0,t)\cap\{\eta<2\}} M(s)(d)\mathrm{d}s\\ &> \int_{(0,t)\cap\{\eta<2\}} \frac{4\mathrm{d}t}{\eta(t)(\eta(t)+w_1(t))(1+\frac{2}{C_8})} = +\infty\end{aligned}$$

put this into 4.2.4, we have $w_2 < 0$ which gives a contradiction. The rest case is that the second term converges to $-\infty$ and the third term converges to a constant. But notice the second term satisfies

$$\int_{(0,t)\cap\{\eta\geq 2\}} \frac{4\mathrm{d}s}{\eta(s)^2+\eta(s)w_1(s)} < \frac{4\mathrm{d}s}{4+2w_1(s)}$$

when t goes to $+\infty$, right side converges to $+\infty$ as $\dfrac{2}{2+C_7}t$, hence in the equality before, the right side converges to $+\infty$ as $\dfrac{C_7}{2+C_7}t$, we have that w_1 converges to $+\infty$, this contradicts to w_1 is vanishing. □

Remark 155. Equation (4.2.1) may have several vanishing solution which are unbounded. And it is easy to prove that if (4.2.1) has an unbounded solution then the driving function must satisfy $\underline{\lim}_{t\to+\infty}\eta(t) = 0$. This condition will play an important role later. Actually, this property tells us that, except maybe for the smallest one, if x_1 and x_2 are two captured solution which are driven by ξ, then $\lim_{t\to+\infty}|x_1(t)-x_2(t)| = 0$.

We could see that for fixed one driving function, imaginary Loewner equation and the dual equation are both vanishing or both not vanishing in most of the cases. A natural idea is finding the rest cases.

At first, we discuss the case that (4.1.3) is vanishing, but (4.2.1) is not vanishing. In this case, we can only give an example.

Example 156. *We define a piece-wise function*

$$\eta(t) = \sqrt{4 - \frac{1}{2^{n-1}}}, t \in I_n = [t_n, t_{n+1}], n = 1, 2, \ldots$$

where $t_n = 2^{n+2} \ln \sqrt{2} - 8 \ln \sqrt{2}$. *Then for this driving function* η, (4.2.1) *is vanishing, but* (4.1.3) *is not vanishing. Hence for the continuous functions which are sufficient close to* η, (4.2.1) *is also vanishing, but* (4.1.3) *is also not vanishing.*

Proof. We consider equation 4.1.3 at first. It is easy to check, if $y(t_n) \leq 2^{-\frac{n}{2}}$, then y is decreasing on I_n, and we have

$$\frac{\dot{y}(t)}{y(t)} = 1 - \frac{4}{\eta(t)^2 + y(t)^2} < -\frac{2^{-}}{4 - 2^{-n}}.$$

integrate in I_n,

$$\frac{\ln y(t_n)}{\ln y(t_n + 1)} < -\frac{2^{-n}}{4 - 2^{-n}}(t_{n+1} - t_n) < \ln \sqrt{2}$$

hence we have$y_{n+1} < y_n/\sqrt{2}$. By induction, the solution with initial value $y_0 < 1/\sqrt{2}$ is vanishing.

Then we consider equation 4.2.1. Since the driving function is increasing, hence if the solution y stop decreasing in sometime, then y will increasing to $+\infty$. For equation 4.2.1, if y is decreasing, then $y(t_n) < y_n = (4 - \eta^2)/\eta = 2^{-n+1}$ in I_n. Notice that $y_n/y_{n+1} = 2$. But by the lower bound function, we have that for any solution y_n, $y(t_n)/y(t_{n+1}) > \frac{1}{2}$. This implies that whatever the initial value is, the decreasing speed of $y(t_n)$ is slow than the decreasing speed of y_n,hence after a sufficient large time, we have $y(t_n) > y_n$, hence all the solutions are not vanishing. □

Now we see the second case, that is (4.2.1) is vanishing, but(4.1.3) is not vanishing. We have the following conclusion :

Proposition 157. *If a bounded driving function* η *is vanishing in equation (4.2.1) but is not in equation (4.1.3), then* $\lim_{t \to +\infty} \eta(t) = 0$.

Proof. We assume that η has a positive lower bound $c > 0$. From lemma 154, we can choose a vanishing solution of (4.2.1) $w(t)$ s.t. $w(t) < c \leq \eta(t)$. Then we consider the solution y of (4.1.3) with initial value $y(0) = w(0)$. We have

$$\frac{\mathrm{d}(\ln y)}{\mathrm{d}t} = 1 - \frac{4}{\eta(t)^2 + y(t)^2} < 1 - \frac{4}{\eta(t)^2 + cy(t)} \leq 1 - \frac{4}{\eta(t)^2 + \eta(t)y(t)}$$

hence y is smaller than w and y is a vanishing solution of (4.1.3), this finishes the proof. □

4.3 Curve generation problem

4.3.1 angular limit and imaginary Loewner equation

We consider the connection between curve generation problem and the imaginary Loewner equation in this section.By theorem 77 and its remark, we need to consider whether the angular limit $\lim_{y\to 0^+} g_t^{-1}(\lambda(t)+y\mathrm{i})$ exists. We assume $t=1$ wlog.

At first, we consider the easy case that is if λ is captured at time then we have :

Proposition 158. *If the driving function λ is captured at time 1, then the limit $\lim_{y\to 0^+} g_1^{-1}(\lambda(1)+y\mathrm{i})$ exists, and the belong to $\mathbb{R}$.*

Proof. We give a proof by the boundary behaviour of the conformal mapping. By the symmetry extension, g_t^{-1} can be extended to a conformal mapping from $\mathbb{C}_\infty \setminus I_1$ to $\mathbb{C}_\infty \setminus (K_1 \cup \hat{K_1})$ with $\lambda(1) \in I_1$. Since equation has a captured solution X, we assume $X(0) > \lambda(0)$ wlog, then we have 1 is a left extremal point of λ. Actually, $\lambda(1)$ is the right endpoint of I_1. We consider the solution X_ε of the real Loewner equation (3.1.1), the initial value condition is $X_\varepsilon(1) = \lambda(1)+\varepsilon$, where $\varepsilon > 0$. Then we have

$$\lim_{\varepsilon\to 0^+} g_1^{-1}(\lambda(1)+\varepsilon) = \lim_{\varepsilon\to 0^+} X_\varepsilon(0).$$

Since $X_\varepsilon(0)$ is decreasing by ε, the limits exists and equals to a real number. By remark 39, the angular limit exists. □

Conversely, if the limit $\lim_{y\to 0^+} g_1^{-1}(\lambda(1)+y\mathrm{i})$ exists, set it equals z_0. The we consider the solution of Loewner equation with initial value z_0, set the real and imaginary parts of the solution are X_t and Y_t respectively, then we have Y_t is a vanishing solution at time 1, and the corresponding driving function i $|X_t - \ lambda(t)|$. Hence if for all X_t, there is no vanishing solution, then we can obtain that the limit does not exists. For instance, we have the following simple proposition :

Proposition 159. *If the driving function λ is continuous at $[0,1]$. Then for $x_0 \neq \lambda(1)$, limit $\lim_{y\to 0^+} g_1^{-1}(x_0+y\mathrm{i})$ does not belong to J_1.*

Proof. If we have $z_0 \in J_1$, assume the real and imaginary parts of the solution of Loewner equation are X_t and Y_t respectively, then $|X_t - \lambda(t)|$ as a driving function of the imaginary Loewner equation, it is vanishing at time 1. But $X_1(1) = x_0 \neq \lambda(1)$, hence the driving function after the local transformation converges to infinity, such driving function is not vanishing apparently. □

By this proposition, we have the following conclusion which is a part of proposition 80 :

Proposition 160. *If the driving function λ is continuous in $[0,1]$, then there is at most one point of J_1 is 1-accessible.*

Proof. By the property above, since λ is above, then only at $\lambda(t)$, the limit of g_1^{-1} belongs to J_1. But by remark 39 and theorem 38, if limit by some curves at $\lambda(1)$ exist, then the principal points set has only one element. □

Hence we have, if J_1 has at least two points are 1-accessible, then the driving function is not continuous at time 1.

Consider such λ, it is continuous at $[0, 1)$, but not continuous at 1. We assume the limitsup and limit inf are $a, -a$. As the first subfigure of 4.3.1, g_1 maps the blue line to $[-a, a]$. Then fro all $x_0 \in (-a, a)$, limit $\lim_{y \to 0^+} g_1^{-1}(x_0 + y\mathrm{i})$ exists. But if λ is closet to $-a$ and a at 1 sufficiently, then for all $x_0 \in (-a, a)$, the limit above does not exist. Because if the limit exists, then we use the local time transformation at x_0 and time 1, then then driving function which induced by $|X_t - \lambda(t)|$ will be large at most of the times, so the corresponding Y_t is not vanishing, this gives a contradiction. In this case, J_1 has only two accessible points, others are not accessible.

4.3.2 proof of theorem 5

In this section, we prove theorem 5. Solving the curve generation problem is linked to the study of the case $\lim_{z \to \lambda(T)} g_T^{-1}(z)$ does not exist. When the limit does not exists, we define the limit se :$C(g_T^{-1}, \lambda(T))$. Since the time is fixed, hence if we say a point is accessible, it means that this point is t-accessible. We have four cases of the limit $\lim_{z \to \lambda(T)} g_T^{-1}(z)$ does not exist :

(a) There are at least two limit points which are accessible.

(b) There is no accessible limit point.

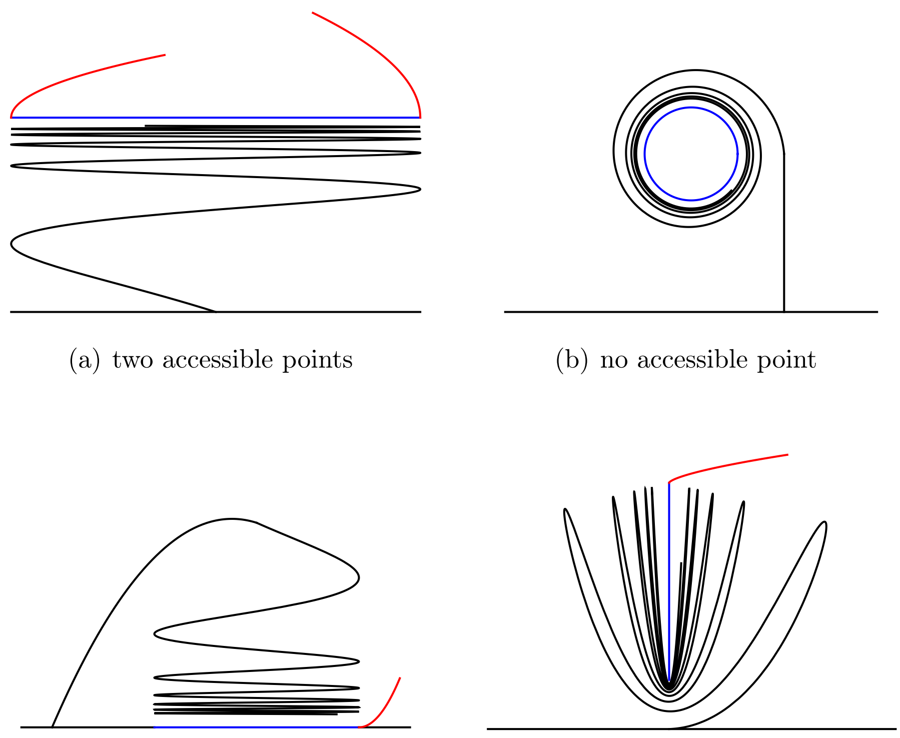
(c) There is only one accessible point x and $x \in \mathbb{R}$.

(d) There is only one accessible point z and $z \in \mathbb{H}$.

Figure 4.3.2 illustrates these 4 cases. In this figure, the blue lines are the set of limit points and the red line is the path β of the definition of accessible point. Actually, the limit set can be very complicated, as we mentioned in last subsection, the limit set maybe not local connected. A natural idea is that if we have a condition of the driving function can exclude the 4 cases above, then the limit exists.

We has already proved in last subsection that the first case of the figure corresponds to the driving function is not continuous at time T. Hence for the continuous driving function, the first case will not happen. For the second case, as we mentioned in theorem 94, if the curve circle around a set infinitely many times by one direction, then the corresponding limits $a(t) = b(t) = 4$ or -4. But for the rest cases of (b), the analogous property of driving function is still unknown.

Our purpose is to discuss (c) and (d),like before, we may, wlog, assume $T = 1$. About (c), the angular limit $\lim_{y \to 0^+} g_1^{-1}(\lambda(1)+y\mathrm{i})$ exists and belongs to real axis. Hence the driving function is captured at time 1. We assume all the solutions are larger



(a) two accessible points

(b) no accessible point

(c) one accessible point in $\mathbb{R}$

(d) one accessible point in $\mathbb{H}$

FIGURE 4.3.1 – examples of the Loewner equation is not generated by curve

than the driving function wlog. For any limit point x in $\mathbb{R}$, we solve the real Loewner equation (3.1.1) with x being the initial value, then we have a captured solution at time 1. And a captured solution corresponds to a driving function η of equation (4.2.1) after the local time transformation. Notice that the relation between η and the driving function is (4.2.3), or $\xi = H(\eta)$, where H is the operator defined before. Then if ξ is captured, there may be several ηs satisfying $H(\eta) = \xi$. And every $\eta \in A$ corresponds to a x in the real axis with $T_x = 1$. It is easy to check that, x and η have

$$\eta \mapsto \lambda(0) - \eta(0)$$

We write $A = \{\eta | H(\eta) = \xi\}$ and $X = \{\lambda(0) - \eta(0) | \eta \in A\} = \{x \in \mathbb{R} | T_x = 1\}$, then X is a single point or a right closed interval($(x_1, x_2]$ or $[x_1, x_2]$).

If $\eta \in A$, we have the properties of η :

Lemma 161. *ξ is the driving function of equation* (3.2.1)*, A is defined as above. If there exists $\eta_0 \in A$ s.t. $\varliminf_{t\to+\infty} \eta_0(t) > 0$, and η_0 is a vanishing driving function of equation (4.2.1), η_0 is not the largest element of A, then we have $\varliminf_{t\to+\infty} \eta(t) > 0, \forall \eta \in A$.*

Proof. If A consists of only one function, then the lemma is true. So we assume there are at least two functions in A. As we mentioned before, every function in A corresponds to an interval in $X = [x_1, x_2]$ or $(x_1, x_2]$, and let $x_0 \in X$ be the limit point corresponding to η_0. Because η_0 is a vanishing driving function of (4.2.1), we have $x_0 < x_2$. We choose two points $\hat{x}_1, \hat{x}_2$ from X and $\hat{x}_1 < \hat{x}_2 < x_2$, $\hat{\eta}_1$ and $\hat{\eta}_2$ are the corresponding functions in A. Let x be the solution of (4.2.1) with initial value $\hat{x}_2 - \hat{x}_1$, and $\hat{\eta}_1$ the corresponding driving function. Then x is a captured solution, and $\hat{\eta}_1(t) = \hat{\eta}_2(t) + x(t)$. By lemma 154, we have $\lim_{t \to +\infty} x(t) = 0$. Notice that $\hat{x}_1$ and $\hat{x}_2$ can be all the points in $X \setminus \{x_2\}$ which includes x_0, which finishes the proof. $\square$

The following lemma is very important :

Lemma 162. *Let $X = \{x \in \mathbb{R} | T_x = 1\}$. If X is an interval $[x_1, x_2]$ or $(x_1, x_2]$, and there exists $x_0 \in X \setminus \{x_2\}$ and $c > 0$ s.t. the corresponding function η_0 satisfies $\eta_0(t) > c, \forall t$, then for all $x \in X \setminus \{x_2\}$, $\exists \varepsilon > 0$ s.t. $B(x, \varepsilon) \cap \bar{\mathbb{H}} \subset J_1 \doteq K_1 \setminus \cup_{t<1} K_t$, where $B(x, \varepsilon)$ is the disk of center x and radius ε.*

Proof. We consider the solutions $X_\varepsilon(t), Y_\varepsilon(t)$ of (2.3.3) with initial value ($X_\varepsilon(0) = x, Y_\varepsilon(0) = \varepsilon$). Performing the same time change as before we have

$$\dot{x}_\varepsilon = x_\varepsilon - \frac{4(\xi - x_\varepsilon)}{(\xi - x_\varepsilon)^2 + y_\varepsilon^2}$$

where $x_\varepsilon, y_\varepsilon, \xi$ correspond to $X_\varepsilon, Y_\varepsilon, \lambda$ after time change respectively.

We assume $\hat{x}$ is the captured solution of (3.2.1) with initial value $\lambda(1) - x$. Define $\eta = \xi - \hat{x}$: by last lemma, we may assume $\eta(t) > C > 0$. Set $w_\varepsilon = x_\varepsilon - \hat{x}$, then the equation becomes

$$\dot{w}_\varepsilon = w_\varepsilon - \frac{4w_\varepsilon}{(\eta - w_\varepsilon)\eta} + \frac{4}{(\eta - w_\varepsilon)^2 + y_\varepsilon^2} \frac{y_\varepsilon^2}{\eta - w_\varepsilon} \tag{4.3.1}$$

$$= w_\varepsilon - \frac{4w_\varepsilon}{(\eta + w_\varepsilon)\eta} - \frac{4}{\eta - w_\varepsilon} \left(\frac{2w_\varepsilon^2}{\eta^2 + \eta w_\varepsilon} - \frac{y_\varepsilon^2}{y_\varepsilon^2 + (\eta - w_\varepsilon)^2} \right) \tag{4.3.2}$$

with the initial value $w_\varepsilon(0) = 0$. Let w^ε be the solution of (4.2.1) with initial condition $w^\varepsilon(0) = \varepsilon$: we claim that for sufficient small ε, $w_\varepsilon(t) < w^\varepsilon(t) < \eta(t), \forall t \in [0, +\infty)$.

Let y^ε be the solution of (4.1.3) with initial value $y^\varepsilon(0) = \varepsilon$: by lemma 148 and proposition 157, for sufficient small ε, $y^\varepsilon(t) < w^\varepsilon(t) < C/5, \forall t$. Notice that y^ε is driven by $\eta = \xi - \hat{x}$ and y_ε by $|\xi - x_\varepsilon|$. Since $x_\varepsilon(t) > \hat{x}(t), \forall t > 0$, we see that $y_\varepsilon < y^\varepsilon$ holds before the first time t s.t. $\xi(t) = \hat{x}(t)$, which is equivalent to saying that $w_\varepsilon(t) = \eta(t)$.

At time 0, $w_\varepsilon(t) < w^\varepsilon(t) < C/5$. Define t_1 to be the first time when $w_\varepsilon(t) = w^\varepsilon(t)$. If $t_1 < +\infty$, then $w_\varepsilon(t_1) = w^\varepsilon(t_1) > y^\varepsilon(t_1) > y_\varepsilon(t_1)$, and $y_\varepsilon^2(t_1) + (\eta(t_1) - w_\varepsilon(t_1))^2 > 16\eta^2(t_1)/25 > 3\eta^2(t_1)/5 > \eta(t_1)^2/2 + \eta(t_1)w_\varepsilon(t_1)/2$. This implies that

$$\dot{w}_\varepsilon(t_1) < w_\varepsilon(t_1) - \frac{4w_\varepsilon(t_1)}{(\eta(t_1) + w_\varepsilon(t_1))\eta(t_1)} = \dot{w}^\varepsilon(t_1)$$

in contradiction with the fact that t_1 is first time when $w_\varepsilon = w^\varepsilon$. This proves the claim.

Now, as we mentioned, $y_\varepsilon < y^\varepsilon$ will hold before $w_\varepsilon = \eta$ happens. By the claim, it will never happen. This gives us that $\lim_{t\to+\infty} y_\varepsilon(t) = 0$, and then $\lim_{t\to 1} Y_\varepsilon(t) = 0$, so $(x, \varepsilon) \in J_1$ for sufficient small ε. And the estimation of ε depends continuously on η, and η depends continuously on x, this finishes the proof. □

Actually, lemma 162 proves that, except for x_2 which corresponds to the maximal captured solution, there is no limit point in $\mathbb{R}$ if the driving function satisfies this condition, Thus we only need to prove that there is no limit point in $\mathbb{H}$ as well.

There are many driving functions which satisfy the hypothesis of lemma 162 which may lead the existence of $\lim_{z\to\lambda(1)} g_1^{-1}(z)$, 5 is just an example. Now we only need to prove that the driving function in theorem 5 satisfies the hypothesis of the lemma 162.

Lemma 163. *If* λ, *the driving function, satisfies* $\forall t \geq 0, a < \frac{\lambda(1)-\lambda(t)}{\sqrt{1-t}} < b$ *and* $a > \max\{4, (b+\sqrt{b^2-16})/2\}$, *then there exists infinitely many captured solution at time* 1, *s.t. the corresponding* η *satisfies* $\underline{\lim}_{t\to+\infty} \eta(t) > 0$.

Proof. We consider the solution x of (3.2.1) with initial value x_0 satisfying $(b + \sqrt{b^2-16})/2 < x_0 < a$. It is easy to check that the solution is decreasing when $x(t) > (b+\sqrt{b^2-16})/2$ and increasing when $x(t) < (a+\sqrt{a^2-16})/2$. So the solution will lie between $(b+\sqrt{b^2-16})/2$ and $(a+\sqrt{a^2-16})/2$ after a positive time, which implies that x is a captured solution and the corresponding $\eta > a-(b+\sqrt{b^2-16})/2$ after that time. This finishes the proof of this lemma. □

Now theorem 5 can be proved easily.

Proof of theorem 5. At first, it is easy to see that that the condition in theorem 5 is equivalent to $a > \max\{5, (b+\sqrt{b^2-16})/2\}$, after some times, the driving function will satisfy the condition of lemma 163. And the driving function also satisfies the hypothesis of lemma 162, hence there is an unique limit point x_2 in $\mathbb{R}$, and the captured solution which corresponds to x_2 is the minimal captured solution. So we only need to show that there is also no limit point in $\mathbb{H}$.

If there is a limit point in $\mathbb{H}$, we can get the corresponding solution of the Loewner equation after time change. Denote x and y to be its real and imaginary part. Same as lemma 162, we can define η and $\tilde{x}$, and set $w = x - \tilde{x}$. In all the captured solutions of ξ, $\tilde{x}_2$ is the corresponding solution of x_2,and $\tilde{x}_2$ is the minimal captured solution. At first, since y is a vanishing solution which driven by $|\xi - x|$, from lemma 113, $\xi - \tilde{x}_2 > 2$, hence $x < \tilde{x}_2$ can not always happen. And all the captured solutions except $\tilde{x}_2$ are converge to $\tilde{x}$, hence x will be greater than $\tilde{x}$ at some time, and after that, $x > \tilde{x}$ always hold.

We can see that η have not only a positive lower bound but also an upper bound, and this upper bound decreasing to 0 as a increasing to $+\infty$. It is easy to check that, when $a > 5$, this upper bound is less than 2. Hence we consider the imaginary Loewner equation of y, since y is a vanishing solution which driven by $\eta - w$, there are only two cases : the first case is that after a sufficent large time, y has a positive lower bound, and this lower bound increasing to 2 as a increasing to $+\infty$. Now we consider equation (4.3.1), for a solution w with positive initial value,

$$\begin{aligned}\dot{w} &= w - \frac{4w}{(\eta - w)\eta} + \frac{4}{(\eta - w)^2 + y^2}\frac{y^2}{\eta - w} \\ &= w - \frac{4}{\eta - w}\left(\frac{y^2}{(\eta - w)^2 + y^2)} - \frac{w}{\eta}\right)\end{aligned}$$

when $w < \eta$, the last term is negative unless $\eta > 2y$. This upper bound of η is sufficiently small when a is sufficiently large,but the lower bound of y is large, hence $\eta > 2y$ will not happen, actually, $a \leq 5$ can make sure that w is increasing and $\dot{w} > w$. Then at some finite time, $w = \eta$ will happen,that is $x = \xi$. By the real Loewner after the time change,we obtain that when $x > \xi$, x is still increasing and $\dot{x} > x$. Thus x will increasing to $+\infty$ exponentially, this makes a contradiction to the fact that y is vanishing.

In the second case, y decreasin to 0 exponentially. Now we consider the small circles of lemma 162, after a finite time, (x, y) will be in one of them. But the points of the circles are the inner point of J_1, which gives us a contradictiopn. □

Chapitre 5

Self-similar Driving Function

In this chapter we introduce the author's results on self-similarity functions and self-similarity curves. Our main idea comes from the construction of the driving function.

5.1 Definition of Self-similar

An important property of the chordal Loewner equation is the scale property, the third of 73. Taking Brownian motion as an example, From the propsition 49, Brownian motion also satisfies the scale property, that is, B_{r^2t}/r and B_t have the same distribution. Hence the stochastic curves which are generated by them also have the same distribution. On the other hand, by the scale property, for a $r > 1$, enlarge the curves of B_t is by r times, they are the curves generated by B_{r^2t}/r. Therefore we have :

Proposition 164. *For $r > 0$, after the SLE curve is enlarged by r times, it still has the same distribution with the original curve has.*

Then consider about the general driving function, if the driver satisfies that for all $t, \lambda_1(t) = \lambda_2(r^2t)/r, r < 1$, then the image generated by λ_1 is enlarged r times to be the image of λ_2. For many curves, such as snowflake curves, they are all self-similar, which means there exists a constant r, and we can choose a point and part of the curve, enlarge the part r times with the point as the centre, then it and the original curve are the same. We can define :

Definition 165. *If the driving function λ satisfies that $\forall t, \lambda(t) = \lambda(r^2t)/r$, $r > 1$, then we say λ is r right self-similar.*

For example, we can define λ on $[1, r]$, if λ is a continues function and satisfies $\lambda(1) = \lambda(r)/r$, then we can use the relation $\lambda(t) = \lambda(r^2t)/r$ to define the λon $[0, +\infty)$ iteratively. So if the Loewner process is generated by a curve, then the curve must be r self-similar. Conversely, if a curve is r self-similar, then the corresponding driving function must also be r right self-similar.

If a driving function is self-similar for any $r > 1$, then we call it absolute self-similarity on the right side. It is easy to see that the image is a line. At the same time, you can use$\lambda(t) = \lambda(r^2 t)/r$to get driving function $\lambda(t) = c\sqrt{t}$.

Because Brownian motion has the symmetry property, so we have : for any$0 < r < 1$, $B(T) - B(t)$ has the same distribution with$(B(T) - B(T - r(T-t)))/\sqrt{r}$. So we can consider the image of the $g_{(1-T)r}$ of the SLE curves on$[(1-r)T, T]$, if we enlarge it by r times, then it has the same distribution as the SLE curves on $[0, T]$. If we consider the general driving function, we can define :

Definition 166. *For $T > 0$, if the driving function satisfies*

$$\frac{\lambda(T) - \lambda(T - r(T-t))}{\sqrt{r}} = \lambda(T) - \lambda(t), 0 < r < 1 \tag{5.1.1}$$

Then we say λ is r left self-similar at time T.

In this case, if the driving function λ satisfies it is r left self-similar at time T, if λ is generated by a curveγ, then $g_{(1-r)T}(\gamma[(1-r)T, T])$ and $\gamma([0, T])$ are similar. We will discuss the left self-similar function in this chapter.

If for all r, the driving function is r left self-similar at T, then we call it absolute left self-similar at T. Similarly, in this case, the driving function is $c\sqrt{T-t}$. In[19], the absolute self-similarity is used to find the solutions of the Loewner equation with driving function $c\sqrt{T-t}$.

Without loss of generality, we set $T = 1$. And in the rest of this chapter, we say the driving function is self-similar if it is left self-similar at time 1. Then we can do a local time transformation on the driving function, and put it into (5.1.1), then we have the following conclusions :

Theorem 167. $0 < r < 1$*, if driving function λ is r self-similar, then the driving function ξ of* (3.2.1) *is a periodic function in $[0, +\infty)$, the period is $-\frac{\ln r}{2}$.*

After the time transformation, the Loewner equation (2.3.1) becomes :

$$\dot{g}(t) = g(t) - \frac{4}{\xi(t) - g(t)}, g(0) = z \tag{5.1.2}$$

where $\xi$is a periodic function. Without causing confusion, we assume the period of ξ is T, then we can use the dynamic system to solve the problems related to this equation.

5.2 Real periodic equation

First, we study the one-dimension case, which is the equation (3.2.1) where the driving function ξ is a periodic function with a period of T. We assume that the supremum andÂ infimum of ξ are a and b respectively, then we can define an iterative map f on the real axis, for any initial value $x_0 \neq \xi(0)$, the solution x of the corresponding equation either does not exist on $[0, T]$, or exists on $[0, T]$. We define : $f : x_0 \mapsto x(T)$ if the solution of the equation exists in $[0, T]$; $f(x_0) = \xi(T) = \xi(0)$, if the equation The solution does not exist in $[0, T]$.

Like the the Fatou set and he Julia set in complex dynamic system, we define the following sets :

$$F \doteq \{x_0 : f^n(x_0) \to \infty, n \to +\infty\}$$
$$J \doteq \{x_0 : \exists n \text{ s.t. } f^n(x_0) = \xi(0)\}$$
$$P \doteq \mathbb{R} \setminus (F \cup J)$$

then F, P and invariant sets of f. Where F can be divided into F^+ and F^-, which are sets of initial values that tend to be $+\infty$ and $-\infty$ respectively, which are also invariant sets. J is called the vanish set, ie the solution corresponding to the initial value in J will be captured by the driving function in limited time. P is called the captured set. We can see that the solution corresponding to any initial value in the capture set is the capture solution of the equation (3.2.1). And $J \cup P$ is bounded under f mapping. So we can conclude that :

Proposition 168. *F^+, F^- are open sets, P is an empty set or closed set or takes $\xi(0)$ as a boundary point of it.*

Proof. For any initial value $x_0 \ inF^+$, since x_0 is unbounded under iteration, there is time t_1 such that the solution x of the equation with initial value x_0, satisfies $x(t_1) > b, 0$. By the continuity of the ordinary differential equation solutions to the initial values, we have that for the initial values which is closed to x_0 sufficiently, the corresponding solutions are also larger than b and 0 at t_1. Then by equation (3.2.1), such solutions will increasing after t_1, and will tend to infinity. Hence, x_0 is an inner point of F^+, which proves the conclusion.

As for P, if P is not an empty set, then we can consider x_0, which is the boundary point of P. and we only need to discuss the case that $x_0 \neq \xi(0)$ and $x_0 \in J$. Since F is an open set, x_0 belongs to J or P. IF x_0 belongs to J, then the solution x with initial value x_0 will be captured in finite time t_1, $t_1 > 0$. It is easy to see that the points between x_0 and $\xi(0)$ are also belong to J. Similarly, according to the continuity of the solution of the differential equation, for the initial value sufficient close to x_0, if these initial values are points in P, the solutions corresponding to these initial values will be close to x_0 at time t_1.Then these points all belong to J, which makes a contradiction. Therefore P can only have three cases as stated in the proposition. □

Now the situations of F, J, P are clear. First of all, on the real axis, there is an open infinite interval on each side, which is F^+ and F^-, and the part between them is $J \cup P$, we have the following cases :

(1) P is empty, and equation (3.2.1) is not captured, J is a close set between F^+ and F^- ;

(2) P is a half open half closed interval, adjacent with J at $\xi(0)$, J is a closed interval ;

(3) P is a single point, J is a half open half closed interval with a boundary point P ;

(4) P is a close set, J is a half open half closed interval.

We give two simple examples. When ξ is always 0, it is case (1). When ξ is 4 constantly, it is case (2). As for case (3), (4) We will discuss later.

Now we consider the case where P is not an empty set, that is, the case where the driving function is captured. Let us assume that $\xi > 0$, then on the real axis, from left to right, there are F^-, P, J, F^+ consecutively. It's easy to see that f has no fixed point on F, and f will always equal the fixed point $\xi(0)$ after iterated finitely on J, but f is the orientation-preserving self-homeomorphism from P to P, so there must be a fixed point on P. We discuss each situation separately, first of all, the situation that P is a single point set, which is relatively easy.

Proposition 169. *If P is a single point set, then P is an unstable fixed point.*

If P is not a single point set, since f is an orientation preserving homeomorphism, then the left endpoint of P must be a fixed point. If the left endpoint of P is atrracting, we take any point other than the left endpoint on P to get a corresponding capture solution x. By this capture solution, we can find the corresponding dual equation (4.2.1). Then by lemma 154, we can conclude that except for at most one solution, which is actually corresponding to the left endpoint, the rest of the solutions are converged to x. So if there is one more fixed point except the left endpoint of P, then there must be a third fixed point between these two fixed points, and we obtain the dual equation by the rightmost fixed point, then the other two fixed points correspond to two periodic vanishing solutions of the dual equation, which contradicting the lemma 154. So P has only one fixed point at this situation, which is the left endpoint of P, and it is unstable on the left but stable on the right.

If the left endpoint of P is repelling, there is a fixed point at P. As mentioned above, there are at most two fixed points in P, so either P is half-open and half-closed, and has an inner point that is attracting, corresponding to the situation (3), or P is a closed interval, the other end point is the fixed point unstable on the left side and stable on the right side, which corresponds to the situation (4). If latter case holds, we denote the fixed point on the right as x_0, and the get the dual virtual equation (4.2.2) with its corresponding capture solution, then the other fixed point corresponds to a solution w of an equation. This solution is a periodic solution, so it satisfies that :

$$0 = \int_0^T 1 - \frac{4}{\eta^2 + \eta w} \mathrm{d}t$$

where η is also periodic. So for a very small initial value of the equation (4.2.2), the integration above must be negative. From this, the derivative of f at x_0 is less than 1, so x_0 must be attracting, this makes a contradiction. Hence we have proved that the situation (4) does not exist. So we have :

Proposition 170. *When $\xi > 0$, if P is not a single point set, then it is a right open left closed interval. Either the left endpoint of P is unstable fixed and there is another inner point of P is stable fixed, or it is unstable fixed on the left side but stable fixed on the right side and there is only this one fixed point in P.*

Remark 171. ξ equals constant $c > 4$ is the first case above, and ξ equals 4 is the second case above.

Every fixed point x corresponds to a periodic captured solution x, hence we can write the driving function of the corresponding dual equation 4.2.1, $\eta = \xi - x$ is also periodic. Then can compute the lower bound function $L(t)$ of η by (4.1.5). Conversely, if there is a periodic η, we get a driving function ξ by (4.2.3). It is easy to prove :

Proposition 172. *If $L(T) = 0$,the driving function is the second case of proposition 170. If $L(T) < 0$, then it is the first case. If $L(T) > 0$, then it is the first case or P is a single point set.*

5.3 Complex periodic equation

Next, we consider the complex periodic equation (5.1.2). We will prove the following theorem 6. Similarly, for any $z \in \mathbb{C}$, we can define $f(z)$ as $g_T(z)$. If the solution for g does not exist in $[0,T]$, we define $g_T(z) = \xi(0)$. Then consider this dynamic system, and again, we can define the set

$$F \doteq \{z_0 : f^n(z_0) \to \infty, n \to +\infty\}$$

$$J \doteq \{z_0 : \exists n \text{ s.t. } f^n(z_0) = \xi(0)\}$$

$$P \doteq \mathbb{R} \setminus (F \cup J)$$

We can prove that F is still an open set, J corresponds to $\cup_{t<1} K_t$, P corresponds to $K_1 \setminus \cup_{t<1} K_t$. So unlike the real equation, P will not be an empty set. But F, P are still invariant sets under f.

Although f is not a holomorphic function in the general sense, we can consider about f^{-1}, since g_1 is a conformal mapping of $\mathbb{H}/ \setminus K_1$ to $\mathbb{H}$, f^{-1} is a holomorphic function of the upper half plane to itself. If we define $J_0 \doteq \{z_0 : f(z_0) = \xi(0)\}$, then we have :

$$J = \bigcup_{n=1}^{+\infty} f^{-n}(J_0)$$

And we only need to prove :

$$P_0 \doteq \bigcap_{m=1}^{+\infty} \overline{\bigcup_{n=m}^{+\infty} f^{-n}(J_0)}$$

is a single point set. It is easy to see that $P_0 \subset P$ is also the invariant set under f.

The hyperbolic metric of $\mathbb{H}$ is decreasing strictly under f. By Wolff-Denjoy theorem of the hyperbolic dynamic system, we have : $\forall z \in \mathbb{H}$, f^{-n} converges to a point of $\bar{\mathbb{H}}$ locally and uniformly. According to the hypothesis of the theorem 6, we have $f^{-1}(J_0) \subset \mathbb{H}$, then P_0 is a single point set. This finishes the proof of theorem 6.

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Henshui Zhang

Analyse locale de l'équation de Loewner

Résumé :
Dans cette thèse nous étudions le problème de la génération d'une courbe par l'équation de Loewner generalisée. Nous utilisons une transformation locale dans l'équation chordal de Loewner, analysons la solution de l'équation de Loewner, et obtenons trois résultats.
En premier lieu, nous analysons la limite supérieure et la limite inférieure de l'ordre 1/2 à gauche de la fonction pilotant l'équation, nous prouvons ensuite un lemme basique qui assure que la courbe générée ne s'auto-intersecte pas localement. Ce lemme nous conduit à trois conclusions. Premièrement Lind a prouvé que lorsque la norme Hölder-1/2 est inférieure à 4, alors l'équation de Loewner est générée par une courbe simple. Nous nous intéressons au cas où la norme Hölder-1/2 est supérieure à 4, et donnons une condition suffisante pour que la courbe générée soit simple. Deuxièmement, la limite inférieure de l'ordre 1/2 du mouvement brownien tend vers 0 localement, nous donnons un estimé de la vitesse à laquelle il tend vers 0. Troisièmement, nous prouvons que pour la fonction de Weierstrass d'ordre 1/2 dont le coefficient est inférieur à une certaine constante, l'équation de Loewner correspondante est générée par une courbe simple.
Dans la deuxième partie, nous définissons l'équation de Loewner imaginaire et son équation duale, et nous procédons à la transformation locale de ces deux équations. Après analyse de leurs propriétés d'annulation, nous construisons le lien entre ces dernières et le problème de génération de courbe. Nous donnons ensuite une conditions suffisante pour que l'équation de Loewner soit localement générée par une courbe.
Finalement, nous définissons et nous intéressons au cas où la fonction pilotant l'équation est auto-similaire à gauche, et utilisons des connaissances en dynamique complexe pour prouver que si elle est localement générée par une courbe dans le demi-plan supérieur, alors elle est entièrement générée par une courbe.

Local analysis of Loewner equation

Abstract :
This thesis studies the curve generation problem of the general Loewner equation. We use a local transformation in the chordal Loewner equation, and analyse the solution of the Loewner equation, obtain three results.
At first, we analyse the Limit superior and limit inferior of the left 1/2 order of the driving function, then we prove a basic lemma about that the generation curves do not intersect with itself locally. By this lemma, we have three conclusion. Firstly, Lind proved that when 1/2-Hölder norm is less than 4, then the Loewner equation is generated by a simple curve. We discuss the case that the 1/2-Hölder norm is greater than 4, and give a sufficient condition of the generation curve is simple. Secondly, the limit inferior of the 1/2 order of the Brownian motion will tends to 0 locally, we give a estimation of the speed of it tends to 0. Thirdly, we proof that for the 1/2 order Weierstrass function with coefficient less that a constant, the Loewner equation which is driven by it is generated by a simple curve.
In the second part, we define the imaginary Loewner equation and its dual equation, and we do the local transformation for these two equation, after analyse their vanishing property, we build the connection between it with the curve generation problem. And then we give a sufficient condition on that the Loewner equation is generated by a curve locally.
At last, we define and discuss the left self-similar driving function, and use the knowledge of complex dynamic to prove that if it is generated by a curve in the upper-half plane locally, then it is generated by a curve entirely.
Keywords : Loewner equation, hull, driving function.

MAPMO UMR 6628 - CNRS
Rue de Chartres - BP6759
45067 ORLÉANS CEDEX