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Spectral Methods In SubRiemannian Geometry

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Dédicace

I dedicate this thesis to my parents and siblings, whose love and laughter brighten my world.

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Aperçu de la thèse

0.1 Motivation

Soit M, une variété lisse de dimension d. Un opérateur P est dit elliptique d'ordre p dans M si $Pu \in H^s(M) \implies u \in H^{s+p}(M)$. Les opérateurs elliptiques généralisent les opérateurs de laplace, dont le cadre naturel est la géométrie riemannienne. Une variété riemannienne est une variété lisse munie d'un produit scalaire défini positif sur chaque espace tangent appelé la métrique riemannienne. On peut la définir à l'aide d'un ensemble de champs de vecteurs qui engendrent l'espace tangent en chaque point.

L'ellipticité entraîne que $Pu \in \mathcal{C}^\infty(M) \implies u \in \mathcal{C}^\infty(M)$ (car u sera alors dans $H^p(M)$ pour tout p). Cependant, l'ellipticité est une condition suffisante mais non nécessaire pour la régularité des solutions : un contre-exemple est l'équation de la chaleur, qui est une équation aux dérivées partielles parabolique mais qui possède des solutions régulières). Ainsi, un opérateur hypoelliptique est défini comme un opérateur P tel que $Pu \in \mathcal{C}^\infty(M) \implies u \in \mathcal{C}^\infty(M)$. Le premier à introduire un tel opérateur fut Kolmogorov en 1934 dans [59], alors qu'il travaillait sur le mouvement de particules en collision. En étudiant l'équation

$$\partial_t u - (x\partial_y + \partial_x^2)u = f,$$

il observa que l'opérateur $x\partial_y + \partial_x^2$ est hypoelliptique. On définit aussi une troisième classe d'opérateurs : les opérateurs sous-elliptiques. On dit qu'un opérateur P est sous-elliptique d'ordre p avec une perte ϵ de dérivées si $Pu \in H^s(M) \implies u \in H^{s+p-\epsilon}(M)$ pour un certain $\epsilon \in]0,1[$. Une relation entre ces différentes classes est la suivante: :

$$\text{Elliptique } \Rightarrow \text{Sous-elliptique } \Rightarrow \text{Hypoelliptique}.$$

Les opérateurs sous-elliptiques, qui ont été étudiés intensivement depuis le vingtième siècle, sont donc intéressants car ils généralisent les opérateurs elliptiques tout en continuant à satisfaire l'hypoellipticité.

Un exemple typique d'opérateur subelliptique est l'opérateur de Grushin. Les opérateurs de Grushin ont été définis dans [44] en 1970, comme les opérateurs de la forme :

$$P = -\partial_x^2 - |x|^{2k}\partial_y^2, \tag{1}$$

pour certains $k \in \mathbb{N}$, agissant sur $L^2(\mathbb{R}^2_{x,y})$. Ce fut le point de départ pour définir un opérateur de type Grushin qui fut ensuite largement étudié par de nombreux chercheurs (par exemple dans [21][28][37][53] ...). Un opérateur de type Grushin intéressant est le suivant (il peut être considéré comme la généralisation pour le cas $k = 1$). Sur $\mathbb{R}^2$, nous définissons l'opérateur de Baouendi Grushin généralisé (simplement opérateur de Baouendi Grushin) par :

$$P = -\partial_x^2 - V(x)\partial_y^2, \tag{2}$$

pour V satisfaisant les conditions suivantes :

$$V \in \mathcal{C}(\mathbb{R}), V(x) \geqslant 0, V(0) = V'(0) = 0, V''(0) > 0, \lim_{|x|\to+\infty} V(x) = +\infty. \tag{3}$$

Certains chercheurs ont étudié ce type d'opérateurs de Baouendi Grushin (et d'autres types) notamment du point de vue de leur contrôlabilité et/ou observabilité dans différents domaines ([9][10][21][26][31][57]). La contrôlabilité et l'observabilité sont un type "fort" d'inégalités de concentration qui impliquent d'autres inégalités "plus faibles" comme les inégalités de concentration pour les fonctions propres.

L'étude des inégalités de concentration pour les fonctions propres a une histoire riche qui s'étend sur plusieurs décennies, où plusieurs façons de mesurer les concentrations possibles ont été utilisées (voir [20][22]). Les inégalités de concentration pour les fonctions propres sont des estimations de la probabilité que les fonctions propres d'un certain type d'opérateur soient concentrées dans une région particulière de l'espace sous-jacent. Plus précisément, on se demande si la masse des fonctions propres sur la variété peut être contrôlée par sa masse sur un sous-domaine.

Le lien direct entre les propriétés de concentration des fonctions propres et la contrôlabilité/observabilité peut varier en fonction du système spécifique et du cadre de contrôle. Dans certains cas, on peut montrer que l'une est une condition suffisante ou nécessaire pour la validité de l'autre. Par exemple, dans [12], l'auteur mentionne que sous certaines conditions, l'inégalité de concentration des valeurs propres est une condition suffisante pour l'observabilité exacte de l'équation des ondes associée à l'opérateur de laplace (voir page 1030 de [12]). D'un autre côté, de manière plus générale, les auteurs de [24] (où l'inégalité de concentration des fonctions propres de l'opérateur de Schrödinger est prouvée sur un tore) mentionnent que l'inégalité de concentration des fonctions propres peut être obtenue à l'aide de la formule de Duhamel en partant de l'observabilité de l'équation de Schrödinger. On renvoie à [23][64] pour des études détaillées sur le lien entre l'observabilité et les inégalités de concentration des fonctions propres.

Plusieurs auteurs ont étudié la concentration des fonctions propres de l'opérateur de laplace, par exemple [17][20][23][33][46]. Ces études relient ce type de de concentration à la validité de certains critères géométriques. Une condition suffisante bien connue est ainsi la condition de contrôle géométrique, également appelée Condition de Contrôle Géométrique de Bardos-Lebeau-Rauch, introduite dans [12]. Cette condition demande que toutes les trajectoires du flot géodésique généralisé rentrent dans la région de contrôle. Il est connu que cette condition est équivalente à l'observabilité dans certains cas, tels que l'équation des ondes (voir [12][19]), et suffisante dans d'autres cas, comme pour l'équation de Schrödinger (voir [60]). Cependant, cette condition n'est pas nécessaire en général. Dans [46], par exemple, les auteurs ont montré que, sur un domaine polygonal arbitraire, la norme L^2 de la fonction propre sur le domaine est contrôleée par la masse sur un voisinage U des sommets alors que U ne vérifie pas la condition de contrôle géométrique. Pour obtenir ce résultat, les auteurs introduisent une autre condition géométrique appelée condition cylindrique (voir [46]).

Dans notre travail, nous ne nous préoccupons pas des conditions géométriques et nous examinons une condition purement spectrale qui sert de condition suffisante pour la validité des inégalités de concentration pour les fonctions propres des opérateurs de Baouendi Grushin sur une bande horizontale arbitraire du cylindre infini. Plus précisément, notre principal intérêt ici est de relier l'inégalité de concentration à une certaine condition spectrale, puis d'étudier cette condition pour l'opérateur généralisé de Baouendi Grushin sur un cylindre. En fait, l'inégalité de concentration est déjà connue dans de nombreux cas, en utilisant des méthodes géométriques sophistiquées. Ainsi, dans [62, Chapitre 3], par exemple, l'auteur étudie le cadre suivant. Soit $M = (-1,1)_x \times \mathbb{T}_y$ où $\mathbb{T}$ est cercle, et soit $\gamma \in \mathbb{R}^{*+}$. Définissons l'opérateur de type Baouendi Grushin suivant

$$\Delta_\gamma = \partial_x^2 + |x|^{2\gamma}\partial_y^2, \tag{4}$$

avec le domaine

$$D(\Delta_\gamma) = \{u \in D'(M), \partial_x^2 u \in L^2(M), |x|^{2\gamma}\partial_y^2 u \in L^2(M), u = 0 \text{ sur } \partial M\}.$$

Il montre alors le théorème.

Theorem 0.1 (Letrouit [62]). *Soit $\gamma \geqslant 1$ et soit $w = (-1,1)_x \times I$ pour un certain intervalle I. Alors, il existe $C, h_0 > 0$ tels que pour tout $u \in D(\Delta_\gamma)$, et tout $0 < h \leqslant h_0$,*

$$\|u\|_{L^2(M)} \leqslant C\left(\|u\|_{L^2(w)} + h^{-(\gamma+1)}\left\|(h^2\Delta_\gamma + 1)u\right\|_{L^2(M)}\right). \tag{5}$$

Ce théorème entraîne le contrôle des fonctions propres :

Corollary 0.1.1. *Soit $\gamma \geqslant 1$ et soit $w = (-1,1)_x \times I$ pour un certain intervalle I. Alors, il existe $C > 0$ tels que pour toute fonction propre u de Δ_γ,*

$$\|u\|_{L^2(M)} \leqslant C\|u\|_{L^2(w)}. \tag{6}$$

Nous chercherons ici à montrer des résultats similaires pour ce type d'opérateur, avec des méthodes spectrales.

La définition (2) montre que l'opérateur généralisé de Baouendi Grushin (et tout opérateur de type Baouendi Grushin) peut être écrit comme la somme du carré de deux champs vectoriels lisses $X_1 = \partial_x$ et $X_2 = \sqrt{V(x)}\partial_y$. On l'appelle donc parfois opérateur sous-laplacien de Grushin. Plus généralement, les sous-laplaciens sont des généralisations de l'opérateur de laplace sur une variété riemannienne. On les définit de la façon suivante.

Soient $X_1, ..., X_p$ des champs de vecteurs lisses sur une variété lisse M. Nous disons que $X_1, ..., X_p$ sont générateurs par crochets itérés (ou satisfont la condition de Hörmander) de rang r si les champs $X_1, ..., X_p$ complétés par leurs crochets itérés $[X_i, X_j], [X_i, [X_j, X_k]]$... jusqu'à une longueur r engendrent l'espace tangent en chaque point $m \in M$ (voir [50]). Dans ce cadre, le sous-laplacien par rapport à une mesure lisse ω est alors défini comme

$$\Delta = -\sum_{i=1}^{p} X_i^* X_i = \sum_{i=1}^{p} X_i^2 + \operatorname{div}_\omega(X_i)X_i, \tag{7}$$

où $\operatorname{div}_\omega(X_i)$ désigne la divergence de X_i relativement à ω. Ces opérateurs sont aussi appelés opérateurs de type 2 de Hörmander (le type 1 étant simplement la somme des carrés) car ils ont été largement étudiés par Hörmander. Il a montré que sous la condition qui porte son nom,, Δ est hypoelliptique (voir [50]). Hörmander a prouvé cela en montrant que Δ est sous-elliptique et satisfait l'estimation suivante (qui est la définition de sous-ellipticité) : $\exists s, C > 0, \forall u \in \mathcal{C}_c^\infty(M)$,

$$\|u\|_{H^s(M)} \leqslant C \left(\langle \Delta u, u \rangle_{L^2(M)} + \|u\|_{L^2(M)} \right). \tag{8}$$

et Rotschild et Stein ont montré dans le théorème 17 de [75] que $s = \frac{1}{r}$ est optimal. Lorsque la variété M est compacte, alors (8) implique que $(\Delta, \mathcal{C}^\infty(M))$ est essentiellement auto-adjoint sur les fonctions lisse, et la seule extension auto-adjointe unique a une résolvante compacte. Il en découle que le sous-laplacien a un spectre discret dans ce cadre.

Le cadre général pour de tels opérateurs est la géométrie sous-Riemannienne.
La géométrie sous-riemannienne est une généralisation de la géométrie riemannienne, dans laquelles toutes les directions ne jouent pas le même rôle et où certaines contraintes sont imposées pour se déplacer le long de la variété. Les variétés sous-riemanniennes servent ainsi dans l'étude des systèmes contraints en mécanique classique, tels que le mouvement de véhicules sur une surface, le mouvement des bras de robots et la dynamique orbitale des satellites. Plus précisément, si on munit une variété de champs de vecteurs qui n'engendrent pas l'espace tangent, alors on ne peut pas définir une métrique riemannienne. Cependant, lorsque ces vecteurs satisfont la condition de Hörmander, alors on dit que la variété est sous-riemannienne et on peut définir la métrique sous-riemannienne associée à $\mathscr{C} = \{X_1, ..., X_p\}$, sur TM par

$$g_m(X_m) = \inf \left\{ |u|^2_{\mathbb{R}^p}; u \in \mathbb{R}^p, \sum_{i=1}^{p} u_i X_i(m) = X(m) \right\}, \tag{9}$$

avec la convention que $\inf \varnothing = +\infty$. La structure $(M, \mathscr{C}, g)$ est appelée une structure sous-riemannienne. La géométrie sous-riemannienne est très étudiée depuis le milieu des années 80, en commençant par l'étude des groupes de Heisenberg et en se concentrant sur les propriétés géométriquees des boules et des géodésiques.

Il existe de nombreuses questions qui se posent en géométrie sous-riemanienne, notamment des questions de théiore du contrôle (voir [78]). Une autre approche consiste à considérer la structure sous-riemannienne comme une limite singulière des structures riemanniennes (voir [4][27][36][40][43][67][76][83]). Ces approximations riemanniennes, lorsqu'elles sont associées à des estimations uniformes, permettent d'étendre certains résultats riemanniens aux contextes sous-riemanniens. Par exemple, les auteurs de [27] ont utilisé de telles approximations pour généraliser certaines estimations riemanniennes connues (propriété de doublement, inégalité de Poincaré, estimations gaussiennes, etc.) aux variétés sous-riemanniennes.

Cette convergence est souvent considérée en termes de convergence des distances. Plus précisément, si $(M, \mathscr{C}, g)$ est une structure sous-riemannienne, le théorème de Chow-Rashevskii, (aussi appelé de théorème de Chow), garantit que deux points quelconques d'une variété sous-riemannienne connexe, dotée d'une distribution génératrice par crochets itérés, sont reliés par un chemin horizontal dans la variété (voir [3][52]). Ainsi, la métrique sous-riemannienne g définit une distance d sur M. Habituellement, dans le cadre d'une approximation par une famille de métriques riemanniennes g^h avec sa distance d^h associée, on montre le théorème suivant.

Theorem 0.2. *La famille de distances d^h converge uniformément vers d sur chaque ensemble compact de la variété M.*

Ainsi, la géométrie de la structure sous-riemannienne est la limite de la suite des géométries riemanniennes et on aimerait savoir ce qu'il en est au niveau spectral.

Maintenant, il est clair, d'après son expression, que l'opérateur sous-laplacien dépend de la mesure choisie dans (7). Dans le cadre riemannien, une mesure canonique est obtenue à l'aide de la métrique riemannienne. Ce n'est pas le cas dans les variétés sous-riemanniennes, car les métriques sous-riemanniennes ne sont pas définies sur l'ensemble de l'espace tangent et il n'existe pas de moyen canonique de l'étendre à l'ensemble de l'espace tangent. Ainsi, une question très naturelle se pose ici : comment définir une mesure canonique sur une variété sous-riemannienne, de façon à avoir un sous-laplacien canonique ?

La question a été initialement soulevée par Brockett en 1982 dans son article (see [18]). Sa motivation venait du désir de construire un opérateur de laplace sur une variété sous-riemannienne tridimensionnelle, qui serait intrinsèquement lié à la structure métrique, analogue à l'opérateur de laplace-

Beltrami sur une variété riemannienne. Plus récemment, Montgomery a abordé ce problème dans un contexte plus général [66]. Le volume de Popp, le volume de Hausdorff et le volume sphérique de Hausdorff sont quelques exemples intéressants de mesures sur une variété sous-riemannienne (voir [2][42][65] et les références à l'intérieur).
Le volume de Popp, par exemple, a d'abord été défini par Octavian Popp mais introduit seulement par Montgomery dans [66]. Le volume de Popp est défini en induisant un produit scalaire canonique sur l'espace vectoriel gradué définie par la structure des crochets de Lie, puis en utilisant un isomorphisme non canonique entre l'espace vectoriel gradué et l'espace tangent pour définir un produit scalaire sur l'ensemble de l'espace tangent. En 2013, les auteurs de [13] ont trouvé une expression pour le volume de Popp en termes d'une base adaptée, et depuis lors, cette formule a parfois été utilisée dans ce contexte comme la définition du volume de Popp. Plus précisément, on considère une base adaptée $Z_1, ..., Z_d$ (définie au sens de [13]) et on définit récursivement les sous-espaces $D_i = D_{i-1} + [D_0, D_{i-1}]$, où D_0 est le sous-espace engendré par les champs vectoriels initiaux. On définit ensuite les constantes de structure adaptées qui, de façon informelle, sont les coefficients des crochets des Z_i dans D_i modulo D_{i-1} (en ne considérant que les crochets de longueur i).

Theorem 0.3 (Barilari-Rizzi[13]). *Dans la base $Z_1, ..., Z_d$, le volume de Popp est donné par*

$$d\mathcal{P} = \frac{1}{\sqrt{\prod_j \det(B_j)}} d\nu_1 \wedge ... \wedge d\nu_d, \tag{10}$$

où B_j sont des matrices définies à l'aide des constantes de structure adaptées et $\nu_1, ..., \nu_d$ forment la base duale à la base adaptée.

Avec la famille de structures riemanniennes (et de métriques riemanniennes) approchant la structure sous-riemannienne (resp. la métrique sous-riemannienne), on peut définir la famille d'opérateurs de laplace associés Δ_h. Ces opérateurs Δ_h sont des opérateurs elliptiques, et donc, si M est compacte, ils ont une résolvante compact et, par conséquent, un spectre discret. On peut donc s'intéresser à la convergence de ce spectre.

Dans le cas où la structure riemannienne dégénère sur une structure sous-riemannienne, peu de choses sont connues sur la convergence du spectre des laplaciens (voir [39][40][76]). Dans certains cas spécifiques, il a été démontré que la famille Δ_h converge vers Δ, et que chaque valeur propre de Δ_h converge vers une valeur propre de Δ. Cela a d'abord été observé par Fukaya dans

[39] puis démontré par Ge dans [40] (voir également [76] pour le cas des variétés de contact). Plus précisément, soit M une variété compacte munie d'une métrique riemannienne g. Soit H une distribution sur M de dimension constante, et soit $H^{\perp}$ la distribution orthogonale à H. Écrivons $g = g_H \oplus g_{H^{\perp}}$. On définit la famille de métriques riemanniennes pour $h > 0$,

$$g_h = g_H \oplus h^{-2} g_{H^{\perp}}.$$

Theorem 0.4 (Ge [40])**.** *Soit Δ_h le laplacien associé à g_h. Alors, Δ_h converge lorsque $h \to 0$ vers un opérateur sous-elliptique de second ordre*

$$\Delta_H = -\sum_i e_i^2,$$

où e_i est une base orthonormale pour H. De plus, si $\lambda_1(h) \leqslant \lambda_2(h) \leqslant ....$ et $\lambda_1 \leqslant \lambda_2 \leqslant ....$ désignent les valeurs propres de Δ_h et de Δ_H respectivement, alors $\lambda_k(h)$ converge lorsque $h \to 0$ vers λ_k.

On cherche à généraliser ce type de résultat pour une approximation riemannienne mieux adaptée à la structure et une question se posera alors relativement à la mesure utilisée.

Dans la section suivante, on présente les résultats de ce travail.

0.2 Résultats Principaux

Notre travail sera réparti en trois chapitres. Par conséquent, nous divisons cette section en trois sous-sections, chacune contenant les résultats d'un chapitre.

0.2.1 Inégalités de Concentration

Désignons par $X = \mathbb{R} \times \mathbb{S}^1$ le cylindre infini et par $w = \mathbb{R} \times [a, b]$, une bande horizontale le long de X. Notons

$$L_0^2(X) = \left\{ u : X \to \mathbb{R}; \int_X |u(x,y)|^2 dxdy < \infty, \int_{\mathbb{S}^1} u(x,y)dy = 0 \right\}.$$

Soit $V \in \mathbb{V} = \{x^2\tilde{W}, \tilde{W} \in \mathcal{C}_b^0(\mathbb{R}), \tilde{W} \geqslant 1\}$ équipé de la norme $\left\| x^2\tilde{W} \right\|_{\mathbb{V}} = \left\| \tilde{W} \right\|_{\infty}$. Pour $V \in \mathbb{V}$, nous désignons par

$$P_V = -\partial_x^2 - V(x)\partial_y^2,$$

l'opérateur généralisé de Baouendi Grushin sur le domaine D défini par

$$D = \{u \in L_0^2(X); \partial_x^2 u \in L^2(X), V(x)\partial_y^2 u \in L^2(X)\}.$$

Nous disons que l'inégalité de concentration est vérifiée pour P_V s'il existe une constante $c = c(w)$ telle que pour toute fonction propre u de P_V, on a

$$\|u\|_{L^2(M)} \leqslant c\|u\|_{L^2(w)}.$$

Tout d'abord, nous donnons une condition suffisante pour que P_V vérifie l'inégalité de concentration. Nous montrons que

Theorem 0.5. *Si mult$(E) = 2$ pour chaque valeur propre E de P_V, alors l'inégalité de concentration est vérifiée.*

Ce théorème sera reformulé en tant que théorème 2.2. Observons qu'une valeur propre de P_V est toujours de multiplicité au moins 2

Nous montrons ensuite que la condition de la proposition 0.5 n'est pas vraie en général en étudiant les opérateurs obtenus en posant $V(x) = x^2 + s^2$. Ceux-ci ne sont pas à proprement parler dans la classe définie précédemment (pour $s > 0$, l'opérateur est elliptique) mais ils préfigurent la partie sur l'approximation riemannienne. Nous montrons les résultats suivants (Cor. 2.4.1 et Prop. 2.5) .

Theorem 0.6. *Si s^2 est rationnel, alors la multiplicité des valeurs propres de P_s n'est pas uniformément bornée.*

Theorem 0.7. *Si s^2 est irrationnel, alors mult$(E) = 2$ pour tous les $E \in$ spec$(P_{x^2+s^2})$.*

En tant que corollaire des théorèmes 0.5 et 0.7, nous obtenons que

Corollary 0.7.1. *Si s^2 est irrationnel, alors l'inégalité de concentration est vérifiée pour $P_{x^2+s^2}$.*

Ces résultats amènent à se poser la question de la validité des inégalités de concentration de manière générique. L'idée d'étudier la condition spectrale de manière générique réside dans le fait que cette condition est une condition de 'simplicité' sur les valeurs propres de l'opérateur non elliptique P_V. Un résultat général de simplicité des valeurs propres pour les opérateurs elliptiques a été introduit pour la première fois par Albert dans sa thèse

[6], puis généralisé ultérieurement pour le cas bidimensionnel dans son article [7]. Plus tard, Uhlenbeck a montré que le théorème était valable dans toutes les dimensions [81][82]. Nous montrons une variation du théorème d'Albert [8] et d'Uhlenbeck pour donner un résultat similaire pour l'opérateur sous-elliptique P_V. Ainsi, nous étudions les perturbations de l'opérateur de Baouendi Grushin et montrons le résultat suivant (Th. 2.20) Désignons par $\mathbb{V}_b = \{V \in \mathbb{V}; \exists E \in \operatorname{spec}(P_V), \operatorname{mult}(E) > 2\}$ les *mauvais* potentiels (pour lesquels la condition spectrale n'est pas vérifiée).

Theorem 0.8. *Le complément de $\mathbb{V}_b$ est résiduel dans l'espace topologique $(\mathbb{V}, \|.\|_{\mathbb{V}})$.*

Ce théorème signifie que, génériquement, les valeurs propres d'un opérateur de Baouendi Grushin sont de multiplicité 2. Selon le théorème 0.5, cela implique que l'inégalité de concentration est valable pour un opérateur de Baouendi Grushin générique.

0.2.2 Etude d'une approximation riemannienne

Soient $X^{01}, ..., X^{0p}$ des champs de vecteurs lisses sur M tels que $X^{01}, ..., X^{0p}$ véridient la condition de Hörmander de rang r, et supposons que

$$T_m M = \operatorname{span}\{X^{ij}\}_{0 \leqslant i \leqslant r, 1 \leqslant j \leqslant N_i},$$

où les vecteurs $(X^{ij})_{j \leqslant N_i}$ sont une famille particulière de crochets de longueur i des $X^{01}, ..., X^{0p}$. Soit $N = N_0 + ... + N_r$. Nous définissons la métrique sous-riemannienne g^0 comme suit

$$g^0_m(X_m) = \inf\left\{ |u|^2_{\mathbb{R}^p}; u \in \mathbb{R}^p, \sum_{j=1}^{p} u_j X^{0j}(m) = X(m) \right\}, \tag{11}$$

et nous désignons par d^0 sa distance sous-riemannienne associée. Nous définissons notre schéma d'approximation. Pour $u \in \mathbb{R}^N$, nous écrivons $u = (u_0, u_1, ..., u_r)$, où chaque u_i est de longueur N_i. Pour tout $h \in \mathbb{R}\backslash\{0\}$ et tout $u \in \mathbb{R}^N$, nous définissons la dilatation δ_h comme suit

$$\delta_h(u) = (u_0, h^{-1}u_1, h^{-2}u_2, ..., h^{-r}u_r). \tag{12}$$

Nous définissons la famille de métriques riemanniennes g^h comme suit

$$g_m^h(X(m)) = \inf \left\{ |\delta_h u|_N^2; u \in \mathbb{R}^N, \sum_{i=0}^{r} \sum_{j=1}^{N_i} u_{ij}(m) X^{ij}(m) = X(m) \right\}. \quad (13)$$

Désignons par d^h sa distance riemannienne associée. Comme nous l'avons mentionné précédemment, il est bien connu que la distance sous-riemannienne est la limite des distances riemanniennes, uniformément sur les ensembles compacts de la variété. Comme nous cherchons à étudier les mesures de volume, il est naturel d'étudier la mesure associée à la famille de métriques riemanniennes, $\mathrm{dvol} g^h$. Si nous fixons un certain repère, disons $Z_1, ..., Z_d$, alors

$$\mathrm{dvol} g^h = \sqrt{|\det(G_h)|} d\nu_1 \wedge ... \wedge d\nu_d,$$

où G_h est la matrice de représentation de g^h dans le repère et $\nu_1, ..., \nu_d$ est le repère dual à $Z_1, ..., Z_d$. C'est pourquoi nous nous concentrons sur l'étude du déterminant de G_h. Pour tout $0 \leqslant i \leqslant r$, désignons par A_i la matrice de représentation des vecteurs X^{ij} dans ces coordonnées. Nous démontrons la formule suivante qui exprime ${G_h}^{-1}$ en termes des A_i:

Proposition 0.9. *Nous avons*

$${G_h}^{-1} = \sum_{i=0}^{r} h^{2i} A_i^t A_i.$$

Cela sera reformulé en tant que théorème 3.10. Nous étudions ensuite le déterminant de ${G_h}^{-1}$. L'expression précédente implique que le déterminant de ${G_h}^{-1}$ est un polynôme en h. Cependant, cette information n'est pas suffisante pour étudier le comportement limite du déterminant. Nous décrivons alors le spectre de ${G_h}^{-1}$ en montrant le théorème suivant (Thm. 3.13) .

Theorem 0.10. *Fixons un point $m \in M$ et notons $(n_i)_{0 \leqslant i \leqslant r}$ le vecteur de croissance de la structure sous-riemannienne en m. Pour tout $0 \leqslant i \leqslant r$, il existe $n_i - n_{i-1}$ branches propres $\{\lambda_i^j(h)\}_{1 \leqslant j \leqslant n_i - n_{i-1}}$ de ${G_h}^{-1}$ telles que*

$$\lambda_i^j(h) = h^{2i} \eta_i^j(h),$$

avec $\lim_{h \to 0} \eta_i^i(h) \neq 0$ pour tout i, j.

En conséquence, nous obtenons que

$$\det(G_h^{-1}(m)) = f_h(m)h^{2\varsigma(m)},$$

avec $\varsigma(m) = \sum_1^r i(n_i(m) - n_{i-1}(m))$ (où $(n_0(m), n_1(m), ..., n_{r-1}(m), d)$ est le vecteur de croissance en un point m) et f_h converge ponctuellement vers une fonction positive f (à déterminer). Ensuite, nous énoncons le théorème ci dessous.

Theorem 0.11. *Sous l'hypothèse d'équirégularité, la fonction $m \mapsto f(m)$ est lisse, strictement positive, et indépendante du choix des coordonnées.*

En particulier, $(1/\sqrt{f})$ définit une densité $(1/\sqrt{f})dx$, sur M (voir Cor. 3.19.1).

Enfin, nous prouvons, en utilisant les propriétés intéressantes d'un repère adapté, que ce volume obtenue par cette approximation est relié au volume de Popp par le résultat suivant.

Theorem 0.12. *Le volume $d\mathcal{P}_o$ obtenue par l'approximation précédente coïncide avec le volume de Popp $d\mathcal{P}$ au un facteur $\sqrt{2^r}$ près, c'est-à-dire*

$$d\mathcal{P} = \frac{1}{\sqrt{2^r}} d\mathcal{P}_o.$$

Ce résultat sera énoncé dans le corollaire 3.64.

0.2.3 Sur la Convergence du Spectre

Nous supposons maintenant que M est compact et nous considérons le cadre précédent. Définissons l'espace de Hilbert $L^2_\omega(M)$ par rapport à une mesure de volume fixée ω comme suit :

$$L^2_\omega(M) := \{u : M \to \mathbb{R};\ \|u\|^2_{L^2_\omega} := \int_M |u|^2 d\omega < \infty\}.$$

Sur $L^2_\omega(M)$, nous définissons le sous-laplacien comme suit

$$\Delta_0 = \sum_{j=0}^{p} (X^{0j})^{*_\omega} X^{0j}, \tag{14}$$

où $*_\omega$ désigne l'adjoint par rapport à $d\omega$.

Définissons maintenant l'espace de Hilbert associé à $\mathrm{dvol}g^h$ comme suit :

$$L^2_h(M) := \{u : M \to \mathbb{R};\ \|u\|^2_{L^2_h(M)} := \int_M |u|^2 h^{2\varsigma} \mathrm{dvol}g^h < \infty\}. \tag{15}$$

Pour tout $h > 0$, nous définissons sur $L^2_h(M)$ la famille d'opérateurs elliptiques :

$$\tilde{\Delta}_h = \sum_{i=0}^{r} \sum_{j=1}^{N_i} h^{2i} (X^{ij})^{*_h} X^{ij}, \tag{16}$$

où $*_h$ désigne l'adjoint par rapport à $\mathrm{dvol}g^h$.

Nous commençons par adapter la preuve de Kohn pour démontrer une version uniforme indépendante du paramètre de l'estimation sous-elliptique locale célèbre. En utilisant le calcul pseudo-différentiel, nous démontrons (Prop. 4.12).

Theorem 0.13. *Sous les hypothèses précédentes.* $\exists \epsilon > 0, \forall s \in \mathbb{R}, \exists C(s) > 0, \forall h \in [0,1], \forall u \in \mathcal{C}^\infty(M)$,

$$\|u\|_{H^{\epsilon+s}_\omega(M)} \leqslant C(s) \left(\left\| \tilde{\Delta}_h u \right\|_{H^s_\omega(M)} + \|u\|_{H^s_\omega(M)} \right). \tag{17}$$

D'une part, ce théorème entraîne les faits bien connus sur le sous-laplacien; il est sous-elliptique, hypoelliptique, essentiellement auto-adjoint et a une résolvante compacte, et donc un spectre discret. D'autre part, il implique une estimation uniforme sur les fonctions propres de $\tilde{\Delta}_h$. En utilisant cette estimation et des théorèmes spectraux standards, nous prouvons ce qui suit (Thm. 4.20).

Theorem 0.14. *Pour une structure sous-riemannienne équirégulière. Soit $(h_n)_{n\geqslant 0}$ une suite qui tend vers 0 et $(u_n)_{n\geqslant 0}$ une suite de fonctions propres normalisées de $\tilde{\Delta}_{h_n}$. Soit $(\mu_n)_{n\geqslant 0}$ la suite associée de valeurs propres. Supposons que la suite $(\mu_n)_{n\geqslant 0}$ est bornée. Alors, les assertions suivantes sont vraies.*

1. *Il existe une sous-suite $(\mu_{n_k})_{k\geqslant 0}$ qui converge vers une valeur propre de Δ_0, disons λ.*

2. *En extrayant éventuellement une sous-suite, $(u_{n_k})_{k\geqslant 0}$ (correspondant à $(\mu_{n_k})_{k\geqslant 0}$) converge vers v_0 dans $H^l_\omega(M)$ pour tout l, et v_0 est une fonction propre de Δ_0 associée à λ.*

En tant que corollaire, nous obtenons alors le résultat (Thm. 4.21.)

Theorem 0.15. *Désignons par* $(\lambda_k)_{k\geqslant 0}$ *et* $(\tilde{\lambda}_k(h))_{k\geqslant 0}$ *le spectre ordonné de* Δ_0 *et* $\tilde{\Delta}_h$ *respectivement, compté avec multiplicité. Alors, pour tout* $k \geqslant 0$ *fixé, nous avons*

$$\lim_{h\to 0} \tilde{\lambda}_k(h) = \lambda_k. \tag{18}$$

Ce théorème dit donc que, dans le cas équirégulier, le spectre du laplacien riemannien associé à g^h converge vers les spectre du laplacien sous-riemannien relativement à la mesure de Popp. Dans le chapitre, nous montrerons un résultat similaire pour la suite des laplaciens relativement à une mesure ω fixée. Ce dernier ne nécessite alors pas l'hypothèse d'équirégularité.

0.3 Commentaires

Nous donnons quelques commentaires sur nos résultats. Ceux-ci seront précisés dans le manuscrit.

Nous soulignons que le théorème 0.5 (condition spectrale suffisante) ne dépend pas directement de l'opérateur de Baouendi Grushin généralisé, mais des propriétés de séparation des variables satisfaites par cet opérateur. Ainsi, ce théorème pourrait être généralisé en tant que condition suffisante pour la validité des inégalités de concentration pour d'autres opérateurs généraux.

En ce qui concerne le théorème principal (théorème 0.8), la théorie des perturbations analytiques de Kato est nécessaire, car nous utilisons le théorème de Hellmann-Feynman. Nous donnerons les éléments principaux de cette théorie pour le cas de dimension finie dans les annexes, et nous prouvons une généralisation dans le cas de dimension infinie adaptée à notre cadre. Nous verrons également que ce théorème fonctionne sur un tore.

En ce qui concerne la deuxième partie, il convient de noter que la mesure induite par le schéma d'approximation (qui dépend du choix des champs de vecteurs) n'est pas canonique (ou intrinsèque) à la structure sous-riemannienne. Nous verrons que cette mesure dépend de notre choix des champs de vecteurs qui engendrent l'espace tangent.

Enfin, commentons un peu sur la convergence du spectre. Tout d'abord, notre principal théorème concerne le spectre ordonné, et non les branches analytiques de valeurs propres (bien que, nous verrons que dans le cas de la mesure de volume fixe, la convergence des branches propres est vraie).

Maintenant, pour prouver les théorèmes sur la convergence du spectre de $\tilde{\Delta}_h$, nous prouvons d'abord les résultats pour Δ_h qui est défini par rapport à $d\omega$ au lieu de dvolg^h. Traiter avec dvolg^h est une tâche plus difficile en raison de la présence de fonctions dépendant de h. Cette difficulté sera surmontée grâce à l'hypothèse d'équiregularité. Sans cette hypothèse, les choses deviennent beaucoup plus difficiles et nos résultats sur la convergence du spectre riemannien ne sont valides que dans le cadre équiregulier.

Dans le cadre non-équiregulier, où l'ensemble singulier $\mathcal{Z}$ n'est pas vide, de nombreuses choses cessent de fonctionner. Par exemple, $M\backslash\mathcal{Z}$ n'est plus vide, et il n'est pas clair si l'estimation sous-elliptique (17) reste vraie. Par conséquent, nous n'avons aucune idée si $(\Delta_0, \mathcal{C}_0^\infty(M\backslash\mathcal{Z}))$ est essentiellement auto-adjoint. Il convient de noter que certains auteurs ont étudié cette question et ont donné des conditions sur la structure sous-riemannienne qui impliquent que l'opérateur $(\Delta_0, \mathcal{C}_0^\infty(M\backslash\mathcal{Z}))$ est essentiellement autoadjoint (voir [15][16][35][72]).

0.4 Comparaison Avec des Résultats Antérieurs

Bien que similaires, il existe plusieurs différences entre le corollaire 0.1.1 de Letrouit [62] et nos résultats. Tout d'abord, notre approche est purement spectrale, tandis que l'approche de Letrouit est plus géométrique. Deuxièmement, les paramètres sont différents. En effet, ici, nous traitons l'opérateur généralisé de Baouendi Grushin, pour une classe générale de potentiels $\mathbb{V}$, et nous prouvons la validité de l'inégalité de concentration sur le cylindre infini, un domaine non borné, ce qui est généralement plus difficile que de traiter avec des domaines bornés (nous verrons également que nos résultats s'appliquent à un tore, par exemple).

En ce qui concerne le schéma d'approximation, comme indiqué précédemment, bien que la mesure obtenue à la limite ne soit pas canonique pour la structure sous-riemannienne, mais soumise à un bon choix de champs de vecteurs, elle donne une mesure "naturelle" avec laquelle travailler.

De plus, le théorème 0.12 donne en réalité une manière de calculer le volume de Popp, en plus de celle donnée par les auteurs dans [13]. Il suffit en effet d'écrire les matrices A_i, d'écrire la matrice G_h^{-1} en utilisant la proposition 0.9 et de calculer le déterminant. Ensuite, en utilisant le vecteur de croissance, on en déduit la fonction f_h puis la mesure de Popp en utilisant le théorème 0.12 après avoir pris la limite lorsque $h \to 0$. Il est intéressant de noter que ce calcul peut se faire dans n'importe quel système de coordonnées, pas forcément adapté à la structure sous-riemannienne.

Notre résultat de convergence est aussi plus général que celui de Ge dans [40]. Notre construction est aussi plus adaptée à la configuration sous-riemannienne ; par exemple, la dilatation effectuée est associée au drapeau des sous-espaces vectoriels D_i, $0 \leqslant i \leqslant r$. Dans [40], la configuration a été adaptée d'une certaine manière à la preuve ; la forme de volume à laquelle il aboutit est, exprimée dans nos notations, $h^2 d\omega$. Une autre différence, qui peut ne pas avoir une grande influence, est que dans [40], il était supposé que les champs de vecteurs initiaux sont toujours un ensemble linéairement indépendant (base), ce qui signifie que D_0 a un rang constant. Ici, nous n'avions aucune hypothèse sur le rang de D_0 qui peut dépendre du point.

0.5 Plan du Manuscrit

Dans cette section, nous présentons le plan du manuscrit, où nous expliquons le contenu de chaque chapitre.

- **Chapitre 2:** Ce chapitre est consacré à l'étude de l'inégalité de concentration pour l'opérateur généralisé de Baouendi Grushin.
  - Dans la section 2.1, nous donnons quelques définitions et notations. Nous reformulons ensuite et prouvons le théorème 0.5.
  - Dans la section 2.2, nous examinons l'exemple $P_{x^2+s^2}$ et prouvons les théorèmes 0.6 et 0.7 (et déduisons le corollaire 0.7.1).
  - Dans la section 2.3, nous étudions le cas général et prouvons le théorème 0.8.
  - Dans la section 2.4, nous expliquons brièvement pourquoi notre résultat est valable sur un tore.
- **Chapitre 3:** Ce chapitre est dédié à l'étude de l'approximation riemannienne et de la mesure limite.
  - Dans la section 3.1, nous rappellons quelques définitions importantes et nous présentons le cadre.
  - Dans la section 3.2, nous introduisons notre schéma d'approximation. En particulier, nous prouvons que g^h est une famille de métriques

riemanniennes, puis nous adaptons la preuve de [40] pour démontrer la convergence des distances.

  - Dans la section 3.3, nous étudions la forme de volume $\mathrm{dvol}g^h$. En particulier, nous prouvons la proposition 0.9 et le théorème 0.10 ainsi que leurs conséquences.
  - Dans la section 3.4, nous définissons la forme de volume de Popp et la comparons à notre forme de volume. En particulier, nous prouvons le théorème 0.3.

- **Chapitre 4:** Ce chapitre est consacré à l'étude de la convergence du spectre.
  - Dans la section 4.2, nous étudions les estimations sous-elliptiques dans les deux cas : forme de volume fixe et $\mathrm{dvol}g^h$.
  - Dans la section 4.3, nous prouvons les théorèmes 0.14 et 0.15.

Nous verrons que notre travail repose sur la théorie de la perturbation en dimension finie et utilise certains théorèmes fondamentaux de la théorie spectrale. Pour cela, nous écrivons les annexes qui sont divisées en :

- Chapitre A: Dans ce chapitre, nous donnons quelques préliminaires de base en théorie spectrale fréquemment utilisés dans le manuscrit.
  - Dans la section A.1, nous présentons quelques théorèmes bien connus.
  - Dans la section A.2, nous étudions l'opérateur de Schrödinger.
  - Dans la section A.3, nous énonçons quelques théorèmes bien connus de bornitude pour les opérateurs pseudo-différentiels.
- Chapitre B: Dans ce chapitre, nous donnons brièvement la théorie de la perturbation de Kato dans le cadre de la dimension finie.
- Tout au long des chapitres 3 et 4, je donne des exemples qui sont courants dans le contexte de la géométrie sous-riemannienne. J'applique essentiellement mes résultats aux cas Grushin, Heisenberg et Martinet, qui sont tous des structures presque riemanniennes typiques (voir la définition 3.1.1.2).

Chapter 1

Introduction

1.1 Motivation And Previous Results

Consider a smooth manifold M of dimension d. We say that an operator P is an elliptic operator of order p in M if whenever $Pu \in H^s(M)$ it implies that $u \in H^{s+p}(M)$. Elliptic operators are a generalization of Laplace operators and their natural framework is the Riemannian geometry. A Riemannian manifold is a smooth manifold equipped with a positive-definite inner product called the Riemannian metric, usually defined using a set of vector fields that span the tangent space at every point.

If P is elliptic, then whenever $Pu \in \mathcal{C}^\infty(M)$ it implies that $u \in \mathcal{C}^\infty(M)$ (as u will be in $H^p(M)$ for any p). However, ellipticity is a sufficient condition and not necessary for the smoothness of solutions (one can take as a counter-example the heat equation which is a parabolic partial differential equation but has smooth solutions).

So, a hypoelliptic operator is defined as the operator P such that whenever $Pu \in \mathcal{C}^\infty(M)$ it gives that $u \in \mathcal{C}^\infty(M)$. The first one to notice such an operator was Kolmogorov back in 1934 in [59], while he was working on the motion of colliding particles when he wrote down the equation

$$\partial_t u - (x\partial_y + \partial_x^2)u = f,$$

and observed that the operator $x\partial_y + \partial_x^2$ is hypoelliptic.

A third class of operators also introduced are subelliptic operators. An operator P is subelliptic of order p with ϵ loss of derivatives, if whenever

$Pu \in H^s(M)$, it implies that $u \in H^{s+p-\epsilon}(M)$ for some $\epsilon \in]0,1[$. These classes are related as follows:

$$\text{Elliptic} \Rightarrow \text{Subelliptic} \Rightarrow \text{Hypoelliptic.}$$

Subelliptic operators, which have been studied intensively since the 20th century, are interesting as they are a generalization of elliptic operators that implies hypoellipticity.

A typical example of a subelliptic operator is a Grushin operator. Grushin operators were first defined in [44] in 1970, as the subelliptic operators of the form

$$P = -\partial_x^2 - |x|^{2k}\partial_y^2, \tag{1.1}$$

for some $k \in \mathbb{N}$ acting on $L^2(\mathbb{R}^2_{x,y})$. It was the starting point to define a Grushin-type operator that was then studied widely by many investigators (for instance in [21][28][37][53] and many others). An interesting Grushin-type operator is the following (it can be seen as the generalization for the case $k = 1$). On $\mathbb{R}^2$, the generalized Baouendi Grushin Operator (sometimes, we will say Baouendi Grushin Operator) is defined as

$$P = -\partial_x^2 - V(x)\partial_y^2, \tag{1.2}$$

for V satisfying the following:

$$V \in \mathcal{C}(\mathbb{R}), V(x) \geqslant 0, V(0) = V'(0) = 0, V''(0) > 0, \lim_{|x|\to+\infty} V(x) = +\infty. \tag{1.3}$$

Some investigators handled this kind of generalized Grushin operators (and other types) and most of them studied their controllability and/or observability in different domains ([9][10][21][26][31][57]). Controllability and observability are a 'strong' type of concentration inequalities that seems to imply another 'weaker' inequalities like the concentration inequalities for eigenfunctions.

The study of concentration inequalities for eigenfunctions has a rich history that spans several decades where several ways of measuring possible concentrations were raised (see [20][22]). Concentration inequalities for eigenfunctions are estimates of the probability that the eigenfunctions of a certain type of operator will be concentrated in a particular region of the underlying space, more specifically whether the magnitude of the eigenfunctions on the manifold can be controlled by its magnitude on some sub-domain.

The direct link between concentration properties for eigenfunctions and controllability/observability may vary depending on the specific system and control framework, however, under some settings, some investigators showed that one can be sufficient or necessary for the validity of the other. For instance, in [12], the author mentions that under some conditions the concentration inequality for eigenvalues is a sufficient condition for the exact observability of the wave equation associated with the Laplace operator (see page 1030 in [12]). On the other hand, more generally, the authors in [24] (where the concentration inequality for eigenfunctions of the Schrödinger operator is proved on a torus) mention that the concentration inequality for eigenfunctions can be derived (using Duhamel formula) from the observability of the Schrd̈inger equation. See [23][64] for detailed studies about the link between observability and concentration inequalities for eigenfunctions.

Anyway, several authors studied this type of concentrations on eigenfunctions of Laplace operator as in [17][20][23][33][46]. These studies depend usually on studying the validity of some geometric criteria called the geometric control criteria or conditions. Usually, a well-known sufficient condition for the validity of the concentration inequality for Grushin-type operators is the so-called Geometric Control Condition of Bardos-Lebeau-Rauch introduced in [12]. This condition says that all the trajectories of the generalized geodesic flow will enter the control region before some time. Although it is well known that this condition is equivalent to observability in many cases such as the wave equation (see [12][19]) and sufficient in others such as for the Schrödinger equation (see [60]), however, this condition is not necessary in general. In [46] for instance, the authors proved that the eigenfunction mass of the Dirichlet (or Neumann) Laplacian on an arbitrary polygonal domain, where the later condition fails to be true, cannot concentrate away from the vertices; that is the L^2 norm of the eigenfunction on a neighborhood U of the vertices is controlled by a constant $c = c(U)$ multiplied by the norm on the polygonal domain. However, they introduced another geometric condition called the cylindrical condition (see [46]). These all share the dependence of the methods used on a certain geometric control condition.

Here, we do not care about geometric conditions. We investigate a purely spectral condition that serves as a sufficient condition for the validity of concentration inequalities for eigenfunctions of the Baouendi Grushin operators on an arbitrary horizontal strip of the infinite cylinder. Specifically, our main interest here is to link the concentration inequality to some spectral condition, and then study this condition for the generalized Baouendi

Grushin operator. In fact, the concentration inequality could be known for $V(x) = x^2$ using some involved geometric methods. In [62, Chapter 3] for instance, the author considered a different case and proved the following: Let $M = (-1, 1)_x \times \mathbb{T}$ where $\mathbb{T}$ is the 1 dimensional torus in the y-variable, and let $\gamma \in \mathbb{R}^{*+}$. Define the Baouendi Grushin type operator

$$\Delta_\gamma = \partial_x^2 + |x|^{2\gamma}\partial_y^2, \tag{1.4}$$

with domain

$$D(\Delta_\gamma) = \{u \in D'(M), \partial_x^2 u \in L^2(M), |x|^{2\gamma}\partial_y^2 u \in L^2(M), u = 0 \text{ on } \partial M\}.$$

Theorem 1.1 (Letrouit [62])**.** *Let $\gamma \geqslant 1$ and let $w = (-1, 1)_x \times I$ for some interval I. Then, there exists $C, h_0 > 0$ such that for any $u \in D(\Delta_\gamma)$, and any $0 < h \leqslant h_0$,*

$$\|u\|_{L^2(M)} \leqslant C\left(\|u\|_{L^2(w)} + h^{-(\gamma+1)}\left\|(h^2\Delta_\gamma + 1)u\right\|_{L^2(M)}\right). \tag{1.5}$$

His proof for this theorem is geometric and depends on the validity of the geometric control condition. Because we are interested in eigenfunctions, we write a corollary of theorem 1.1.

Corollary 1.1.1 (Letrouit [62])**.** *Let $\gamma \geqslant 1$ and let $w = (-1, 1)_x \times I$ for some interval I. Then, there exists $C > 0$ such that for any eigenfunction u of Δ_γ, we have*

$$\|u\|_{L^2(M)} \leqslant C\|u\|_{L^2(w)}. \tag{1.6}$$

Here, we are interested in proving similar results using analysis on the spectrum of the operator, in a more general (or different) setup.

Observe from (1.2) that the generalized Baouendi Grushin operator (and any Grushin type operator) can be written as the sum of the square of two smooth vector fields $X_1 = \partial_x$ and $X_2 = \sqrt{V(x)}\partial_y$, and it is called some times the Grushin sublaplace operator. The model operator for these operators (and subelliptic operators in general) is the sublaplacian. It is a generalization of the Laplace operator in a Riemannian manifold.

Let $X_1, ..., X_p$ be smooth vector fields on a smooth manifold M. We say that $X_1, ..., X_p$ are bracket generating (or satisfy the Hörmander condition) of step r if $X_1, ..., X_p$ with their iterative brackets $[X_i, X_j], [X_i, [X_j, X_k]]$... up to length r span the tangent space at every point $m \in M$ (see [50]). The sublaplace operator with respect to a smooth volume form (a differential form of degree equal to the degree of the manifold M) ω is defined as

$$\Delta = -\sum_{i=1}^{p} X_i^* X_i = \sum_{i=1}^{p} X_i^2 + \operatorname{div}_\omega(X_i) X_i, \tag{1.7}$$

where $\operatorname{div}_\omega(X_i)$ denotes the divergence with respect to ω of X_i. These are called type 2 Hörmander operators (type one is just the sum of squares) as they were extensively studied by Hörmander. He proved that under the bracket-generating condition, Δ is hypoelliptic (see [50]). Hörmander proved this by proving that Δ is subelliptic and satisfies the following estimate (which is the definition of subellipticity): $\exists s, C > 0, \forall u \in \mathcal{C}_c^\infty(M)$,

$$\|u\|_{H^s(M)} \leqslant C \left(\langle \Delta u, u \rangle_{L^2(M)} + \|u\|_{L^2(M)} \right). \tag{1.8}$$

In fact, Rothschild and Stein proved in theorem 17 of [75] that $s = \frac{1}{r}$ is optimal. If M is compact, then (1.8) implies that $(\Delta, \mathcal{C}^\infty(M))$ is essentially selfadjoint, and the unique selfadjoint extension has compact resolvent. It follows that it has a discrete spectrum.

The general framework for such operators is subriemannian geometry. subriemannian geometry is a generalization of Riemannian geometry, where not all directions play the same role and some constraints are put for moving along the manifold. Subriemannian manifolds often occur in the study of constrained systems in classical mechanics, such as the motion of vehicles on a surface, the motion of robot arms, and the orbital dynamics of satellites (the motion is always forced by some constraints). More precisely, if a manifold is equipped with vector fields that don't span the tangent space, then they do not define a Riemannian metric. However, if these vectors satisfy the Hörmander condition, then we say that the manifold is subriemannian and we can define the subriemannian metric associated to $\mathscr{C} = \{X_1, ..., X_p\}$, defined on TM as[1]

$$g(m, X_m) = \inf \left\{ |u|_{\mathbb{R}^p}^2 ; u \in \mathbb{R}^p, \sum_{i=1}^{p} u_i X_i(m) = X(m) \right\}, \tag{1.9}$$

with the convention that $\inf\{\varnothing\} = +\infty$. The structure $(M, \mathscr{C}, g)$ is called a subriemannian structure. An interest in the study of subriemannian geom-

[1] We could define the subriemannian metric with respect to a general metric on $\mathbb{R}^p$, but this will slightly make any difference throughout the manuscript.

etry was increased in the mid-'80s, when it started by studying Heisenberg groups focusing on geometries of balls and geodesics.

In general, there are many techniques for studying subriemannian geometry, including Subriemannian calculus and control theory (see [78]). Another approach that a lot of investigators use is to see the sublaplace structure as a singular limit of Riemannian ones (see [4][27][36][40][43][67][76][83]). These Riemannian approximations when gathered with uniform estimates allow to extend some Riemannian results into subriemannian settings. For example, the authors in [27] considered such approximations to generalize some known Riemannian estimates (doubling property, Poincare inequality, Gaussian estimates...) to subriemannian manifolds.

This convergence is seen in terms of convergence of distances. More precisely, let $(M, \mathscr{C}, g)$ be a subriemannian structure. The Chow–Rashevskii theorem, known as Chow's theorem, ensures that any two points of a connected subriemannian manifold, endowed with a bracket generating distribution, are connected by a horizontal path in the manifold (see [3][52]). So, g defines a distance d on M. Usually, a family of Riemannian metrics g^h with its corresponding family of Riemannian distances d^h are introduced, and then d^h is proved to satisfy the following.

Theorem 1.2. *The family of distances d^h converges uniformly to d on every compact set of the manifold M.*

It is clear from its expression, that the sublaplace operator depends on the chosen volume in (1.7). In a Riemannian setting, a canonical volume form can be obtained using the Riemannian metric. However, this is not the case in subriemannian manifolds as subriemannian metrics are not defined on the whole tangent space and there is no canonical way to extend it to the whole tangent space. So, a very natural question arises here: Can we define a canonical volume form on a subriemannian manifold?

The question was initially brought to attention by Brockett in 1982, in his paper (see [18]). His motivation stemmed from the desire to construct a Laplace operator on a three-dimensional Subriemannian manifold, which would be intrinsically connected to the metric structure, analogous to the Laplace-Beltrami operator on a Riemannian manifold. In more recent times, Montgomery tackled this problem in a more general context[66]. Popp's volume, Hausdorff volume and spherical Hausdorff volume are some interesting examples of canonical volume forms on a subriemannian manifold (for information on Hausdorff volume and spherical Hausdorff volume, see [2][42][65]

and the references within).
Popp's volume for instance was first defined by Octavian Popp but introduced only by Montgomery in [66]. Popp's volume is defined by inducing a canonical inner product on the graded vector space using Lie brackets, and then using a non-canonical isomorphism between the graded vector space and the tangent space to define an inner product on the whole tangent space. In 2013, the authors in [13] proved an expression for Popp's volume in terms of an adapted frame, and since then, this formula has sometimes been taken in this context as the definition of Popp's volume. More precisely, Consider an adapted frame $Z_1, ..., Z_d$ (defined in the sense of [13]) and define recursively the subspaces $D_i = D_{i-1} + [D_0, D_{i-1}]$, where D_0 is the span of the initial vector fields. Informally, the adapted structure constants are the coefficients of the vector fields in D_i modulo D_{i-1} (only consider the coefficients of the vector fields obtained by bracketing of length i).

Theorem 1.3 (Barilari-Rizzi[13])**.** *In the frame $Z_1, ..., Z_d$, Popp's volume is given by*

$$d\mathcal{P} = \frac{1}{\sqrt{\prod_j \det(B_j)}} d\nu_1 \wedge ... \wedge d\nu_d, \tag{1.10}$$

where B_j are matrices that are defined using the adapted structure constants and $\nu_1, ..., \nu_d$ the frame dual to the adapted frame.

Usually, With the family of Riemannian structures (and Riemannian metric) approximating the subriemannian structure (resp. subriemannian metric), one can define naturally an associated family of Laplace operators Δ_h. These Δ_h's are elliptic operators, and so, if M is compact, they have compact resolvent and consequently a discrete spectrum.

In this case, where the Riemannian structure collapses to a subriemannian one, and where the limiting operator of the singular perturbation is hypoelliptic, only a little is known about the convergence of the spectrum of the Laplacians (see [39][40][76]). In some specific settings, it was shown that the family Δ_h converges to Δ, and that each eigenvalue of Δ_h converges to those of Δ. This was first observed by Fukaya in [39] and then proved by Ge in [40] (See also [76] for contact manifold case). More precisely, let M be a compact manifold equipped with a Riemannian metric g. Let H be a distribution on M of constant dimension and let $H^\perp$ denote the distribution orthogonal to H. Write $g = g_H \oplus g_{H^\perp}$. Define the family of Riemannian metric for $h > 0$,

$$g_h = g_H \oplus h^{-2} g_{H^\perp}.$$

Theorem 1.4 (Ge [40]). *Let Δ_h be the Laplacian associated to g_h. Then, Δ_h converges as $h \to 0$ to a second order subelliptic operator*

$$\Delta_H = -\sum_i e_i^2,$$

where e_i is an orthonormal frame for H. Moreover, if $\lambda_1(h) \leqslant \lambda_2(h) \leqslant ....$ and $\lambda_1 \leqslant \lambda_2 \leqslant ....$ denote the eigenvalues of Δ_h and Δ_H respectively, then $\lambda_k(h)$ converges as $h \to 0$ to λ_k.

In the next section, I state the results of my work.

1.2 Main Results

Our work will be distributed into three chapters. So, we will divide this section into three subsections, each will contain the results of a chapter.

1.2.1 About Concentration Inequality

Denote by $X = \mathbb{R} \times \mathbb{S}^1$ the infinite cylinder in $\mathbb{R}^2$ and by $w = \mathbb{R} \times [a, b]$, a horizontal strip along X. Denote by

$$L_0^2(X) = \left\{ u : X \to \mathbb{R}; \int_X |u(x,y)|^2 dxdy < \infty, \int_{\mathbb{S}^1} u(x,y)dy = 0 \right\}.$$

Let $V \in \mathbb{V} = \{x^2\tilde{W}, \tilde{W} \in \mathcal{C}_b^0(\mathbb{R}), \tilde{W} \geqslant 1\}$ equiped with the norm $\left\| x^2\tilde{W} \right\|_{\mathbb{V}} = \left\| \tilde{W} \right\|_{\infty}$. For $V \in \mathbb{V}$, we denote by

$$P_V = -\partial_x^2 - V(x)\partial_y^2,$$

the generalized Baouendi Grushin Operator on D defined by

$$D = \{u \in L_0^2(X); \partial_x^2 u \in L^2(X), V(x)\partial_y^2 u \in L^2(X)\}.$$

We say that the concentration inequality holds for P_V if there exist a constant $c = c(w)$ such that for any eigenfunction u of P_V, we have

$$\|u\|_{L^2(M)} \leqslant c\|u\|_{L^2(w)}.$$

First, we give a sufficient condition for P_V to satisfy the concentration inequality. We prove that

Theorem 1.5. *If mult$(E) = 2$ for every eigenvalue E of P_V, then the concentration inequality holds.*

This theorem will be restated as theorem 2.2.

However, this is not always the case; the condition in proposition 1.5 is not true in general. This will be ensured by studying the special case, where $V(x) = x^2 + s^2$. For instance, we prove the following.

Theorem 1.6. *If s^2 is rational, then the multiplicity of the eigenvalues of P_s is not uniformly bounded.*

This theorem will be restated as corollary 2.4.1.

Theorem 1.7. *If s^2 is irrational, then mult$(E) = 2$ for all $E \in$ spec$(P_{x^2+s^2})$.*

This theorem will be restated as proposition 2.5. As a corollary of theorems2.2 and 1.7, we get that

Corollary 1.7.1. *If s^2 is irrational then the concentration inequality holds for $P_{x^2+s^2}$.*

This gave the inspiration to ask about the validity of concentration inequalities generically. The idea of studying the spectral condition generically is the fact that this condition is a 'simplicity' condition on the eigenvalues of the non-elliptic operator P_V. A general result of simplicity of eigenvalues for elliptic operators was first introduced by Albert in his thesis [6] and proved later for two dimensional case in his paper [7]. Later, Uhlenbeck showed that the theorem does hold in all dimensions [81][82]. We prove a variation of Albert's [8] and Uhlenbeck's to prove a similar result for the subelliptic operator P_V. So, we study the perturbation of the Baouendi Grushin operator and prove the following result. Denote by $\mathbb{V}_b = \{V \in \mathbb{V}; \exists E \in \mathrm{spec}(P_V), \mathrm{mult}(E) \neq 2\}$.

Theorem 1.8. *The complement of $\mathbb{V}_b$ is residual in $(\mathbb{V}, \|.\|_{\mathbb{V}})$.*

This theorem will be restated as theorem 2.20. This theorem says that generically, the eigenvalues of a Baouendi Grushin operator have multiplicity 2. By theorem 1.5, it implies that the concentration inequality is valid for a generic Baouendi grushin operator.

1.2.2 About Approximation Scheme

Let $X^{01}, ..., X^{0p}$ be smooth vector fields on M such that $D_0 = \text{span}\{X^{01}, ..., X^{0p}\}$ satisfies the Hörmander condition of step r. For $0 \leqslant i \leqslant r$, we define recursively

$$D_i = D_{i-1} + [D_0, D_{i-1}],$$

and we let $n_i = \dim(D_i)$. Then by Hörmander condition, $n_r = d$ and $D_r = T_m M$.

For some enumeration of the vector fields, we have

$$T_m M = \text{span}\{X^{ij}\}_{0 \leqslant i \leqslant r, 1 \leqslant j \leqslant N_i},$$

where X^{ij} is a vector obtained by i iterative brackets of $X^{01}, ..., X^{0p}$.

Let $N = N_0 + ... + N_r$. We define the subriemannian metric g^0 following 1.9 as

$$g(m, X_m) = \inf \left\{ |u|^2_{\mathbb{R}^p}; u \in \mathbb{R}^p, \sum_{j=1}^{p} u_j X^{0j}(m) = X(m) \right\},$$

and denote by d^0 its associated subriemannian distance.

We define our approximation scheme. For $u \in \mathbb{R}^N$, we write $u = (u_0, u_1, ..., u_r)$, where each u_i is of length N_i. For all $h \in \mathbb{R} \backslash \{0\}$ and all $u \in \mathbb{R}^N$, define the dilation δ_h as

$$\delta_h(u) = (u_0, h^{-1}u_1, h^{-2}u_2, ..., h^{-r}u_r).$$

We define the family of Riemannian metrics g^h as

$$g^h_m(X_m) = \inf \left\{ |\delta_h u|^2_N; u \in \mathbb{R}^N, \sum_{i=0}^{r} \sum_{j=1}^{N_i} u_{ij}(m) X^{ij}(m) = X(m) \right\}. \tag{1.11}$$

Denote by d^h its associated Riemannian distance. As we said earlier, it is well-known that the subriemannian distance is the limit of a family of Riemannian distances uniformly on compact sets of the manifold. As we aim at studying volume forms, it is natural to study the volume form associated to the family of Riemannian metrics, $\text{dvol}g^h$. If we fix some frame, say $Z_1, ..., Z_d$, then

$$\text{dvol}g^h = \sqrt{|\det(G_h)|} d\nu_1 \wedge ... \wedge d\nu_d,$$

where G_h is the representation matrix of g^h in the frame and $\nu_1, ..., \nu_d$ is the frame dual to $Z_1, ..., Z_d$. That is why we focus on studying the determinant of G_h. For any $0 \leqslant i \leqslant r$, denote by A_i the representation matrix of the vectors X^{ij} obtained by bracketing of length i in these coordinates. Then we prove an expression of G_h^{-1} in terms of the $A_i's$:

Proposition 1.9. *We have*

$$G_h{}^{-1} = \sum_{i=0}^{r} h^{2i} A_i^t A_i.$$

This will be restated as theorem 3.10. This is a nice expression of G_h^{-1}, and so we study the determinant of G_h^{-1} which conventionally implies information about the determinant of G_h. This expression implies that the determinant of G_h^{-1} is a polynomial in h. However this information is not enough to study the limiting behavior of the determinant, and so we aim to describe the spectrum of G_h^{-1}.

Recall that for $0 \leqslant i \leqslant r$, $n_i = \dim(D_i)$, and set $n_{-1} = 0$. We prove that

Theorem 1.10. *Fix a point $m \in M$. For any $0 \leqslant i \leqslant r$, there are $n_i - n_{i-1}$ eigenbranches $\{\lambda_i^j(h)\}_{1 \leqslant j \leqslant n_i - n_{i-1}}$ of G_h^{-1} such that*

$$\lambda_i^j(h) = h^{2i} \eta_i^j(h),$$

with $\lim_{h \to 0} \eta_i^i(h) \neq 0$ *for any* $0 \leqslant i \leqslant r, 1 \leqslant j \leqslant n_i - n_{i-1}$.

This theorem will be restated as theorem 3.13. As a corollary, we get that

$$\det(G_h^{-1}(m)) = f_h(m) h^{2\varsigma(m)},$$

with $\varsigma(m) = \sum_1^r i(n_i(m) - n_{i-1}(m))$ (where $(n_0(m), n_1(m), ..., n_{r-1}(m), d)$ is the growth vector at a point m) and $f_h(m)$ converges pointwisely, as $h \to 0$, to a positive function $f(m)$ (to be determinied). Also, we prove the following.

Theorem 1.11. *Under the equiregularity assumption, the function $m \mapsto f(m)$ is smooth, non-vanishing, and independent of the choice of coordinates.*

In particular, $(1/\sqrt{f})$ defines a volume form, $(1/\sqrt{f})dx$ on M. This theorem will be restated as corollary 3.19.1.

We then recover the same result using the nice properties of an adapted frame. We prove moreover, that this volume form, induced from the approximation scheme, satisfies the following:

Theorem 1.12. *The volume form $d\mathcal{P}_o$ induced from the approximation scheme coincides with Popp's volume $d\mathcal{P}$ up to a multiplying by $\sqrt{2^r}$; that is,*

$$d\mathcal{P} = \frac{1}{\sqrt{2^r}} d\mathcal{P}_o.$$

This will be stated in corollary 3.64.

1.2.3 About Convergence Of Spectrum

Suppose M is compact and orientable. Consider the previous framework, and suppose the subriemannian structure is equiregular. Define the Hilbert space $L^2_\omega(M)$ with respect to a fixed volume form ω as follows:

$$L^2_\omega(M) := \{u : M \to \mathbb{R};\ \|u\|^2_{L^2_\omega} := \int_M |u|^2 d\omega < \infty\}.$$

On $L^2_\omega(M)$, we define the sublaplace operator as

$$\Delta_0 = \sum_{j=0}^{p} (X^{0j})^{*_\omega} X^{0j}, \tag{1.12}$$

where the star denote the adjoint with respect to $d\omega$.

Now, define the Hilbert space associated to $h^{2\varsigma}\mathrm{dvol}g^h$, by

$$L^2_h(M) := \{u : M \to \mathbb{R};\ \|u\|^2_{L^2_h(M)} := \int_M |u|^2 h^{2\varsigma}\mathrm{dvol}g^h < \infty\}.$$

For any $h > 0$, we define on $L^2_h(M)$, the family of elliptic operators:

$$\tilde{\Delta}_h = \sum_{i=0}^{r} \sum_{j=1}^{N_i} h^{2i} (X^{ij})^{*_h} X^{ij},$$

where the star denotes the adjoint with respect to $\mathrm{dvol}g^h$.

We adapt Kohn's proof to prove a uniform parameter-dependent version of the famous local subelliptic estimate using pseudo-differential calculus. For instance, we prove the following:

Theorem 1.13. *The following holds true: $\exists \epsilon > 0, \forall s \in \mathbb{R}, \exists C(s) > 0, \forall h \in [0, h_1], \forall u \in \mathcal{C}^\infty(M)$,*

$$\|u\|_{H^{\epsilon+s}_\omega(M)} \leqslant C(s) \left(\left\| \tilde{\Delta}_h u \right\|_{H^s_\omega(M)} + \|u\|_{H^s_\omega(M)} \right). \tag{1.13}$$

This will be restated as proposition 4.12.

On one hand, this theorem covers the well-known facts about the sublaplacian; it is subelliptic, hypoelliptic, essentially selfadjoint, and has compact resolvent, and thus a discrete spectrum. On the other, it implies a uniform estimate on the eigenfunctions of $\tilde{\Delta}_h$. Using this estimate, and some standard spectral theorems, we prove the following.

Theorem 1.14. *Let $(h_n)_{n\geqslant 0}$ be a sequence that goes to 0 and $(u_n)_{n\geqslant 0}$ be a sequence of normalized eigenfunctions of $\tilde{\Delta}_h$. Let $(\mu_n)_{n\geqslant 0}$ be the associated sequence of eigenvalues. Assume that the sequence $(\mu_n)_{n\geqslant 0}$ is bounded. Then, the following assertions hold true.*

1. *There exist a subsequence $(\mu_{n_k})_{k\geqslant 0}$ that converges to an eigenvalue of Δ_0, say λ.*

2. *Up to extracting a subsequence, $(u_{n_k})_{k\geqslant 0}$ (that corresponds to $(\mu_{n_k})_{k\geqslant 0}$) converges to v_0 in $H^l_\omega(M)$ for any l, and v_0 is an eigenfunction of Δ_0 associated to λ.*

This theorem will be restated as theorem 4.20. As a corollary, we get

Theorem 1.15. *Denote by $(\lambda_k)_{k\geqslant 0}$ and $(\tilde{\lambda}_k(h))_{k\geqslant 0}$ the ordered spectrum of Δ_0 and $\tilde{\Delta}_h$ respectively, counted with multiplicities. Then, for any $k \geqslant 0$ fixed, we have*

$$\lim_{h\to 0} \tilde{\lambda}_k(h) = \lambda_k. \tag{1.14}$$

This theorem will be restated as theorem 4.21.

1.3 Some Comments On The Results

We give some comments on our results. All these comments will be made precise in the sequel.

We point out that theorem 1.5 (spectral sufficient condition) does not depend directly on the Baouendi Grushin operator, but on some properties satisfied by this operator. So, this theorem could be generalized as a sufficient condition for the validity of concentration inequalities for other general operators.

Now, concerning the main theorem (theorem 1.8), Kato's theory of analytic perturbation is needed, as we use the Hellmann–Feynman theorem.

However, what we do here, is that we give the theory for finite-dimensional case in the appendices, and prove a generalization in the infinite-dimensional case adapted to our framework. We shall see also, that this theorem works on a torus.

Regarding the second part, we should point out, that the volume form induced from the approximation scheme (which is in some sense adapted to the subriemannian structure) is not canonical (or intrinsic) to the subriemannian structure. We shall see that this volume form depends on our choice of the vector fields that span the tangent space.

Finally, let's comment a little about the convergence of the spectrum. First notice that our main theorem is about ordered spectrum, and not about eigenbranches (though, we shall see that in the fixed volume form case, the convergence of eigenbranches is true).

Now, To prove the theorems on the convergence of spectrum of $\tilde{\Delta}_h$, we first prove the results for Δ_h which is defined with respect to $d\omega$ instead of $\mathrm{dvol}g^h$. Dealing with $\mathrm{dvol}g^h$ is a harder task because the presence of h-dependent function f_h will interrupt some uniform estimates. This difficulty will be surpassed due to the equiregularity assumption mainly. Without this assumption, things get much more difficult. Indeed, our results for the convergence of spectrum works only in an equiregular settings.

In the non-equiangular setting, where the singular set $\mathcal{Z}$ is non-empty, many things will fail to work. For instance, $M\backslash\mathcal{Z}$ is not compact anymore, and it is not clear whether the subelliptic estimate (1.13) remains true. As a consequence, we have no idea now if $(\Delta_0, \mathcal{C}_0^\infty(M\backslash\mathcal{Z}))$ is essentially self-adjoint. It is worth noting that some authors have studied this question, and proved that basically, some conditions on the singular set $\mathcal{Z}$ implies the essential selfadjointness of $(\Delta_0, \mathcal{C}_0^\infty(M\backslash\mathcal{Z}))$ (see [15][16][35][72]).

1.4 Comparizon With Previous Results

Although similar, there are several differences between corollary 1.1.1 by Letrouit [62] and our result. First of all, as explained to be the crucial point here, is that our approach is purely spectral, while his approach is geometric (depends on geometric conditions). Second, the settings are different. Indeed here, we deal with the generalized Baouendi Grushin operator, for a general class of potentials $\mathbb{V}$, and prove the validity of the concentration inequality on the infinite cylinder, an unbounded domain, which dealing with usually is harder than dealing with bounded domains (we will also see that our results hold on a torus for instance).

Concerning the approximation scheme, as said earlier, although the volume form induced from it is not canonical to the subriemannian structure, but subjected to a good choice of vector fields, it gives a 'nice' volume form to deal with.

Moreover, a particular (a very good) choice of the complement of D_0 implies that this volume form induced from the approximation scheme is the Popp's volume. As we can see, theorem 1.12 actually gives a way to compute Popp's volume, other than that given by the authors in [13]. What we have to do, is to write the matrices A_i, write the matrix G_h^{-1} using proposition 1.9, and compute the determinant. Then, using the growth vector we deduce the function f_h. We deduce Popp's volume using theorem 1.12 after taking the limit as $h \to 0$.

Finally, our convergence result is much more general than that of Ge in [40]. In fact, our setup is more adapted to the subriemannian setting; for instance, the dilation taken is associated to the vector space D_i, $0 \leqslant i \leqslant r$. It implies the existence of the function f_h which needs a special treatment. In [40], the setting was adapted somehow to the proof; the volume form he ended up with is, expressed in our notations, $h^c d\omega$ for some constant c. This is because in [40], the author supposed that the initial vector fields are always a spanning linearly independent set (basis) which means that D_0 is of constant rank. Here, we had no assumption on the rank of D_0, which can be spanned with a random number of vector fields.

1.5 Plan Of The Manuscript

Here we give the plan of the manuscript, where we explain the contents of every chapter.

- **Chapter 2:** This chapter is dedicated to studying the concentration inequality for the generalized Baouendi Grushin operator.
  - In section 2.1, we give some definitions and notations. We then restate and prove theorem 1.5.
  - In section 2.2, we investigate the example $P_{x^2+s^2}$ and prove theorems 1.6 and 1.7 (and deduce corollary 1.7.1).

  - In section 2.3, we study the general case, and prove theorem 1.8.
  - In section 2.4, we briefly explain why our result holds true on a torus.

- **Chapter 3:** This chapter is dedicated to studying the approximation scheme, and the volume form it induces.
  - In section 3.1, I recall some important definitions, that form the keywords for the chapter. I then introduce my framework.
  - In section 3.2, we introduce our approximation scheme. In particular, we prove that g^h is a family of Riemannian metrics, and then adapt the proof of [40] to prove the convergence of distances.
  - In section 3.3, we study the volume form $\mathrm{dvol}g^h$. In particular, we prove proposition 1.9 and theorem 1.10 and their consequences.
  - In section 3.4, we define Popp's volume and compare it to our volume form. In particular, we prove theorem 1.3.

- **Chapter 4:** This chapter is dedicated to studying the convergence of the spectrum.
  - In section 4.2, we study the subelliptic estimates in both cases: fixed volume form and $\mathrm{dvol}g^h$.
  - In section 4.3, we prove theorems 1.14 and 1.15.

We will see that our work is based on the finite dimensional perturbation theory, and will use some fundamental theorems from spectral theory. For that, we write the appendices that are divided into:

- Chapter A: In this chapter, we give some basic preliminaries in spectral theory that are frequently used in the manuscript.
  - In section A.1, we give some general well-known theorems.
  - In section A.2, we study the Schrödinger operator.
  - In section A.3, we state some well-known boundedness theorems for pseudo-differential operators.

- Chapter B: In this chapter, we briefly give Kato's perturbation theory in the finite dimensional setting.

- Throughout chapters 3 and 4, I give examples that are standard in the context of subriemannian geometry. I basically apply my results to the Grushin case, Heisenberg case, and the Martinet case which are all typical almost Riemannian structures (see definition 3.1.1.2).

Table 1.1: Table of Notation

card$\{\}$	Cardinal of the set
$f = O(g)$ at x_0	$\lim_{x\to x_0}(f(x)/g(x)) \leqslant c$
$f = o(g)$ at x_0	$\lim_{x\to x_0}(f(x)/g(x)) = 0$
$f \sim g$ near x_0	$f(x) = g(x) + o(g(x))$ at x_0
$\mathrm{mult}(E)$	multiplicity of an eigenvalue E
T	will always denote an operator
$\mathrm{spec}(T)$	spectrum of T
$\ker(T)$	kernel of a function T
$\mathrm{Im}(T)$	Image of a function T
$\delta_{i,j}$	kronecker delta
$\mathrm{supp}(u)$	support of a function u
A^t	transpose of a matrix A
$\det(A)$	determinant of a matrix A
$\mathbb{M}_{i\times j}$	space of $i \times j$ matrices
$\dim(V)$	dimension of a vector space V
$\exists$	there exist(s)
$\forall$	for all
$\mathscr{D}(M)$	space of test functions on M
$\mathscr{D}'(M)$	space of distributions on M (dual space of $\mathscr{D}(M)$)
$\lvert u\rvert_n$	Euclidean norm of a vector $u \in \mathbb{R}^n$ given by $(\sum_{i=1}^n u_i^2)^{\frac{1}{2}}$

Chapter 2

Eigenvalue Multiplicity And Concentration Properties Of Baouendi-Grushin Operators

We prove a generic simplicity result on the multiplicity of the eigenvalues of the generalized Baouendi Grushin operator that implies the validity of concentration inequality for eigenfunctions.

2.1 Introduction

Studying concentration inequalities for eigenfunctions is studying whether the magnitude of the eigenfunctions on the manifold can be controlled by its magnitude on some sub-domain. Mathematically speaking, for a smooth connected manifold M, an operator T defined on $L^2(M)$ is said to satisfy the concentration inequality on a subset N of M (control region) if

$$\exists c(N) > 0, \forall E \in \mathrm{spec}(T), \forall \phi \in \ker(T - E), \quad \|\phi\|_{L^2(M)} \leqslant c(N) \|\phi\|_{L^2(N)} . \tag{2.1}$$

Here, we consider the generalized Baouendi Grushin operator on an infinite cylinder and study the validity of the concentration inequality on a horizontal strip of the infinite cylinder. We prove that a certain spectral condition is sufficient for the concentration inequality and that this spectral condition holds for a generic Baouendi-Grushin operator. First, we introduce our setup and define the Generalized Baouendi Grushin Operator.

2.1.1 Definitions And Notations

Denote by X the infinite cylinder in $\mathbb{R}^2$, $X = \mathbb{R} \times \mathbb{S}^1$, with fundamental domain $\mathbb{R} \times [-\pi, \pi]$, and by $w = \mathbb{R} \times [a, b]$, a horizontal strip along X. We define the following (sub)spaces:

- Denote by $L^2_0(X)$, the Hilbert space defined as

$$L^2_0(X) = \left\{ u : X \to \mathbb{R}; \int_X |u(x,y)|^2 dxdy < \infty, \int_{\mathbb{S}^1} u(x,y)dy = 0 \right\}, \tag{2.2}$$

  equipped with the usual L^2 norm. Note that the last condition in the definition of $L^2_0(X)$ is equivalent to saying that $L^2_0(X)$ is the orthogonal complement, in $L^2(X)$, of the functions that only depend on x. We will see later that this condition is necessary for P_V to have a discrete spectrum.

- Denote by $\mathcal{C}^\infty_{c,0}(X)$ the set of smooth functions of compact support on X that are in $L^2_0(X)$, equipped with the supremum norm on X.

- Define the uniform norm on $\mathbb{R}$ as follows

$$\|W\|_\infty := \|W\|_{L^\infty(\mathbb{R})} := \sup_{x \in \mathbb{R}} |W(x)|. \tag{2.3}$$

- Denote by $\mathcal{C}^0_b(\mathbb{R})$ the space of continuous bounded functions on $\mathbb{R}$, equipped with the uniform norm on $\mathbb{R}$.

- Denote by $\mathcal{C}^\infty_c(\mathbb{R})$ the space of smooth functions with compact support on $\mathbb{R}$, equipped with the uniform norm on $\mathbb{R}$.

- Denote by $\mathbb{W}$ the subspace of $\mathcal{C}^\infty_c(\mathbb{R})$, $\mathbb{W} = \{W \in \mathcal{C}^\infty_c(\mathbb{R}); W \geqslant 0\}$, equipped with the uniform norm defined by (2.3).

2.1.1.1 The Generalized Baouendi Grushin Operator

We introduce the set

$$\mathbb{V} := \{V = x^2 \tilde{W}; \tilde{W} \in \mathcal{C}^0_b(\mathbb{R}), \tilde{W} \geqslant 1\}.$$

For $V \in \mathbb{V}$, we set $X_1 = \partial_x$ and $X_2 = \sqrt{V}\partial_y$. We define the generalized Baouendi-Grushin (hypoelliptic) operator as

$$P_V := -X_1^2 - X_2^2 = -\partial_x^2 - V(x)\partial_y^2. \tag{2.4}$$

The operator P_V with domain $\mathcal{C}_{c,0}^\infty(X)$ is essentially self-adjoint on $L_0^2(X)$ and the unique self-adjoint extension with domain

$$D = \{u \in L_0^2(X); \partial_x^2 u \in L^2(X), V(x)\partial_y^2 u \in L^2(X)\}, \tag{2.5}$$

has compact resolvent (see remark 2.1 below). Consequently, its spectrum consists of an increasing sequence of positive (as the operator is self-adjoint and bounded from below by 0) eigenvalues with finite multiplicity, that converges to $+\infty$.

Although it is not our interest in this chapter, it is good to note that if we denote by g the subriemannian metric associated to $\mathscr{C} = \{X_1, X_2\}$, then $(X, \mathscr{C}, g)$ is a 2 almost-Riemannian structure (see subsection 3.1.1.2, Chapter 3).

2.1.1.2 One Dimensional Schrödinger Operator

For a non-negative continuous function V satisfying $\lim_{|x|\to+\infty} V(x) = +\infty$, define the one dimensional (parameter dependent) Schrödinger operator

$$P_V^k := -\partial_x^2 + k^2 V(x),$$

with domain $\mathcal{C}_c^\infty(\mathbb{R})$, defined on $L^2(\mathbb{R})$ (the usual Sobolev space on $\mathbb{R}$). It is well-known that the operator $(P_V^k, \mathcal{C}_c^\infty(\mathbb{R}))$ is essentially self-adjoint, and that the domain of the self-adjoint extension satisfies

$$\mathbb{D}_V \subset \{u \in H^1(\mathbb{R}), V^{\frac{1}{2}} u \in L^2(\mathbb{R})\},$$

where $H^1(\mathbb{R}) = \{u \in L^2(\mathbb{R}); u' \in L^2(\mathbb{R})\}$. For $k \neq 0$, this operator has compact resolvent. For sake of completion, we give a proof in Appendix A.2. Consequently, the spectrum of P_V^k consists in an increasing sequence of positive (as the operator is self-adjoint and bounded from below by 0) eigenvalues with finite multiplicity, that converges to $+\infty$, and $L^2(\mathbb{R})$ has an orthonormal basis that consists of eigenfunctions.

2.1.2 Geometric Control Condition

A well-known sufficient condition for the concentration inequality is the so-called Geometric Control Condition of Bardos-Lebeau-Rauch introduced in [12] (see also [23]). This condition says that all the trajectories of the generalized geodesic flow will enter the control region before some time.
This condition is not necessary in general. In [46] for instance, the authors proved that the L^2 norm of the eigenfunction on a neighborhood U of the vertices, where the geometric control condition fails, is controlled by a constant $c = c(U)$ multiplied by the norm on the polygonal domain (see also [51]).
In our case, the Hamiltonian associated to P_s is

$$H(x, y, \xi, \eta) = -\xi^2 - (x^2 + s^2)\eta^2,$$

where (ξ, η) are the coordinates dual to (x, y) (that is (x, y, ξ, η) are the coordinated of the cotangent bundle T^*X). The geodesics between the points (x_0, y_0) and (x_1, y_1) are the projections onto the $(x, y)-$plane of the solutions to the Hamiltonian system

$$\begin{cases} \dot{x} = H_\xi = -2\xi \\ \dot{y} = H_\eta = -2(x^2 + s^2)\eta \\ \dot{\xi} = -H_x = -2x\eta^2 \\ \dot{\eta} = -H_y = 0, \end{cases} \tag{2.6}$$

with the boundary conditions $x(0) = x_0, y(0) = y_0, x(1) = x_1$ and $y(1) = y_1$, where the dot denotes the variation with respect to the time parameter t. System (2.6) implies that $\eta = \text{cst} = \eta_0$ and $\dot{y} = -2(x^2 + s^2)\eta_0$. If $\eta = 0$, then $y_0 = y_1$, and

$$x(t) = t(x_1 - x_0) + x_0, \qquad y(t) \equiv y_0, \qquad t \in [0, 1]$$

is the unique geodesic joining (x_0, y_0) to (x_1, y_0) (for detailed information about geodesics of Grushin operator, see [28][29]).
Thus, if $y_0 \notin [a, b]$, then the control domain w doesn't satisfy the geometric control condition (it will never enter w for any $t > 0$).

Here, we investigate a purely spectral condition that serves as a sufficient condition for the validity of concentration inequalities for eigenfunctions of the Baouendi Grushin operators on an arbitrary horizontal strip of the infinite cylinder. To be more specific, our interest here is not the concentration inequality itself (that could be known in the case where $V(x) = x^2$ by using some involved geometric methods), but in deriving a spectral condition leading to it.

2.1.3 Spectral Sufficient Condition

Let's first note that there exists a basis of eigenfunctions of P_V of the form

$$\phi(x, y) = \varphi_k(x)e^{iky} \qquad\qquad k \in \mathbb{Z}^*,$$

with $\varphi_k \in L^2(\mathbb{R})$. Indeed, it is well-known that if two operators T and $\tilde{T}$ commute, that is $T\tilde{T} = \tilde{T}T$, then one can find a joint eigenfunction for the two operators (see [11][41]). As the potential V is independent of y, then P_V and ∂_y^2 commutes. The eigenfunctions of ∂_y^2 have the form $\varphi_k(x)e^{iky}$ with $k \in \mathbb{Z}$, so there exists $k \in \mathbb{Z}^*$ (the Hilbert space is $L_0^2(X)$ and so functions that are independent of y are excluded, which is the case when $k = 0$) such that $\varphi_k(x)e^{iky}$ is an eigenfunction of P_V. Substituting in the eigenvalue equation, this implies that $\varphi_k(x)$ is an eigenfunction of the one-dimensional Schrödinger operator P_V^k. Then $\varphi_k(x) = \varphi_{k,j}(x)$ corresponds to the j^{th} eigenvalue of P_V^k for some $j \in \mathbb{N}$. For a fixed k, we can choose a family of orthonormal eigenfunctions $\{\varphi_{k,j}(x)\}_{j\in\mathbb{N}}$ that form a basis to $L^2(\mathbb{R})$.
Now, since $\{e^{iky}\}_{k\in\mathbb{Z}^*}$ is an orthogonal basis to $L^2(\mathbb{S}^1)$, we get that

$$\{\varphi_{k,j}(x)e^{iky}\}_{j\in\mathbb{N},k\in\mathbb{Z}^*}$$

is an orthogonal basis to $L^2(\mathbb{R} \times \mathbb{S}^1)$. Indeed, for any $u \in L^2(\mathbb{R} \times \mathbb{S}^1)$, we can write $u(x, y) = \sum_{k\in\mathbb{Z}^*} u_k(x)e^{iky}$, with

$$u_k(x) = \mathcal{F}(u)(x, k) = \frac{1}{2\pi}\int_{\mathbb{S}^1} u(x, y)e^{-iky}dy,$$

where $\mathcal{F}(u)$ here denotes the Fourier transform of u on the circle. Suppose now, that for any $j \in \mathbb{N}, k \in \mathbb{Z}^*$,

$$\langle u, \varphi_{k,j}(x)e^{iky}\rangle_{L^2(\mathbb{R}\times\mathbb{S}^1)} = 0. \tag{2.7}$$

We compute

$$\begin{aligned}\langle u_k(x), \varphi_{k,j}\rangle_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} u_k(x)\varphi_{k,j}(x)dx \\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{S}^1} u(x,y)e^{-iky}\varphi_{k,j}(x)dxdy \\ &= 0,\end{aligned}$$

where the last equality is by (2.7). Consequently, since for any $k \in \mathbb{Z}^*$, $(\varphi_{k,j})_{j\in\mathbb{N}}$ is an orthonormal basis, we get that $u_k = 0$ for any k which implies that $u = 0$.
So, $\{\varphi_{k,j}(x)e^{iky}\}_{k\in\mathbb{Z}^*,\ j\in\mathbb{N}}$ is an eigenbasis for $L^2(\mathbb{R}\times\mathbb{S}^1)$, and so they cover all the eigenvalues of P_V. The eigenvalue equation implies that

$$\operatorname{spec}(P_V) = \bigcup_{k\in\mathbb{Z}^*} \operatorname{spec}(P_V^k). \tag{2.8}$$

This method is called separation of variables and it works for any second order operator satisfying the above description, which shows the importance of V being independent of y.

Also, observe that $P_V^k = P_V^{-k}$, so if we denote by $\varphi_l(x)$ an eigenfunction corresponding to the l^{th} eigenvalue of P_V^k, then since we can write the eigenfunctions of P_V using separation of variables, $\varphi_l(x)e^{iky}$ and $\varphi_l(x)e^{-iky}$ are both eigenfunctions that correspond to the same eigenvalue of P_V and so the multiplicity of any eigenvalue of P_V is at least two (we recall that $k \neq 0$).

Remark 2.1. *Using this transition between P_V and P_V^k, one can deduce that P_V with domain $C_0^\infty(X)$ is essentially self-adjoint. Indeed, P_V is positive (semi-bounded from below), so it is enough to prove that*

$$\ker(P_V^* + 1) = \{0\}. \tag{2.9}$$

Let $u \in L_0^2(X)$ be such that for any $\varphi \in C_{c,0}^\infty(X)$

$$\langle u, (P_V + 1)\varphi\rangle_{L_0^2(X)} = 0.$$

We write $u = \sum_{k\in\mathbb{Z}^*} u_k(x)e^{iky}$ and take the test function of the form $\psi(x)e^{iky}$, then we have that

$$\forall\psi \in C_c(\mathbb{R}),\quad \langle u_k, (P_V^k + 1)\psi\rangle_{L^2(\mathbb{R})} = 0.$$

As P_V^k is essentially self-adjoint, this implies that $u_k = 0$ which gives that $u = 0$. Then (2.9) holds true. Therefore, $(P_V, C_0^\infty(X))$ is essentially self-adjoint.

Moreover, since (P_V, D) is self-adjoint and has a discrete spectrum (which is the union of the spectrum of P_V^k), then it has a compact resolvent (using the spectral theorem, the resolvent operator can be written as a limit of finite rank operators).

A sufficient condition for P_V to satisfy the concentration inequality is given in the following proposition.

Theorem 2.2. *Suppose that all the eigenvalues of P_V are of multiplicity 2. Then the concentration inequality holds true for $w = \mathbb{R}\times]a,b[$:*

$$\exists C > 0,\ \forall E \geqslant 0,\ \forall \phi \in \ker(P_V - E), \quad \|\phi\|_{L^2(X)} \leqslant C\|\phi\|_{L^2(w)}.$$

Proof. Let E be an eigenvalue of P_V. If $\mathrm{mult}(E) = 2$, then any eigenfunction of E can be written as

$$\phi(x,y) = \alpha\varphi_k(x)e^{iky} + \beta\varphi_{-k}(x)e^{-iky} = \varphi_k(x)(\alpha e^{iky} + \beta e^{-iky}),$$

where the last equality is because $\varphi_k = \varphi_{-k}$ (see the paragraph before this proposition).
We will explicitly compute $\|\phi\|^2_{L^2(w)}$. First, write

$$\alpha = \alpha_0 + i\alpha_1 \text{ and } \beta = \beta_0 + i\beta_1, \qquad \alpha_0, \alpha_1, \beta_0, \beta_1 \in \mathbb{R}.$$

Observe that $\phi(x,y)$ has the following expression:

$$\begin{aligned}\phi(x,y) &= \varphi_k(x)\big[(\alpha_0 + i\alpha_1)(\cos(ky) + i\sin(ky)) \\ &\qquad\qquad + (\beta_0 + i\beta_1)(\cos(ky) - i\sin(ky))\big] \\ &= \varphi_k(x)\big[(\alpha_0 + \beta_0)\cos(ky) + (\beta_1 - \alpha_1)\sin(ky) \\ &\qquad\qquad + i\big((\alpha_1 + \beta_1)\cos(ky) + (\alpha_0 - \beta_0)\sin(ky)\big)\big].\end{aligned}$$

Now, direct computation for $|\phi(x,y)|$ gives that

$$\begin{aligned}|\phi(x,y)|^2 &= \Re(\phi(x,y))^2 + \Im(\phi(x,y))^2 \\ &= |\varphi_k(x)|^2\left[\kappa_1\cos^2(ky) + \kappa_2\sin^2(ky) + 2\kappa_3\cos(ky)\sin(ky)\right],\end{aligned}$$

where $\kappa_1 = (\alpha_0+\beta_0)^2+(\alpha_1+\beta_1)^2$, $\kappa_2 = (\alpha_0-\beta_0)^2+(\alpha_1-\beta_1)^2$, and

$$\kappa_3 = (\alpha_0+\beta_0)(-\alpha_1+\beta_1)+(\alpha_0-\beta_0)(\alpha_1+\beta_1) = 2(\alpha_0\beta_1-\alpha_1\beta_0).$$

We compute

$$\begin{aligned}\|\phi\|^2_{L^2(w)} =& \|\varphi_k\|^2_{L^2(\mathbb{R})}\left[\kappa_1\int_a^b \cos^2(ky)dy + \kappa_2\int_a^b \sin^2(ky)dy\right.\\ &\left. + 2\kappa_3\int_a^b \cos(ky)\sin(ky)dy\right]\\ =& \|\varphi_k\|^2_{L^2(\mathbb{R})}\left[(\kappa_1-\kappa_2)\frac{f(k)}{4k} + (\kappa_1+\kappa_2)\left(\frac{b-a}{2}\right) + \kappa_3\frac{g(k)}{k}\right],\end{aligned}$$

with $f(k) = \sin(2bk)-\sin(2ak)$ and $g(k) = \cos^2(ak)-\cos^2(bk)$.
Taking $a=-\pi$ and $b=\pi$, we deduce that

$$\|\phi\|^2_{L^2(X)} = \pi\|\varphi_k\|^2_{L^2(\mathbb{R})}(\kappa_1+\kappa_2).$$

So, we get that,

$$\frac{\|\phi\|^2_{L^2(w)}}{\|\phi\|^2_{L^2(X)}} = \frac{\kappa_1-\kappa_2}{\kappa_1+\kappa_2}\frac{f(k)}{4\pi k} + \frac{b-a}{2\pi} + \frac{\kappa_3}{\kappa_1+\kappa_2}\frac{g(k)}{\pi k}. \tag{2.10}$$

Note that the constants κ_1, κ_2 and κ_3 depend on k. The functions $f(k)$ and $g(k)$ are functions of sine and cosine so they are bounded. Also, the term $|\frac{\kappa_1-\kappa_2}{\kappa_1+\kappa_2}|$ is bounded above by 1 and so, the first term of (2.10) converges to 0 as $k\to\infty$.
Now, observe that

$$\kappa_1+\kappa_2 = 2(\alpha_0^2+\beta_0^2+\alpha_1^2+\beta_1^2).$$

Then, we have

$$\frac{\kappa_3}{\kappa_1+\kappa_2} \leqslant \frac{1}{2} \tag{2.11}$$

Indeed, we explicitly write

$$\frac{\kappa_3}{\kappa_1+\kappa_2} = \frac{2(\alpha_0\beta_1-\alpha_1\beta_0)}{2(\alpha_0^2+\beta_0^2+\alpha_1^2+\beta_1^2)},$$

and observe that, as

$$(\alpha_0^2+\beta_0^2+\alpha_1^2+\beta_1^2)-2(\alpha_0\beta_1-\alpha_1\beta_0)=(\alpha_0-\beta_1)^2+(\alpha_1+\beta_0)^2\geqslant 0$$

we get that

$$2(\alpha_0\beta_1-\alpha_1\beta_0)\leqslant \alpha_0^2+\beta_0^2+\alpha_1^2+\beta_1^2.$$

This implies (2.11). Thus, the last term of (2.10) converges to 0 as $k\to\infty$. We deduce that

$$\begin{aligned}\lim_{E\to\infty}\left[\min_{0\neq\phi\in\ker(P_V-E)}\left(\frac{\|\phi\|^2_{L^2(w)}}{\|\phi\|^2_{L^2(X)}}\right)\right]=\lim_{k\to\infty}\Bigg[\min_{\alpha,\beta}\Bigg(&\frac{\kappa_1-\kappa_2}{\kappa_1+\kappa_2}\frac{f(k)}{2\pi k}+\frac{b-a}{2\pi}\\ &+\frac{\kappa_3}{\kappa_1+\kappa_2}\frac{g(k)}{\pi k}\Bigg)\Bigg]=\frac{b-a}{2\pi}>0.\end{aligned}\tag{2.12}$$

It remains to prove that (2.12) implies that the concentration inequality holds true. Suppose that the concentration inequality doesn't hold, that is, for all $c>0$, there exists $E\in\operatorname{spec}(P_V)$ and $0\neq\phi\in\ker(P_V-E)$ such that $c\|\phi\|^2_{L^2(w)}<\|\phi\|^2_{L^2(X)}$. Take $c=c_n=n$. This implies that there exists a sequence of eigenvalues $(E_n)_{n\in\mathbb{N}}$ and a sequence of corresponding eigenfunctions $(\phi_n)_{n\in\mathbb{N}}$ of P_V such that

$$\min_{\phi\in\ker(P_V-E_n)\setminus\{0\}}\left(\frac{\|\phi\|^2_{L^2(w)}}{\|\phi\|^2_{L^2(X)}}\right)\leqslant\frac{\|\phi_n\|^2_{L^2(w)}}{\|\phi_n\|^2_{L^2(X)}}\leqslant\frac{1}{n}.$$

We then observe that

$$\forall n,\quad \min_{\phi\in\ker(P_V-E_n)\setminus\{0\}}\left(\frac{\|\phi\|^2_{L^2(w)}}{\|\phi\|^2_{L^2(X)}}\right)>0$$

so that, necessarily, $E_n\to\infty$ when $n\to\infty$. We get that

$$\lim_{E_n\to\infty}\left[\min_{0\neq\phi\in\ker(P_V-E_n)}\left(\frac{\|\phi\|^2_{L^2(w)}}{\|\phi\|^2_{L^2(X)}}\right)\right]=0,$$

which contradicts (2.12). $\square$

The condition in theorem 2.2 is not true in general. For a better vision of the problem of multiplicity, we study first the simple Grushin operator.

2.2 Study Of An Example

We denote by P_s the operator

$$P_{x^2+s^2} = -\partial_x^2 - (x^2 + s^2)\partial_y^2,$$

defined on D given by (2.5). For $s = 0$, this is a Baouendi-Grushin operator, whereas for $s > 0$, it is elliptic. Since the preceding proposition principally depends on separating the variables of eigenfunctions of the operator and not really on the Baouendi Grushin operator itself, it also applies to any s.

In this section, we investigate the eigenvalues of P_s and the multiplicities of the eigenvalues according to s.

We first compute explicitly the spectrum of P_s. Denote by P_s^k the one dimensional Schrödinger operator defined by

$$P_s^k u(x) := P_{x^2+s^2}^k u(x) = -\partial_x^2 u(x) + k^2(x^2 + s^2)u(x).$$

Recall that P_s^k with domain $\mathcal{C}_c^\infty(\mathbb{R})$ is essentially self-adjoint and that the domain of the unique self-adjoint extension is

$$\mathbb{D}_x \subset \{u \in H^1(\mathbb{R}), (x^2 + s^2)^{1/2}u \in L^2(R)\}.$$

Moreover, $(P_s^k, \mathbb{D}_x)$ has compact resolvent. Its spectrum is discrete and consists of eigenvalues.

Proposition 2.3. *The spectrum of P_s is given by the set*

$$\operatorname{spec}(P_s) = \{E_{k,n}^s = (2n + 1)|k| + k^2 s^2; n \in \mathbb{N}, k \in \mathbb{Z}^*\}. \tag{2.13}$$

An orthonormal basis of eigenfunction corresponding to $E_{k,n}^s$ is given by $\phi_{k,n}(x, y) = \varphi_{k,n}(x)e^{iky}$, with

$$\varphi_{k,n}(x) = |k|^{1/4} H^n\left(x\sqrt{|k|}\right) e^{\frac{-x^2|k|}{2}} e^{iky},$$

where H^n is the Hermite polynomial of degree n, given by

$$H^n(x) = (-1)^n e^{x^2} \frac{\partial^n}{\partial x^n} e^{-x^2}.$$

Proof. Let $\phi(x,y) = \varphi_k(x)e^{iky}$, $k \in \mathbb{Z}^*$, be an eigenfunction of P_s and let E^s be its corresponding eigenfunction. The eigenvalue equation

$$P_s\phi = E^s\phi$$

implies that $E^s = E^s_k$ is an eigenvalue of P^k_s with a corresponding eigenfunction φ_k.
Observe that, the eigenvalue equation implies that $E^s_k = E^0_k + k^2s^2$, where E^0_k is an eigenvalue of the 1-D harmonic oscillator $-\partial^2_x + k^2x^2$, with corresponding eigenfunction φ_k. The spectrum of the harmonic oscillator is well-known and given by $\{(2n+1)|k|; n \in \mathbb{N}, k \in \mathbb{Z}^*\}$. Moreover, the function $\varphi_{k,n}$ given by

$$\varphi_{k,n}(x) = |k|^{1/4}H^n(x\sqrt{|k|})e^{\frac{-x^2|k|}{2}}, \tag{2.14}$$

is an eigenfunction corresponding to $(2n+1)|k|$. Refer to [49, Chapter 11] or [84, Chapter 6] for details about the harmonic oscillator.
Then, for every $k \in \mathbb{Z}^*$, the n^{th} eigenvalue of P^k_s is $E^s_{k,n} = (2n+1)|k| + k^2s^2$, with a corresponding eigenfunction is given by (2.14).

Therefore, the spectrum of P_s is given by (2.13), with a set of corresponding eigenvectors $\{\varphi_{k,n}(x)e^{iky}\}_{k\in\mathbb{Z}^*}$. Since these eigenfunctions span the space $L^2_0(X)$, they cover all the eigenvalues of P_s. We conclude. □

To study the multiplicity of eigenvalues, it is usually helpful to study the Weyl law, which will be described here by studying the asymptotic behavior of the counting function. The counting function N_{P_s} takes a positive real number and counts the number of eigenvalues of P_s less than or equal to this number. In other words, we can write for $E > 0$,

$$N_{P_s}(E) = \sum_{E^s_{k,n} \leqslant E} 1,$$

where the sum is taken over the eigenvalues $E^s_{k,n}$ of P_s.

Proposition 2.4. *[Weyl Law] The following assertions hold true.*

1. *For $s = 0$, $N_{P_0}(E) = E\ln(E) + O(E)$ at infinity.*
2. *For $s \neq 0$, $N_{P_s}(E) = E\ln(\sqrt{E}) + O(E)$ at infinity.*

Proof. Denote by $\lceil . \rceil$ the upper integer part function which takes a real number and gives the first integer greater than or equal to this number. Denote by $\lfloor . \rfloor$ the lower integer part function which takes a real number and gives the first integer less than or equal to this number

1. We compute

$$\begin{aligned} N_{P_0}(E) &= \operatorname{card}\{(n,k) \in \mathbb{N} \times \mathbb{Z}^*/(2n+1)|k| \leqslant E\} \\ &= 2\operatorname{card}\{(n,k) \in \mathbb{N} \times \mathbb{N}^*/(2n+1)|k| \leqslant E\} \\ &= 2 \sum_{0<k\leqslant E} \operatorname{card}\left\{n \in \mathbb{N}/(2n+1) \leqslant \frac{E}{k}\right\} \\ &= 2 \sum_{0<k\leqslant E} \left\lceil \frac{E}{2k} \right\rceil, \end{aligned}$$

Now, since

$$\frac{E}{2k} \leqslant \left\lceil \frac{E}{2k} \right\rceil \leqslant \frac{E}{2k} + 1,$$

then

$$E \sum_{0<k\leqslant E} \frac{1}{k} \leqslant 2 \sum_{0<k\leqslant E} \left\lceil \frac{E}{2k} \right\rceil \leqslant E \sum_{0<k\leqslant E} \frac{1}{k} + 2E.$$

As $E \to \infty$,

$$\sum_{0<k\leqslant E} \frac{1}{k} \;=\; \ln(E) \;+\; O(1).$$

Since E is negligible at infinity compared to $E\ln(E)$, we get $N_{P_0}(E) = E\ln(E) + O(E)$.

2. We set $\alpha = \min\left(E, \frac{\sqrt{E}}{s}\right)$, and we compute

$$\begin{aligned} N_{P_s}(E) &= \operatorname{card}\{(n,k) \in \mathbb{N} \times \mathbb{Z}^*/(2n+1)|k| + k^2s^2 \leqslant E\} \\ &= 2\operatorname{card}\{(n,k) \in \mathbb{N} \times \mathbb{N}^*/(2n+1)|k| + k^2s^2 \leqslant E\} \\ &= 2 \sum_{0<k\leqslant \alpha} \operatorname{card}\left\{n \in \mathbb{N}/(2n+1) \leqslant \frac{E-k^2s^2}{k}\right\} \\ &= 2 \sum_{0<k\leqslant \alpha} \left\lceil \frac{E-k^2s^2}{2k} \right\rceil. \end{aligned}$$

Since

$$\frac{E-k^2s^2}{2k} \leqslant \left\lceil \frac{E-k^2s^2}{2k} \right\rceil \leqslant \frac{E-k^2s^2}{2k} + 1,$$

then

$$E\left(\sum_{0<k\leqslant\alpha}\frac{1}{k}\right)-s^2\left(\sum_{0<k\leqslant\alpha}k\right)\leqslant 2\sum_{0<k\leqslant\alpha}\left\lceil\frac{E-k^2s^2}{2k}\right\rceil$$
$$\leqslant E\left(\sum_{0<k\leqslant\alpha}\frac{1}{k}\right)-s^2\left(\sum_{0<k\leqslant\alpha}k\right)+2E.$$

Finally, we get

$$E\left(\sum_{0<k\leqslant\alpha}\frac{1}{k}\right)-\frac{s^2\lfloor\alpha\rfloor(\lfloor\alpha\rfloor+1)}{2}\leqslant 2\sum_{0<k\leqslant\alpha}\left\lceil\frac{E-k^2s^2}{2k}\right\rceil$$
$$\leqslant E\left(\sum_{0<k\leqslant\alpha}\frac{1}{k}\right)-\frac{s^2\lfloor\alpha\rfloor(\lfloor\alpha\rfloor+1)}{2}+2E.$$

As $E\to\infty$, we have $\alpha=\frac{\sqrt{E}}{s}$, and so we get $N_{P_s}(E)=E\ln(\sqrt{E})+O(E)$.

$\square$

Corollary 2.4.1. *If s is fixed such that s^2 is rational, then the multiplicity is not, in general, uniformly bounded.*

Proof. If $s^2=0$, we write the prime factorisation $E^0_{k,n}=2^{k_0}p_1^{\alpha_1}...p_r^{\alpha_r}$ for an eigenvalue $E^0_{k,n}$. With the convention that $\sum_1^0=\sum_1^{-1}=0$, we have

$$\mathrm{mult}(E)=2\left[\sum_{i=1}^{r}\alpha_i+\sum_{j=1}^{r-1}\alpha_j\left(\sum_{k=j+1}^{r}\alpha_k\right)+1\right].\qquad(2.15)$$

Indeed, for $s=0$, the eigenvalues are of the form $(2n+1)|k|$ with $n\in\mathbb{N}$ and $k\in\mathbb{Z}$. The factor 2 outside the brackets is because of the fact that $E^0_{k,n}=E^0_{-k,n}$. Now, every prime number but 2 (and that's why we distinguished 2) is an odd integer, and the product of two odd integers is odd. So,

$$\text{the term } 2n+1 \text{ can be } \prod_{i=1}^{r}p_i^{j_i} \text{ for any } 0\leqslant j_i\leqslant\alpha_i.$$

The first sum on the right-hand side of (2.15) represents the number of the cases with $j_i = 0$ for all i but one. The second term of the right-hand side covers the number of cases where j_i is not zero at least for two $i's$, and k is not 2^{k_0}. Finally, the 1 is for the case where $k = 2^{k_0}$ ($j_i = \alpha_i$ for all i). This covers all the cases. Formula (2.15) implies that for $s = 0$, the multiplicity is not bounded.

Take now $s^2 = \frac{p}{q}$, with 1 as the greatest common factor of p and q and $p \neq 0$. Then, we have

$$\mathrm{spec}(P_s) = \{(2n+1)|k| + k^2 s^2, (n,k) \in \mathbb{N} \times \mathbb{Z}^*\} \subset \{\alpha + \beta s^2, \alpha, \beta \in \mathbb{Z}\} \subset \frac{1}{q}\mathbb{Z}.$$

Assume, for a contradiction, that the multiplicity is bounded above by some M. Then, for any E, and since the spectrum is a subset of $\frac{1}{q}\mathbb{Z}$, we have

$$N_{P_s}(2E) - N_{P_s}(E) \leqslant MqE.$$

But the previous proposition implies that

$$N_{P_s}(2E) - N_{P_s}(E) \; = \; E \ln \sqrt{E} \; + \; O(E).$$

This yields the contradiction. □

In particular, the multiplicity of eigenvalues of the simple Grushin operator P_{x^2} is not uniformly bounded.

Proposition 2.5. *If s^2 is irrational, then the eigenvalues of P_s are of multiplicity 2.*

Proof. Suppose that P_s has an eigenvalue of multiplicity greater than 2. Then there exists $k, k' > 0, n, n' > 0$ with $k^2 \neq k'^2, n \neq n'$ such that

$$(2n+1)|k| + k^2 s^2 = (2n'+1)|k'| + k'^2 s^2.$$

Then,

$$s^2 = \frac{(2n'+1)|k'| - (2n+1)|k|}{k^2 - k'^2},$$

which contradicts the fact that s^2 is irrational. □

Therefore, as a corollary of theorem 2.2, we have

Corollary 2.5.1. *If s^2 is irrational, then (2.1) holds.*

To sum up, we showed that whenever s^2 is irrational the multiplicity of the eigenvalues of P_s is 2, which implies by the spectral condition that the concentration inequality holds. Moreover, we proved that whenever s^2 is rational, the multiplicity is not uniformly bounded, and thus the spectral condition fails. This means that the spectral condition is not true in general. This example gave the inspiration to study the spectral condition generically for general Grushin operators.

2.3 General Case

Let's first briefly recall our setting. Recall that $\mathbb{W} = \{W \in \mathcal{C}_c^\infty(\mathbb{R}); W \geqslant 0\}$ equipped with the uniform norm on $\mathbb{R}$, and that the set $\mathbb{V}$ is given by

$$\mathbb{V} = \{V = x^2\tilde{W}; \tilde{W} \in \mathcal{C}_b^0(\mathbb{R}); \tilde{W} \geqslant 1\}.$$

On $\mathbb{V}$, we put the following norm: for $V = x^2 W \in \mathbb{V}$, we define the norm $\|V\|_{\mathbb{V}} := \|W\|_\infty$.
For $V \in \mathbb{V}$, consider the generalized Baouendi-Grushin operator $P_V = -\partial_x^2 - V(x)\partial_y^2$, with domain

$$D = \{u \in L_0^2(X); \partial_x^2 u \in L^2(X), V(x)\partial_y^2 u \in L^2(X)\}.$$

Recall that P_V^k denotes the one dimensional Schrödinger operator $P_V^k = -\partial_x^2 + k^2 V(x)$.
Let (P) be the property:

$$\forall k, l \in \mathbb{Z}^*; k^2 \neq l^2 \Rightarrow \mathrm{spec}(P_V^k) \cap \mathrm{spec}(P_V^l) = \varnothing \qquad \text{(P)}.$$

We can see from (2.8) that if (P) holds, then $\mathrm{mult}(E) = 2$ for all $E \in \mathrm{spec}(P_V)$, thus (2.1) holds by theorem 2.2. As said earlier in the general introduction, we are studying the validity of a simplicity result for the non-elliptic Baouendi Grushin operators that is of course not true in general (as shown in section 2.2). However, we prove the validity of the spectral condition for a generic Baouendi Grushin operator.
A general result of simplicity of eigenvalues for elliptic operators was first discussed by Albert in his thesis [6] and published later for two dimensional case in his paper [7]. Later, Uhlenbeck showed that the theorem does hold in

all dimensions [81][82]. This work inspired us to prove a variation on Albert's methods in [8] to prove a similar result for the subelliptic operator P_V. Explicitly speaking, we prove that for a generic V, $\mathrm{mult}(P_V) = 2$, that is, if we denote by $\mathbb{V}_b$ the set of bad $V's$ that do not satisfy the property (P);

$$\mathbb{V}_b := \{V \in \mathbb{V}; \text{(P) doesn't hold }\},$$

then the complement of $\mathbb{V}_b$ in $\mathbb{V}$ is residual in $(\mathbb{V}, \|.\|_{\mathbb{V}})$ (the intersection of open dense sets in $\mathbb{V}$).

Informally speaking, the proof goes by constructing a countable family of open and dense sets in $\mathbb{V}$ such that the complement of $\mathbb{V}_b$ in $\mathbb{V}$ is equal to the countable intersection of these sets. To prove the density, we will need some lemmas whose proofs rely on the Hellmann–Feynman theorem (lemma 2.11 below) applied to the family $P^k_{V+tx^2W}$. However, to use the Hellmann–Feynman theorem, we have to prove that the eigenvalues and eigenvectors of $P^k_{V+tx^2W}$ are analytic in t.

In appendix B, we state and prove the well-known Kato's perturbation theorem in finite dimensional case, which gives the analyticity of eigenquantities of a perturbed finite dimensional operator (references are given there).

Here, we are dealing with infinite dimensional operators. Kato proved that his theory applies to the infinite dimensional problems (see [54, Chapter 7]). However, for the convenience of the reader and self-consistence, we give a proof that is adapted to our settings; we prove a generalization of Kato's theorem into our -infinite dimensional- case, that guarantees the analyticity of eigenvalues (and eigenvectors) of $P^k_{V+tx^2W}$.

Denote by $\mathcal{H}$ the Hilbert space $L^2(\mathbb{R})$, equipped with the usual L^2 norm. For $V \in \mathbb{V}$, denote by $\lambda^k_m(V)$ the m^{th} eigenvalue of P^k_V.

2.3.1 Analyticity Of Eigenvalues And Hellmann–Feynman Theorem

We will prove in this section, that for t positive small enough, the spectrum of $P^k_{V+tx^2W}$ coincides, in an interval, with the spectrum of a finite dimensional analytic operator (theorem 2.10 below).
In this section, we simply write $\lambda_m(V)$ for $\lambda^k_m(V)$ as there will not be any confusion about the corresponding operator.

Proposition 2.6 (Continuity Of Spectrum)**.** *Fix $V \in \mathbb{V}$. Let $(V_n)_{n\geqslant 1}$ be a sequence of functions in $\mathbb{V}$ that converges to V in $\|.\|_{\mathbb{V}}$. Then, for any m and any $\epsilon > 0$ there exist $n_{m,\epsilon}$, such that for all $n \geqslant n_{m,\epsilon}$,*

$$|\lambda_m(V_n) - \lambda_m(V)| < 2\epsilon. \tag{2.16}$$

Proof. Write $V_n = V + x^2 W_n$, where W_n is a sequence of continuous bounded functions that converge uniformly to 0 on $\mathbb{R}$. Observe that

$$\left|\frac{V_n - V}{V}\right| \leqslant |W_n|,$$

since $V(x)/x^2 \geqslant 1$.

For F subset of the domain of P_V^k, denote by Λ_V the following map

$$\Lambda_V(F) = \underset{\substack{u\in F\\ u\neq 0}}{max}\left\{\frac{\langle P_V^k u, u\rangle_{\mathcal{H}}}{\|u\|_{\mathcal{H}}^2}\right\}.$$

Fix m and let F be the subspace spanned by the first m eigenvectors of P_V. We compute

$$\begin{aligned}
\langle P_{V_n}^k u, u\rangle_{\mathcal{H}} &= \langle P_V^k u, u\rangle_{\mathcal{H}} + k^2 \int_{\mathbb{R}} x^2 W_n |u|^2 \\
&\leqslant \Lambda_V(F) + k^2 \int_{\mathbb{R}} V \frac{|V_n - V|}{V} |u|^2 \\
&\leqslant (1 + \|W_n\|_\infty)\lambda_m(V).
\end{aligned}$$

Taking the maximum over all functions $u \in F$, we get that

$$\lambda_m(V_n) - \lambda_m(V) \leqslant \lambda_m(V)\|W_n\|_\infty.$$

Exchanging the roles of V_n and V, we obtain

$$\lambda_m(V) - \lambda_m(V_n) \leqslant \lambda_m(V_n)\|W_n\|_\infty \leqslant \lambda_m(V)\|W_n\|_\infty(1 + \|W_n\|_\infty).$$

Since $\|W_n\|_\infty$ tends to 0, we conclude (2.16). $\square$

A crucial point for proving our main result in this chapter is theorem 2.10 below, which, as explained previously, is a generalization of analyticity result to an infinite dimensional case adapted to our setting. Informally speaking, we prove that the eigenvalues of the operator P_{V+tx^2W} in an interval coincide with those of a finite dimensional analytic operator. For that, we define an isomorphism using the spectral projection for the perturbed operator. So, we need to define this spectral projection. Precisely, fix some $m \in \mathbb{N}$. For $W \in \mathbb{W}$, we need to define the spectral projection Π_W for the operator $P^k_{V+x^2W}$, defined as

$$\Pi_W = \frac{1}{2\pi i}\int_\Gamma (P_{V+x^2W} - \mu)^{-1} d\mu, \tag{2.17}$$

for some Γ containing no eigenvalues of $P^k_{V+x^2W}$.
Denote by κ_m the distance from $\lambda_m(V)$ to the rest of the spectrum of P^k_V, i.e.

$$\kappa_m = \text{dist}\left(\lambda_m(V), \text{spec}(P^k_V)\backslash\lambda_m(V)\right).$$

For any $W \in \mathbb{W}$ satisfying

$$\left\|x^2W\right\|_\infty < \frac{\kappa_m}{|k|^2}, \tag{2.18}$$

we define the intervals J_+ and J_- as

$$J_+ = \left]\lambda_m(V) + |k|\left\|x^2W\right\|_\infty, \lambda_m(V) + \kappa_m\right[,$$

$$J_+ = \left]\lambda_m(V) - \kappa_m, \lambda_m(V) - |k|\left\|x^2W\right\|_\infty\right[.$$

Proposition 2.7. *For any $W \in \mathbb{W}$ satisfying (2.18), for any $J \subset J_- \cup J_+$ we have $\mu \in J$ implies $\mu \notin \text{spec}(P^k_{V+x^2W})$.*

Proof. Let $W \in \mathbb{W}$ satisfying (2.18). P^k_V is an unbounded self-adjoint operator. Then, for $\mu \in J$ (which is a subset of the resolvent set of P^k_V), $(P^k_V - \mu)^{-1}$ is bounded normal operator and the spectral radius coincides with the norm of the resolvent, that is,

$$\left\|(P^k_V - \mu)^{-1}\right\|_{\mathcal{L}(\mathcal{H})} = \sup\left\{|\mu|, \mu \in \text{spec}((P^k_V - \mu)^{-1})\right\} = \frac{1}{\text{dist}(\mu, \text{spec}(P^k_V))}, \tag{2.19}$$

where $\mathcal{L}(\mathcal{H})$ denotes the space of linear bounded functions from $\mathcal{H}$ to $\mathcal{H}$.
Let $u \in \mathcal{H}$, and let $v = (P_V^k - \mu)^{-1}u$. We compute

$$\begin{aligned}\left\|(P_{V+x^2W}^k - P_V^k)(P_V^k - \mu)^{-1}u\right\|_{\mathcal{H}}^2 &= \left\|(P_{V+x^2W}^k - P_V^k)v\right\|_{\mathcal{H}}^2 \\ &= k^4\left\|x^2Wv\right\|_{\mathcal{H}}^2 \leqslant k^4\left\|x^2W\right\|_{\infty}^2\|v\|_{\mathcal{H}}^2 \\ &\leqslant \frac{k^4}{\operatorname{dist}(\mu, \operatorname{spec}(P_V^k))^2}\left\|x^2W\right\|_{\infty}^2\|u\|_{\mathcal{H}}^2.\end{aligned}$$

Now, for $\mu \in J_- \cup J_+$, we have that

$$\operatorname{dist}(\mu, \operatorname{spec}(P_V^k)) > |k|^2\left\|x^2W\right\|_{\infty}. \tag{2.20}$$

Therefore,

$$\left\|(P_{V+x^2W}^k - P_V^k)(P_V^k - \mu)^{-1}\right\|_{\mathcal{L}(\mathcal{H})} \leqslant \frac{k^2}{\operatorname{dist}(\mu, \operatorname{spec}(P_V^k))}\left\|x^2W\right\|_{\infty} < 1.$$

Then by Neumann lemma, $(I + (P_{V+x^2W}^k - P_V^k)(P_V^k - \mu)^{-1})$ is invertible.
Now, we have

$$\begin{aligned}I &= (P_V^k - \mu)(P_V^k - \mu)^{-1} \\ &= (P_V^k - \mu)(P_V^k - \mu)^{-1} + (P_{V+x^2W}^k - \mu)(P_V^k - \mu)^{-1} - (P_{V+x^2W}^k - \mu)(P_V^k - \mu)^{-1} \\ &= (P_{V+x^2W}^k - \mu)(P_V^k - \mu)^{-1} - (P_{V+x^2W}^k - \mu + \mu - P_V^k)(P_V^k - \mu)^{-1} \\ &= (P_{V+x^2W}^k - \mu)(P_V^k - \mu)^{-1} - (P_{V+x^2W}^k - P_V^k)(P_V^k - \mu)^{-1}.\end{aligned}$$

This implies that

$$I + (P_{V+x^2W}^k - P_V^k)(P_V^k - \mu)^{-1} = (P_{V+x^2W}^k - \mu)(P_V^k - \mu)^{-1}.$$

We conclude that $P_{V+x^2W}^k - \mu$ is invertible and thus $\mu \notin \operatorname{spec}(P_{V+x^2W}^k)$. $\square$

Hereafter, whenever we use Π_W, a convenient contour Γ is taken, that is, $\Gamma(t) \cap \mathbb{R} \subset J_- \cup J_+$.
To prove analyticity, and besides defining the spectral projection, we will need the following lemma concerning the convergence of the spectrum of the family $(P_{V+t_nx^2W}^k)_{n\geqslant 1}$.
We need first the following theorem which is standard in spectral theory, and which will be used in this chapter and later in chapter 4.

Theorem 2.8. *Let T be a self-adjoint operator with domain $D(T)$ in a Hilbert space H equipped with the norm $\|.\|_H$. Then $\lambda \in spec(T)$ if and only if there exists a sequence $(u_n)_{n\geqslant 1} \in D(T)$ such that $\|u_n\|_H = 1$ and*

$$\lim_{n\to\infty} \|(T-\lambda)u_n\|_H = 0.$$

Proof. First, let $\lambda \in \mathrm{spec}(T)$. If λ is an eigenvalue of T, we take $u_n = f/\|f\|_H$ for any f in the eigenspace of λ.
If $\ker(T-\lambda) = 0$, then since T is self-adjoint, we have

$$\overline{\mathrm{Ran}(T-\lambda)} = H,$$

and so we define the unbounded operator $B = (T-\lambda)^{-1}$ as: $\mathrm{dom}(B) = \mathrm{Ran}(T)$ and For any $y \in \mathrm{dom}(B)$, $x = By$ is the unique element x such that $Tx = y$. Consequently, there exists a sequence $\{v_n\} \in D(T)$ such that $\|v_n\|_H = 1$ and $\left\|(T-\lambda)^{-1}v_n\right\| \to \infty$. Define

$$u_n = \frac{(T-\lambda)^{-1}v_n}{\|(T-\lambda)^{-1}v_n\|_H}.$$

Clearly, $\{u_n\}$ is the required sequence.

Conversely, let λ be in the resolvent set of T. Then there exists $M > 0$ such that for any $v \in H$,

$$\left\|(T-\lambda)^{-1}v\right\|_H \leqslant M\|v\|_H .$$

let $u = (T-\lambda)^{-1}v \in D(T)$. We have

$$\|u\|_H \leqslant M\|(T-\lambda)u\|_H .$$

The existence of a sequence of normal vectors $(u_n)_n$ in $D(T)$ such that $\|(T-\lambda)u_n\|_H \to 0$ implies that $1 \leqslant 0$ which is a contradiction. □

Lemma 2.9. *Fix $V \in \mathbb{V}$ and $W \in \mathbb{W}$. Let $\{t_n\}_{n\geqslant 1}$ be a sequence in $\mathbb{R}$ that converges to 0. Let $(\varphi_{t_n})_{n\geqslant 1}$ be a sequence of orthonormal eigenfunctions of $P^k_{V+t_n x^2 W}$. Denote by $(\lambda_n)_{n\geqslant 1}$ the corresponding sequence of eigenvalues. Suppose that there exists $M \in \mathbb{R}$ such that for all n, $|\lambda_n| < M$. Then, up to extracting a subsequence,*

$$\lambda_n \to \lambda \in spec(P^k_V). \tag{2.21}$$

Moreover, φ_{t_n} has a subsequence that converges strongly in $\mathcal{H}$ to the eigenfunction of P^k_V corresponding to λ.

Proof. By Bolzano-Weierstrass theorem, $(\lambda_n)_{n\geqslant 1}$ has a subsequence, say $(\lambda_{n_j})_{j\geqslant 1}$ that converges to some $\lambda \in \mathbb{R}$ as $j \to \infty$.
We have: $\varphi_{t_{n_j}} \in \mathbb{D}_{V+t_{n_j}x^2W} = \mathbb{D}_V$ ($tx^2W \in L^\infty(\mathbb{R})$) with $\left\|\varphi_{t_{n_j}}\right\|_{\mathcal{H}} = 1$. Moreover,

$$\begin{aligned}\left\|(P_V^k - \lambda)\varphi_{t_{n_j}}\right\|_{\mathcal{H}} &\leqslant \left\|(P^k_{V+t_{n_j}x^2W} - \lambda)\varphi_{t_{n_j}}\right\|_{\mathcal{H}} + k^2|t_{n_j}|\left\|x^2W\varphi_{t_{n_j}}\right\|_{\mathcal{H}} \\ &\leqslant |\lambda_n - \lambda| + k^2|t_{n_j}|\left\|x^2W\right\|_\infty \to 0.\end{aligned}$$

By theorem 2.8, we get (2.21).

Now, since the sequence $(\varphi_{n_j})_{j\geqslant 1} := (\varphi_{t_{n_j}})_{j\geqslant 1}$ is bounded in $\mathcal{H}$, it has a subsequence that converges weakly to some $\varphi_1 \in \mathcal{H}$. For any $\tilde{\varphi} \in \mathbb{D}_V$, we compute

$$\begin{aligned}\langle P_V^k\varphi_1, \tilde{\varphi}\rangle_{\mathcal{H}} &= \langle (P_V^k - P^k_{V+t_{n_j}x^2W})\varphi_1, \tilde{\varphi}\rangle_{\mathcal{H}} + \langle P^k_{V+t_{n_j}x^2W}(\varphi_1 - \varphi_{n_j}), \tilde{\varphi}\rangle_{\mathcal{H}} + \langle P^k_{V+t_{n_j}x^2W}\varphi_{n_j}, \tilde{\varphi}\rangle_{\mathcal{H}} \\ &= \langle (P_V^k - P^k_{V+t_{n_j}x^2W})\varphi_1, \tilde{\varphi}\rangle_{\mathcal{H}} + \langle (\varphi_1 - \varphi_{n_j}), P^k_{V+t_{n_j}x^2W}\tilde{\varphi}\rangle_{\mathcal{H}} + \lambda_{n_j}\langle \varphi_{n_j}, \tilde{\varphi}\rangle_{\mathcal{H}}.\end{aligned}$$

As $j \to +\infty$, the right-hand side converges to $\langle \lambda\varphi_1, \tilde{\varphi}\rangle_{\mathcal{H}}$ for any $\tilde{\varphi} \in \mathbb{D}_V$, and therefore, for all $\tilde{\varphi} \in \mathbb{D}_V$,

$$\langle (P_V^k - \lambda)\varphi_1, \tilde{\varphi}\rangle_{\mathcal{H}} = 0.$$

This implies that (λ, φ_1) is an eigenpair of the self-adjoint operator P_V^k, provided that $\phi_1 \neq 0$.

Finally, we prove that φ_{n_j} converges to φ_1 strongly in $\mathcal{H}$ (which will also imply that $\|\varphi_1\|_{\mathcal{H}} = 1$). For any $\mu \notin \mathrm{spec}(P^k_{V+t_{n_j}x^2W})$, $(P^k_{V+t_{n_j}x^2W})^{-1}$ is compact which implies that

$$(P^k_{V+t_{n_j}x^2W})^{-1}\varphi_{n_j} \to (P_V^k)^{-1}\varphi_1 \qquad \text{as } j \to \infty.$$

This implies, using the eigenvalue equations, that

$$\frac{1}{\lambda_{n_j} - \mu}\varphi_{n_j} \to \frac{1}{\lambda - \mu}\varphi_1.$$

We conclude by observing that

$$\left\|\varphi_{n_j} - \varphi_1\right\|_{\mathcal{H}} \leqslant |\lambda_{n_j} - \mu| \left(\left\|\frac{1}{\lambda_{n_j} - \mu}\varphi_{n_j} - \frac{1}{\lambda - \mu}\varphi_1\right\|_{\mathcal{H}} + \left|\frac{1}{\lambda_{n_j} - \mu} - \frac{1}{\lambda - \mu}\right|\|\varphi_1\|_{\mathcal{H}}\right).$$

$\square$

Now, we state the theorem that implies the analyticity of eigenvalues of $P^k_{V+tx^2W}$.

Theorem 2.10. *Fix $V \in \mathbb{V}$ and $W \in \mathbb{W}$ and suppose without loss of generality that the support of W is $[-1,1]$. For any $0 < \epsilon < \kappa_m$, denote by I_ϵ the interval*

$$I_\epsilon =]\lambda_m(V) - \epsilon, \lambda_m(V) + \epsilon[.$$

Then, for all $0 < \epsilon < \kappa_m$, there exists $\delta > 0$, and an analytic family of finite dimensional operators, $\tilde{P}^\epsilon_t$ such that for all $|t| < \min\left\{\delta, \kappa_m/(k^2\|W\|_\infty)\right\}$, we have

$$spec(\tilde{P}^\epsilon_t) \cap I_\epsilon = spec(P^k_{V+tx^2W}) \cap I_\epsilon. \tag{2.22}$$

Proof. Denote by $\mathcal{V}_0$ the eigenspace corresponding to the eigenvalue $\lambda_m(V)$, and set $d_0 = \dim(\mathcal{V}_0)$ (which is equal to the multiplicity of $\lambda_m(V)$). For any

$$|t| < \kappa_m/(k^2\|W\|_\infty), \tag{2.23}$$

denote by $\mathcal{V}^\epsilon_t$ the total eigenspace corresponding to the operator $P^k_{V+tx^2W}$ in I_ϵ, and by by Π_0 and Π^ϵ_t the orthogonal projections on $\mathcal{V}_0$ and $\mathcal{V}^\epsilon_t$ respectively. For all t satisfying (2.23), define the operator Ψ^ϵ_t

$$\begin{aligned}\Psi^\epsilon_t : \mathcal{V}_0 &\to \mathcal{V}^\epsilon_t\\ \varphi &\mapsto \Pi^\epsilon_t \Pi_0 \varphi.\end{aligned}$$

We prove that Ψ^ϵ_t is an isomorphism and then construct a family of operators using Ψ^ϵ_t satisfying (2.22).

First, we prove that for each ϵ, the operator Ψ^ϵ_t is an isomorphism from $\mathcal{V}_0$ to $\mathcal{V}^\epsilon_t$. One way to do it is to prove that $\mathcal{V}_0$ and $\mathcal{V}^\epsilon_t$ have the same dimension and that Ψ^ϵ_t maps a basis of the subspace $\mathcal{V}_0$ to a basis of the subspace $\mathcal{V}^\epsilon_t$. Fix some $\epsilon > 0$. We first prove that $\dim(\mathcal{V}^\epsilon_t) \geqslant d_0$. We compute

$$\begin{aligned}\Psi^\epsilon_t &= \Pi_0 \Pi^\epsilon_t\\ &= \frac{1}{(2i\pi)^2}\int_{\Gamma_w}\int_{\Gamma_z}(P^k_{V+tx^2W} - z)^{-1}(P^k_V - w)^{-1}dzdw\\ &= \frac{1}{(2i\pi)^2}\left[\int_{\Gamma_w}\int_{\Gamma_z}\frac{(P^k_{V+tx^2W} - z)^{-1}}{z-w}dzdw - \int_{\Gamma_w}\int_{\Gamma_z}\frac{(P^k_V - w)^{-1}}{z-w}dzdw + \mathcal{R}_t\right],\end{aligned}$$

where

$$\mathcal{R}_t = \int_{\Gamma_w}\int_{\Gamma_z} \frac{(P^k_{V+tx^2W}-z)^{-1}(P^k_{V+tx^2W}-P^k_V)(P^k_V-w)^{-1}}{z-w}dzdw.$$

If we choose Γ_z inside Γ_w (i.e. inside the area encircled by Γ_w), then $(P^k_{V+tx^2W}-z)^{-1}/(z-w)$ is holomorphic and so by Cauchy integral theorem, its integral on Γ_z is 0. Moreover, we apply the Cauchy integral formula on the second term to deduce that

$$\Psi^\epsilon_t = \Pi_0 + \mathcal{R}_t.$$

Now, we compute

$$\begin{aligned}
\|\mathcal{R}_t\|_{\mathcal{L}(\mathcal{H})} &\leqslant |t||k|\left\|x^2W\right\|_\infty \int_{\Gamma_w}\int_{\Gamma_z} \frac{\left\|(P^k_{V+tx^2W}-z)^{-1}\right\|_{\mathcal{L}(\mathcal{H})}\left\|(P^k_V-w)^{-1}\right\|_{\mathcal{L}(\mathcal{H})}}{|z-w|}|dz||dw| \\
&\leqslant |t||k|\left\|x^2W\right\|_\infty \int_{\Gamma_w}\int_{\Gamma_z} \frac{1}{\operatorname{dist}(z,\operatorname{spec}(P^k_{V+tx^2W}))}\frac{1}{\operatorname{dist}(w,\operatorname{spec}(P^k_V))}\frac{|dz||dw|}{|z-w|} \\
&\leqslant |t||k|\left\|x^2W\right\|_\infty \frac{\kappa_m}{2}\frac{\kappa_m}{2}\sup_{z,w}\left(\frac{1}{|z-w|}\right)\mathcal{A}_z\mathcal{A}_w := Ct,
\end{aligned}$$

where $\mathcal{A}_z$ and $\mathcal{A}_w$ are the total arclengths of the contours Γ_z and Γ_w respectively, and C is a constant that doesn't depend on t.
So, for any $\varphi \in \mathcal{H}$, we get

$$\|\mathcal{R}_t\varphi\|_{\mathcal{H}} = o(\|\varphi\|_{\mathcal{H}}) \text{ as } t \to 0.$$

Now, let $\{\varphi^0_1, ..., \varphi^0_{d_0}\}$ be an orthonormal basis for $\mathcal{V}_0$. For $1 \leqslant i \leqslant d_0$, let $\tilde{\varphi}_i(t) = \Psi^\epsilon_t(\varphi^0_i)$. For any $i, j = 1, ..., d_0$, we compute

$$\begin{aligned}
\langle\tilde{\varphi}_i(t), \tilde{\varphi}_j(t)\rangle &= \langle\Psi^\epsilon_t\varphi^0_i, \Psi^\epsilon_t\varphi^0_j\rangle \\
&= \langle\Pi_0\varphi^0_i + \mathcal{R}\varphi^0_i, \Pi_0\varphi^0_j + \mathcal{R}\varphi^0_j\rangle \\
&= \delta_{i,j} + o(1) \qquad\qquad t \to 0.
\end{aligned}$$

Thus, $\{\tilde{\varphi}_i(t)\}_{i=1...d_0}$ is a set of linearly independent vectors in $\mathcal{V}^\epsilon_t$, and so $\dim(\mathcal{V}^\epsilon_t) \geqslant d_0$.
Moreover, there exist δ such that P_{V+tx^2W} has exactly d_0 eigenvalues in I_ϵ for all $t < \delta$. Indeed, suppose to contrary, that for all δ, there exist $t < \delta$ such that $P^k_{V+tx^2W}$ has $d_0 + 1$ eigenvalues in I_ϵ. Let δ_n be a sequence that

converges to 0. Then, there exist a sequence t_n, that converges to 0, such that $P^k_{V+t_nx^2W}$ has d_0+1 eigenvalues in I_ϵ. We denote these eigenvalues and a corresponding set of orthonormal eigenfunctions by

$$\{\lambda_1^n, ..., \lambda_{d_0+1}^n\} \text{ and } \{\varphi_1^n, ..., \varphi_{d_0+1}^n\} \text{ respectively }.$$

For $1 \leqslant j \leqslant d_0+1$, consider the sequence $(\lambda_j^n)_{n\geqslant 1}$. We get d_0+1 bounded (by I_ϵ) sequences which by lemma 2.21, converge (up to a subsequence) to an eigenvalue of the limiting operator P^k_V. If we consider now the set of corresponding orthonormal sequences of eigenfunctions, then for any $j = 1, ..., d_0+1$, again by lemma 2.21, the sequence $(\varphi_j^n)_{n\geqslant 1}$ converges in $\mathcal{H}$ to the eigenfunction of λ_j (the limiting eigenvalue). If we denote by φ_j the eigenfunction corresponding to λ_j for $j = 1, ..., d_0+1$, we get that for any $i, j = 1, ..., d_0+1$,

$$\langle \varphi_i, \varphi_j \rangle_{\mathcal{H}} = \lim_{n\to\infty} \langle \varphi_i^n, \varphi_j^n \rangle_{\mathcal{H}} = \delta_i^j,$$

by the orthonormality of $\{\varphi_1^n, ..., \varphi_{d_0+1}^n\}$, and so, if $i \neq j$, we have $\langle \varphi_i, \varphi_j \rangle_{\mathcal{H}} = 0$ (which implies the orthonormality of $\{\varphi_1, ..., \varphi_{d_0+1}\}$). This implies that $d_0 = d_0+1$, which is a contradiction. Therefore, $\dim(\mathcal{V}_t^\epsilon) = d_0 = \dim(\mathcal{V}_0)$. Finally, we deduce that the set $\{\tilde{\varphi}_i(t)\}_{i=1...d_0}$ is a basis for $\mathcal{V}_t^\epsilon$ and therefore, Ψ_t^ϵ is an isomorphism form $\mathcal{V}_0$ to $\mathcal{V}_t^\epsilon$.
Now, we introduce the family of finite dimensional operators $\tilde{P}_t^\epsilon$ as

$$\begin{aligned} \tilde{P}_t^\epsilon : \mathcal{V}_0 &\to \mathcal{V}_0 \\ \varphi &\mapsto (\Psi_t^\epsilon)^{-1} P^k_{V+tx^2W} \Psi_t^\epsilon \varphi. \end{aligned}$$

Then $\tilde{P}_t^\epsilon$ satisfies (2.22). Indeed, if $\xi(t) \in \operatorname{spec}(\tilde{P}_t^\epsilon) \cap I_\epsilon$ then $\xi(t) \in I_\epsilon$ and there exists $u \in \mathcal{V}_0$ such that $\tilde{P}_t^\epsilon u = \xi(t)u$. So, by definition, we get that $(\Psi_t^\epsilon)^{-1} P^k_{V+tx^2W} \Psi_t^\epsilon u = \xi(t)u$, which implies that

$$P^k_{V+tx^2W} \Psi_t^\epsilon u = \xi(t) \Psi_t^\epsilon u.$$

Thus, $\xi(t)$ is an eigenvalue of $P^k_{V+tx^2W}$ with eigenfunction $\Psi_t^\epsilon u \in \mathcal{V}_t^\epsilon$. In particular, $\xi(t) \in \operatorname{spec}(P^k_{V+tx^2W}) \cap I_\epsilon$. Therefore

$$\operatorname{spec}(\tilde{P}_t^\epsilon) \cap I_\epsilon \subset \operatorname{spec}(P^k_{V+tx^2W}) \cap I_\epsilon.$$

The same argument starting with $\xi(t) \in \operatorname{spec}(P^k_{V+tx^2W}) \cap I_\epsilon$ implies the second direction.

It remains to prove that $\tilde{P}_t$ is analytic. If we denote by $R(t,\zeta)$ the resolvent operator

$$R(t,\zeta) = (P^k_{V+tx^2W} - \zeta)^{-1}$$

for $\zeta \notin \operatorname{spec}(P^k_{V+tx^2W})$, then, for $\zeta_0 \notin \operatorname{spec}(P^k_V)$ we get the relation

$$R(t,\zeta)^{-1} = (1 - (\zeta - \zeta_0 - (P^k_{V+tx^2W} - P^k_V))R(0,\zeta_0))(P^k_V - \zeta_0).$$

So, for $|t|$ small enough and ζ close to ζ_0, $(1-(\zeta-\zeta_0-(P^k_{V+tx^2W}-P^k_V))R(0,\zeta_0))$ is invertible by Neumann's lemma and we can write the inverse as Neumann series. This implies the analyticity of $R(t,\zeta)$ near 0. Moreover, using the formula (2.17), we get that the projection is analytic and thus Ψ_t is (see section B.3.2). Then, $\tilde{P}_t$ is a composition of analytic functions and thus analytic. □

The preceding theorem implies that for t small enough, the eigenvalues of the operator $P^k_{V+tx^2W}$ coincide with those of the finite dimensional analytic operator $\tilde{P}_t$. Using the analytic perturbation theory in finite dimensional case, the eigenvalues of $P^k_{V+tx^2W}$ are analytic.

Finally, we proved in the same theorem, that Ψ_t is an isomorphism from $\mathcal{V}_t$ to $\mathcal{V}_0$. Then Ψ_t^{-1} is an isomorphism from $\mathcal{V}_0$ to $\mathcal{V}_t$, and it maps the basis of $\mathcal{V}_0$ to the basis of $\mathcal{V}_t$. This implies, by the analyticity of Ψ_t that the eigenfunctions of $P^k_{V+tx^2W}$ are analytic.

As we proved the analyticity of the eigenvalues and eigenfunctions of $P^k_{V+tx^2W}$, we can now apply the Hellmann–Feynman theorem, which we now state and prove.

Lemma 2.11 (Hellmann–Feynman)**.** *Let $\lambda(t)$ be an eigenbranch of $P^k_{V+tx^2W}$, and denote by $u(t)$ a normalized eigenfunction branch of $\lambda(t)$. Then,*

$$\frac{d}{dt}\lambda(t) = \left\langle u(t), \left(\frac{d}{dt}P^k_{V+tx^2W}\right)u(t)\right\rangle.$$

Proof. Write the eigenvalue equation:

$$P^k_{V+tx^2W}u(t) = \lambda(t)u(t). \tag{2.24}$$

Multiply by $u(t)$ and then differentiate in t to get

$$\begin{aligned}\frac{d}{dt}\lambda(t) &= \frac{d}{dt}\langle u(t), P^k_{V+tx^2W}u(t)\rangle \\ &= \left\langle \frac{d}{dt}u(t), P^k_{V+tx^2W}u(t)\right\rangle + \left\langle u(t), \frac{d}{dt}\left(P^k_{V+tx^2W}u(t)\right)\right\rangle \\ &= \lambda(t)\frac{d}{dt}\langle u(t), u(t)\rangle + \left\langle u(t), \left(\frac{d}{dt}P^k_{V+tx^2W}\right)u(t)\right\rangle \\ &= \left\langle u(t), \left(\frac{d}{dt}P^k_{V+tx^2W}\right)u(t)\right\rangle.\end{aligned} \tag{2.25}$$

□

Remark 2.12. *Following the same proof, we can show that whenever $u(t)$ and $v(t)$ are two orthogonal eigenfunction branches that correspond to $\lambda(t)$, we have*

$$\langle u(t), \left(\frac{d}{dt}P^k_{V+tx^2W}\right)v(t)\rangle = 0.$$

This is because, by orthogonality, the left-hand side of (2.25) is zero.

Finally, before proving the genericity of the spectral condition for Baouendi Grushin operators, we give a very well-known lemma which will be essential for what follows.

Lemma 2.13. *Any two analytic functions on $\mathbb{R}$ either coincide or intersect on a countable set of points.*

Proof. By identity theorem for analytic functions, if two analytic functions coincide on a subset of $\mathbb{R}$ that has an accumulation point, then the two functions coincide on $\mathbb{R}$. Since any uncountable subset of $\mathbb{R}$ has an accumulation point, we conclude. □

2.3.2 Generic Simplicity Result

To prove our main theorem, we need to give a series of lemmas, that will build the proof at the end.

Lemma 2.14. *Let $g \in L^1_{loc}(\mathbb{R})$ (locally integrable). If for all $W \in \mathbb{W}$,*

$$\int_{\mathbb{R}} W(x)g(x)dx = 0,$$

then $g = 0$ almost everywhere.

Proof. Denote by $\mathbb{1}_{[a,b]}$ the indicator function of the interval $[a,b]$, and by φ the standard mollifier

$$\varphi(x) = \begin{cases} ce^{\frac{1}{x^2-1}} & \text{if } -1 \leqslant x \leqslant 1 \\ 0 & \text{elsewhere} \end{cases},$$

where c is for normalization. For $\epsilon \leqslant 1$, the function

$$W_\epsilon := \varphi_\epsilon * \mathbb{1}_{[a,b]} := \frac{1}{\epsilon}\varphi\left(\frac{x}{\epsilon}\right) * \mathbb{1}_{[a,b]}$$

is in $\mathbb{W}$: smooth, non-negative, its support is subset of $[a-\epsilon, b+\epsilon] \subset [a-1, b+1]$. Moreover, it converges as $\epsilon \to 0$ pointwise to $\mathbb{1}_{[a,b]}$ (this is true by the properties of a mollifier, also called approximation of identity). Now, since g is locally integrable, we apply Lebesgue dominated convergence theorem to deduce that

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}} W_\epsilon(x)g(x)dx = \int_a^b g(x)dx.$$

The left-hand side of the preceding equation is 0 by assumption, so $\int_a^b g(x)dx = 0$. This is true for arbitrary a, b which implies that $g = 0$ a.e. in $\mathbb{R}$. $\square$

It is well-known that the Schrödinger operator has simple eigenvalues on the line (see [70] for instance), but this is not the case on the circle nor on $\mathbb{R}^n$ for $n > 1$. So, for a moment, we forget that we are working on $\mathbb{R}$ and we prove the following lemma on $\mathbb{R}^n$ for $n > 1$, which is a variation of Albert's arguments in [8] (also, it is not hard to see that it remains true on $\mathbb{S}^1$). In the following lemma, x^2 will denote $\|x\|^2$.
Remark that lemma 2.14 holds true on $\mathbb{R}^n$.

Lemma 2.15. *Fix $V \in \mathbb{V}$ and $k \in \mathbb{Z}^*$. Let λ be an eigenvalue of P_V^k of multiplicity m. Then, the following assertions hold true.*

1. *There exists $W \in \mathbb{W}$ such that $P^k_{V+tx^2W}$ has an eigenbranch starting from λ of multiplicity strictly less than m.*

2. *If we denote by $\kappa = \operatorname{dist}(\lambda, \operatorname{spec}(P^k_V))$, then there exists $t_0 > 0$ and $W \in \mathbb{W}$ such that for all $0 < t \leqslant t_0$, $P^k_{V+tx^2W}$ has m simple eigenvalues in $I =]\lambda - \frac{\kappa}{2}, \lambda + \frac{\kappa}{2}[$.*

Proof. 1. Fix $W \in \mathbb{W}$. By analytic perturbation, the eigenvalue λ splits into m eigenbranches (not necessarily distinct) of $P^k_{V+tx^2W}$ (this fact can be extracted from the proof of lemma 2.10 when we proved that $\dim(\mathcal{V}^\epsilon_t) = \dim(\mathcal{V}_0))$.

Suppose that the eigenbranches are identical, and denote this eigenbranch by $\lambda(t)$ ($\lambda(0) = \lambda$). This means that we can find m orthonormal eigenfunction branches $\{u_1(t), ..., u_m(t)\}$ associated to $\lambda(t)$. If we denote by E_λ the eigenspace of λ in P^k_V, then $\mathcal{U} = \{u_1(0), ..., u_m(0)\}$ is an orthonormal basis of E_λ.

Denote by $\dot{q}$ the quadratic form, defined on E_λ by

$$\dot{q}(u) = k^2 \int_{\mathbb{R}^n} x^2 W(x) |u(x)|^2 dx.$$

Hellmann–Feynman theorem (theorem 2.11) implies that at $t = 0$, we have

$$\dot{\lambda}(0) = \dot{q}(u_i(0)), \qquad \forall 1 \leqslant i \leqslant m,$$

where the dot represents the derivative with respect to t. Moreover, using remark 2.12, we get that for any $i \neq j$,

$$0 = \dot{q}(u_i(0), u_j(0)),$$

where we used the same notation for the corresponding symmetric bilinear form. Thus the matrix $A_\mathcal{U} := [\dot{q}(u_i(0), u_j(0))]_{1 \leqslant i,j \leqslant m}$ satisfies $A_\mathcal{U} = \dot{\lambda}(0) I$, where I is the $m \times m$ identity matrix.

So, for any orthonormal basis $\mathcal{V} = \{v_1, ..., v_m\}$ of E_λ, The matrix $A_\mathcal{V}$ is a multiple of the identity matrix. Indeed, the matrices $A_\mathcal{U}$ and $A_\mathcal{V}$ are related as follows: if we denote by P the matrix of change of basis between $\mathcal{U}$ and $\mathcal{V}$, then, as $\mathcal{U}$ and $\mathcal{V}$ are sets of orthonormal vectors, P is orthogonal (that is $P^t P = I$), and

$$A_\mathcal{V} = P^t A_\mathcal{U} P = \dot{\lambda}(0) P^t P = \dot{\lambda}(0) I.$$

Now, fix $\mathcal{V} = \{v_1, ..., v_m\}$ an orthonormal basis of E_λ, and suppose that for any $W \in \mathbb{W}$, the m eigenbranches of $P^k_{V+tx^2W}$ are identical. Then, for any $W \in \mathbb{W}$, there exists a constant that depend on W, $c(W)$, such that

$$k^2 \int_{\mathbb{R}^n} x^2 W(x) v_i(x) v_j(x) dx = c(W)\delta_i^j. \tag{2.26}$$

This implies, by lemma 2.14, that for any $i \neq j$,

$$\forall x \in \mathbb{R}^n,\ |v_i(x)|^2 \ = \ |v_j(x)|^2.$$

Then,

$$\forall x \in \mathbb{R}^n, v_1(x) = \pm v_2(x). \tag{2.27}$$

Moreover, (2.26) implies that

$$\forall x \in \mathbb{R}^n, v_1(x)v_2(x) = 0. \tag{2.28}$$

Thus, (2.27) and (2.28) implies that $\forall x \in \mathbb{R}^n$, $\pm v_1(x)^2 = 0$ which implies that $v_1 = 0$; a contradiction as v_1 is normal.

2. We prove this by induction on m. Assume first that $m = 2$. by part one of this lemma, there is $W \in \mathbb{W}$ such that $P^k_{V+tx^2W}$ has an eigenbranch of multiplicity strictly less than 2. This means that the two eigenbranches are simple[1]. We choose t small enough so that by proposition 2.7, the eigenvalues of P_{V+tx^2W} are simple and in I.

   Suppose this is true for $m-1$. We prove it for m. Using part one of this lemma, there exists a $W_0 \in \mathbb{W}$ such that $P^k_{V+tx^2W_0}$ has an eigenbranch of multiplicity strictly less than m.
   Now, there might be several groups of identical eigenbranches. Lets enumerate them as

$$\Lambda_1 = \{\lambda_1^1(t) = ... = \lambda_1^{m_1}(t)\}, \Lambda_2 = \{\lambda_2^1(t) = ... = \lambda_2^{m_2}(t)\}, ...$$

   where $1 \leqslant m_i \leqslant m - 1$ for all $1 \leqslant i \leqslant \frac{m}{2}$. Now, since two analytic functions are either identical for all t or intersect only on countable set (lemma 2.13), and since the eigenbranch representing Λ_1 is different than those representing Λ_2, then by the analyticity of the eigenbranches, we can choose t_0 small enough so that by proposition 2.7,

[1] For eigenbranches, simple means there are no two identical eigenbranches. They may intersect at a countable set of t however.

$P_{V+t_0x^2W_0}$ has an eigenvalue of multiplicity m_1, an eigenvalue of multiplicity m_2, etc. in I.

By induction hypothesis, there exists $W_1 \in \mathbb{W}$, such that the eigenvalue of multiplicity m_1 of $P_{V+t_0x^2W_0}$ will now split into m_1 distinct eigenbranches of $P_{V+x^2(t_0W_0+tW_1)}$ in I. Now, as these eigenbranches were split by the first place from those in Λ_i for $2 \leqslant i \leqslant \frac{m}{2}$, then again by analyticity of these eigenbranches, they can not, under a perturbation, come back again identical (at least they are different at $t = 0$).

Now, choosing t_1 small enough (so that the condition of proposition 2.7 is now true for $t_0W_0 + t_1W_1$), $P_{V+x^2(t_0W_0+t_1W_1)}$ have m_1 simple eigenvalues and another eigenvalues of multiplicity possibly greater than 1 in I.

Proceeding with the same argument for any set of identical eigenbranches for the resulting perturbed operator, we conclude in the last step that we can choose t_{i_0} small enough and $W_{i_0} \in \mathbb{W}$ such that $P^k_{V+x^2(t_0W_0+...+t_{i_0}W_{i_0})}$ has m distinct eigenvalues in I (where i_0 is definitely less than or equal $\frac{m}{2}$).

in fact, the preceding construction of the $W's$ implies the following statement: $\exists i_0 \leqslant \frac{m}{2}, \exists t_1, ..., t_{i_0}, \forall s_j \leqslant t_j, j \leqslant i_0, \exists W_j(s_j - 1)$, such that

$$P^k_{V+x^2(s_0W_0+s_1W_1(s_0)+...+s_{i_0}W_{i_0}(s_{i_0-1}))} \text{ has simple eigenvalues in } I. \quad (2.29)$$

This concludes the proof. □

Lemma 2.16. *Fix $V \in \mathbb{V}$, and $k, l \in \mathbb{Z}^*$ such that $k^2 \neq l^2$. Let λ be a common simple eigenvalue for P^k_V and P^l_V (simple in both spectrums). Then, there exists $W \in \mathbb{W}$, such that the eigenbranch starting from λ of P^k_{V+tW} and the eigenbranch starting from λ of P^l_{V+tW} are not identical.*

Proof. Suppose that for any $W \in \mathbb{W}$, the eigenbranch of $P^k_{V+x^2tW}$ starting from λ is identical to the eigenbranch of $P^l_{V+x^2tW}$ starting from λ. Denote this eigenbranch by $\lambda(t)$ ($\lambda(0) = \lambda$), and denote by $u(t, W)$ (resp. $v(t, W)$) a corresponding normalized eigenfunction branch for $P^k_{V+x^2tW}$ (resp. $P^l_{V+x^2tW}$). Then, if we denote by E^k_λ and E^l_λ the eigenspace that corresponds to λ in P^k_V and P^l_V respectively, then $u(0, W)$ and $v(0, W)$ are orthonormal basis for E^k_λ and E^l_λ respectively.

Applying Hellmann–Feynman theorem at $t = 0$, we get that

$$\dot{\lambda}(0) = k^2 \int_{\mathbb{R}} x^2W(x)|u(0, W)(x)|^2dx = l^2 \int_{\mathbb{R}} x^2W(x)|v(0, W)(x)|^2dx. \quad (2.30)$$

Since E_λ^k and E_λ^l are one dimensional, then $u(0,W)$ and $v(0,W)$ are independent of W. This implies that for any $W \in \mathbb{W}$,

$$k^2 \int_{\mathbb{R}} x^2 W(x)|u(0)(x)|^2 dx = l^2 \int_{\mathbb{R}} x^2 W(x)|v(0)(x)|^2 dx. \tag{2.31}$$

By lemma 2.14, (2.31) implies that

$$k^2 x^2 |u(0)(x)|^2 = l^2 x^2 |v(0)(x)|^2 \text{ a.e.}$$

Thus, as $u(0)$ and $v(0)$ are normalized, we get that $k^2 = l^2$ which is a contradiction. □

We give now a lemma about the stability of simple eigenvalues under small perturbations.

Lemma 2.17. *Fix $V \in \mathbb{V}$. Suppose that P_V^k has m simple eigenvalues in an interval I. Then, for a small enough perturbation of the operator P_V^k, the m eigenvalues of the perturbed operator in I remain simple.*

Proof. Define the set

$$\mathcal{T} = \{V \in \mathbb{V}; P_V^k \text{ has } m \text{ simple eigenvalues in } I\}.$$

We want to prove that the set $\mathcal{T}$ is open in $\mathbb{V}$. Let $V \in \mathcal{T}$ and let $V_j \in \mathbb{V}$ such that V_j converges to V in $(\mathbb{V}, \|.\|_{\mathbb{V}})$.
We first show that there exists $j_0 > 0$ such that for all $j \geqslant j_0$, $V_j \in \mathcal{T}$. Indeed, by proposition 2.7, there is j_1 small enough such that for all $j \leqslant j_1$, $P_{V_j}^k$ has m eigenvalues in I.
Denote by $\{\lambda_1(V), ..., \lambda_m(V)\}$ and $\{\lambda_1(V_j), ..., \lambda_m(V_j)\}$ the m eigenvalues in I of P_V^k and $P_{V+V_j}^k$ (for $j \leqslant j_1$) respectively.
Moreover, by proposition 2.6 (continuity of the spectrum), for any $\epsilon > 0$ there exists $j_\epsilon > 0$ such that for all $j \geqslant j_\epsilon$ we have

$$|\lambda_i(V_j) - \lambda_i(V)| < 2\epsilon \qquad i = 1, ..., m.$$

Let

$$\delta = \min_{\substack{i \neq l \\ i,l \in \{1,...,m\}}} |\lambda_i(V) - \lambda_l(V)| > 0$$

and take $\epsilon = \dfrac{\delta}{4} > 0$. For any $j > j_0 := \min\{j_1, j_\epsilon\}$, we compute

$$\begin{aligned}\delta &\leqslant |\lambda_i(V) - \lambda_l(V)|\\ &\leqslant |\lambda_i(V) - \lambda_i(V_j)| + |\lambda_i(V_j) - \lambda_l(V_j)| + |\lambda_l(V_j) - \lambda_l(V)|\\ &< \delta + |\lambda_i(V_j) - \lambda_l(V_j)|.\end{aligned}$$

Thus for any $j \geqslant j_0$, $|\lambda_i(V_j) - \lambda_l(V_j)| > 0$ which implies that $\lambda_i(V_j) \neq \lambda_l(V_j)$ for any $i \neq l$. We deduce that for any $j \geqslant j_0$, $V_j \in \mathcal{T}$.
Now, suppose that $\mathcal{T}$ is not open in $\mathbb{V}$. Then for any $V \in \mathcal{T}$, there exists $\epsilon > 0$ such that $B(V, \epsilon) \nsubseteq \mathcal{T}$. Apply the condition repeatedly with $\epsilon = \frac{1}{j}$, we get that there is a sequence V_j such that

$$\left\|V - V_j\right\|_{\mathbb{V}} \leqslant \frac{1}{j} \text{ and } V_j \in \mathcal{T}^c,$$

where $\mathcal{T}^c$ is the complement of $\mathcal{T}$ in $\mathbb{V}$. This implies that there exists a sequence V_j that converges to V but is not in $\mathcal{T}$ for any j which is a contradiction. Therefore, $\mathcal{T}$ is open in $\mathbb{V}$ and we conclude. □

Finally, before (re-)stating and proving our main theorem, we define a Baire space and state Baire's category theorem.

Definition 2.18. *A topological space is called a Baire space if every countable intersection of dense open sets is dense.*

Lemma 2.19 (Baire's Category Theorem)**.** *Every completely metrizable topological space is a Baire space.*

Proof. Refer to [55] for a proof. □

Corollary 2.19.1. *For $V = x^2 W \in \mathbb{V}$, we recall the norm $\|V\|_{\mathbb{V}} := \|W\|_{\infty}$. The space $\mathbb{V}$ equipped with the norm $\|.\|_{\mathbb{V}}$ is a Baire space.*

Proof. We prove that the metric space $(\mathbb{V}, \|.\|_{\mathbb{V}})$ is complete.
Let $(V_n)_{n\geqslant 1} = (x^2 W_n)_{n\geqslant 1}$ be a Cauchy sequence in $(\mathbb{V}, \|.\|_{\mathbb{V}})$. Then, for any $\epsilon > 0$, there exist $n_0 \in \mathbb{N}$ such that for all $m, n \geqslant n_0$, we have

$$\|V_n - V_m\|_{\mathbb{V}} < \epsilon.$$

This implies that for any $\epsilon > 0$, there exist $n_0 \in \mathbb{N}$ such that for all $m, n \geqslant n_0$,

$$\|W_n - W_m\|_{\infty} = \|V_n - V_m\|_{\mathbb{V}} < \epsilon,$$

which gives that $(W_n)_{n\geqslant 1}$ is a Cauchy sequence in $(\mathcal{C}_b^0(\mathbb{R}),\|.\|_\infty)$ which is a complete space. So, W_n converges to some W in $\mathcal{C}_b^0(\mathbb{R})$ uniformly. Finally, we get, for $V = x^2W$ (which is clearly in $\mathbb{V}$), that

$$\|V_n - V\|_{\mathbb{V}} = \|W_n - W\|_\infty \to 0.$$

We conclude that $(V_n)_{n\geqslant 1}$ is convergent in $(\mathbb{V},\|.\|_{\mathbb{V}})$ and therefore the space $(\mathbb{V},\|.\|_{\mathbb{V}})$ is a complete metric space (metric induced from the norm). Using lemma 2.19, $(\mathbb{V},\|.\|_{\mathbb{V}})$ is a Baire space. □

Recall that we say that a subset is residual in a metric space if it is the countable intersection of open dense sets. We now state and prove our main result in this chapter.

Theorem 2.20. *The complement of* $\mathbb{V}_b$ *in* $\mathbb{V}$ *is residual in* $\mathbb{V}$.

Proof. Define the set

$$\mathcal{O}_{k,l,n} = \{V \in \mathbb{V}; \mathrm{spec}_n(P_V^k) \cap \mathrm{spec}_n(P_V^l) = \varnothing\} \subset \mathbb{V},$$

where $\mathrm{spec}_n(P_V^k)$ (resp. $\mathrm{spec}_n(P_V^k)$) denotes the first n eigenvalues of P_V^k (resp. P_V^k) counted without multiplicity. By definition, the set of the good $V's$ satisfying the property (P) is the intersection over all k, l, n of $\mathcal{O}_{k,l,n}$. We prove that $\mathcal{O}_{k,l,n}$ is open and dense in $\mathbb{V}$.

The set $\mathcal{O}_{k,l,n}$ is open in $\mathbb{V}$. Indeed, take $V \in \mathcal{O}_{k,l,n}$. Let $(V_j)_{j\geqslant 1}$ be a sequence in $\mathbb{V}$ that converges to V in $\|.\|_{\mathbb{V}}$. We first prove that there exists $j_0 > 0$ such that for all $j \geqslant j_0$, $V_j \in \mathcal{O}_{k,l,n}$.
Recall that $\lambda_m^k(V)$ and $\lambda_m^l(V)$ denote the m^{th} eigenvalue of P_V^k and P_V^l respectively. Since $(V_j)_{j\geqslant 1}$ converges to V in $\mathbb{V}$, then by proposition 2.6, for any $\epsilon > 0$, there exists $j_k > 0$ and $j_l > 0$ such that for all $j \geqslant j_{k,l} := \max\{j_k, j_l\}$ the following inequalities hold true

$$|\lambda_m^k(V_j) - \lambda_m^k(V)| < 2\epsilon, \text{ and } |\lambda_m^l(V_j) - \lambda_m^l(V)| < 2\epsilon. \tag{2.32}$$

Now, let $I = \{1, ..., n\}$ and denote by

$$\delta = \min_{i_1,i_2\in I} \left\{|\lambda_{i_1}^k(V) - \lambda_{i_2}^l(V)|\right\} > 0.$$

For $\epsilon = \frac{\delta}{4}$, there exists $j_{k,l}$ such that for all $j \geqslant j_{k,l}$, (2.32) implies that

$$\begin{aligned}\delta &\leqslant |\lambda_{i_1}^k(V) - \lambda_{i_2}^l(V)| \\ &\leqslant |\lambda_{i_1}^k(V) - \lambda_{i_1}^k(V_j)| + |\lambda_{i_1}^k(V_j) - \lambda_{i_2}^l(V_j)| + |\lambda_{i_2}^l(V_j) - \lambda_{i_2}^l(V)| \\ &< \frac{\delta}{2} + |\lambda_{i_1}^k(V_j) - \lambda_{i_2}^l(V_j)| + \frac{\delta}{2} = \delta + |\lambda_{i_1}^k(V_j) - \lambda_{i_2}^l(V_j)|.\end{aligned}$$

This implies that $|\lambda^k_{i_1}(V_j) - \lambda^l_{i_2}(V_j)| > 0$ i.e. $\lambda^k_{i_1}(V_j) \neq \lambda^l_{i_2}(V_j)$ for all $i_1, i_2 \in I$, which means that

$$\operatorname{spec}_n(P^k_{V_j}) \cap \operatorname{spec}_n(P^l_{V_j}) = \emptyset.$$

Therefore, for any $j \geqslant j_{k,l}$, $V_j \in \mathcal{O}_{k,l,n}$.
Now, suppose to contrary that $\mathcal{O}_{k,l,n}$ is not open, then for any $V \in \mathcal{O}_{k,l,n}$, there exists $\epsilon > 0$ such that $B(V, \epsilon) \not\subset \mathcal{O}_{k,l,n}$. Applying this condition repeatedly with $\epsilon = \frac{1}{j}$, we get that there is a sequence V_j such that

$$\left\|V_j - V\right\|_{\mathbb{V}} \leqslant \frac{1}{j} \text{ and } V_j \in \mathcal{O}^c_{k,l,n}.$$

This implies that there exists a sequence V_j that converges to V but is not in $\mathcal{O}_{k,l,n}$ for any j which is a contradiction. Therefore, $\mathcal{O}_{k,l,n}$ is open.

Moreover, The set $\mathcal{O}_{k,l,n}$ is dense in $\mathbb{V}$. Indeed, let $V \in \mathbb{V}$ with $V \notin \mathcal{O}_{k,l,n}$. Require to prove that

$$\forall \epsilon > 0, \exists \tilde{V}_\epsilon \in \mathcal{O}_{k,l,n} \text{ such that } \left\|V - \tilde{V}_\epsilon\right\|_{\mathbb{V}} < \epsilon. \tag{2.33}$$

We know that $V \notin \mathcal{O}_{k,l,n}$, so there are some common eigenvalues between P^k_V and P^l_V among the first n eigenvalues of each, some of which maybe be with multiplicity greater than one. Take a common eigenvalue λ of multiplicity m_1 and m_2 in P^k_V and P^l_V respectively. Let

$$\kappa = \min\{\operatorname{dist}(\lambda, \operatorname{spec}(P^k_V)), \operatorname{dist}(\lambda, \operatorname{spec}(P^l_V))\},$$

and denote by $I =]\lambda - \frac{\kappa}{2}, \lambda + \frac{\kappa}{2}[$. By (2.29), $\exists i_0 \leqslant \frac{m_1}{2}(\text{ in } \mathbb{N}^*), \exists t_0, ..., t_{i_0}; \forall s_j \leqslant t_j, 0 \leqslant j \leqslant i_0, \exists W_j = W_j(s_{j-1}) \in \mathbb{W}$;

$$P^k_{V + s_0 W_0 + ... + s_{i_0} W_{i_0}} \text{ has } m_1 \text{ simple eigenvalues in } I.$$

Now, again we do the same to obtain m_2 simple eigenvalues corresponding to l. So, $\exists i_1 \leqslant \frac{m_1}{2} + \frac{m_2}{2}(\text{ in } \mathbb{N}^*), \exists t_0, ..., t_{i_1}; \forall s_j \leqslant t_j, 0 \leqslant j \leqslant i_1, \exists W_j := W_j(s_{j-1}) \in \mathbb{W}$;

$$P^l_{V + s_0 W_0 + ... + s_{i_1} W_{i_1}} \text{ has } m_2 \text{ simple eigenvalues in } I.$$

By stability of simple eigenvalues under small perturbations (lemma 2.17), we get that $\exists i_1 \leqslant \frac{m_1}{2} + \frac{m_2}{2}(\text{ in } \mathbb{N}^*), \exists \tilde{t}_0, ..., \tilde{t}_{i_1}; \forall s_j \leqslant \tilde{t}_j, 0 \leqslant j \leqslant i_1, \exists W_j := W_j(s_{j-1}) \in \mathbb{W}$;

$$P^k_{V + s_0 W_0 + ... + s_{i_1} W_{i_1}} \text{ and } P^l_{V + s_0 W_0 + ... + s_{i_1} W_{i_1}} \text{ has } m_1 \text{ and } m_2 \text{ simple eigenvalues in } I \text{ resp. }.$$

Now, suppose among these simple eigenvalues there are m common eigenvalues. Then, by lemma 2.16, there exists $i_2 \leqslant \frac{m_1}{2}+\frac{m_2}{2}+m($ in $\mathbb{N}^*), \exists \tilde{t}_0, ..., \tilde{t}_{i_2}; \forall s_j \leqslant \tilde{t}_j, 0 \leqslant j \leqslant i_2, \exists W_j := W_j(s_{j-1}) \in \mathbb{W}$;

$P^k_{V+s_0W_0+...+s_{i_2}W_{i_2}}$ and $P^l_{V+s_0W_0+...+s_{i_2}W_{i_2}}$ has no common eigenvalues in I resp. .

Now, this is true for any λ in common between P^k_V and P^l_V. Now, applying the same argument for the second common eigenvalue, then the third, etc..., it yields the following: $\exists \varsigma \in \mathbb{N}^*, \exists t_0, ..., t_\varsigma, \forall s_j \leqslant t_j, j \leqslant \varsigma$, $\exists W_j = W_j(s_j - 1)$, such that

$$\left(\bigcap_{r=k,l} \operatorname{spec}_n \left(P^r_{V+x^2(s_0W_0+s_1W_1(s_0)+...+s_\varsigma W_\varsigma(s_{\varsigma-1}))}\right)\right) = \varnothing.$$

Note that after dealing with the first intersection, we can freely deal with the next one; the eigenbranches that will be extracted from the simple (with respect to k or l) disjoint (with respect to k and l) eigenvalue we dealt with in the previous step won't come back identical because they were split at the first place (by analyticity).

So, what we do to conclude, is that for any $j \leqslant \varsigma$, we choose s_j small enough so that

$$|s_j|\left\|x^2W_j(s_{j-1})\right\|_\infty < \frac{\epsilon}{\varsigma}.$$

This construction implies (2.33).
Therefore, the set

$$\mathbb{V}^c_b := \bigcap_{k,l,n} \mathcal{O}_{k,l,n}$$

is residual in $\mathbb{V}$. □

2.4 Baouendi Grushin Operator On A Torus

Denote by $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ the two dimensional flat torus with fundamental domain $[-\pi,\pi]_x \times [-\pi,\pi]_y$. For any function $f : \mathbb{T}^2 \to \mathbb{R}$ (or $\mathbb{C}$), there corresponds one, and only one (2π-)periodic function $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}$ (or $\mathbb{C}$) given by $\tilde{f}(x, y) = f(e^{ix}, e^{iy})$.

2.4.1 Baouendi Grushin Operator On Torus

In the previous section, when the infinite cylinder X was considered, it was crucial to work on the space $L_0^2(X)$ which is the usual L^2 space excluding the functions that are independent on y. This is because we were working with the one-dimensional parameter-dependent Schrödinger operator P_V^k, instead of the two-dimensional Grushin operator whose spectrum is the union over k of the spectrum of P_V^k, that at $k=0$, doesn't have a compact resolvent. Here, however, we are working on a Torus which is a compact manifold, and this guarantees the compactness of the resolvent. For that, we consider the space

$$L^2(\mathbb{T}^2) = L^2(\mathbb{T}^2, \mathbb{R}) = \left\{ f : \mathbb{T}^2 \to \mathbb{R}; \|f\|_{L^2(\mathbb{T}^2)} := \left\|\tilde{f}\right\|_{L^2([-\pi,\pi])} < \infty \right\}.$$

Denote by $\mathcal{C}^0(\mathbb{S}^1)$ the space of continuous 2π-periodic functions on $\mathbb{R}$;

$$\mathcal{C}^0(\mathbb{S}^1) = \{f : \mathbb{S}^1 \to \mathbb{R}; \tilde{f} \in \mathcal{C}^0(\mathbb{R})\}.$$

We define the Baouendi Grushin operator on $\mathbb{T}^2$ for $V \in \mathcal{C}^0(\mathbb{S}^1)$ that looks like x^2 near 0, and vanishes nowhere in $]-\pi, \pi[$ but on 0. A typical example would be $\sin^2(x)$ (usually, investigators consider $V(x) = 4\sin^2(\frac{x}{2})$ for normalization so that $V''(0) = 2$). For this, it is reasonable to define the set

$$\tilde{\mathbb{V}} = \{V(x) = \sin^2(x)W(x); W \in \mathcal{C}^0(\mathbb{S}^1), W \geqslant 1\}.$$

On $\mathbb{V}$, we put the following norm: for $V = \sin^2(x)W(x) \in \tilde{\mathbb{V}}$, $\|V\|_{\tilde{\mathbb{V}}} = \|W\|_\infty$.

For $V \in \tilde{\mathbb{V}}$, we denote by $\tilde{P}_V$ the operator $\tilde{P}_V = -\partial_x^2 - V(x)\partial_y^2$, defined on

$$\tilde{D} = \{u \in L^2(\mathbb{T}^2); \partial_x^2 u \in L^2(\mathbb{T}^2), V(x)\partial_y^2 \in L^2(\mathbb{T}^2)\}.$$

Take a strip of the torus,

$$w = [-\pi, \pi]_x \times [a, b]_y \subset [-\pi, \pi]_x \times [-\pi, \pi]_y.$$

Finally, we define, for $V \in \tilde{\mathbb{V}}$, the one dimensional Schrödinger operator as $\tilde{P}_V^k = -\partial_x^2 - k^2 V(x)$, with domain

$$\tilde{\mathbb{D}}_V = \{u \in H^1(\mathbb{S}^1); V^{\frac{1}{2}} u \in L^2(\mathbb{S}^1)\}.$$

The operator $(\tilde{P}_V^k, \tilde{\mathbb{D}}_V)$ has compact resolvent. Its spectrum consists of an increasing sequence of positive eigenvalues.

2.4.2 Spectral Condition

As discussed in the introduction of subsection 2.1.3, theorem 2.2 do not really depend on the operator P_V but on the fact that the eigenfunctions of P_V can be written using separation of variables. Also, the same argument about the eigenvalues of P_V being of multiplicity greater than or equal to 2 implies that the multiplicity of the eigenvalues of $\tilde{P}_V$ are even. Thus, it follows that theorem 2.2 is true for the operator $\tilde{P}_V$; that is, the spectral condition is a sufficient condition for the validity of the concentration inequality for the operator $\tilde{P}_V$.

2.4.3 Analyticity Of Eigen-Quantities

The approach of proving the analyticity of the eigenbranches of the perturbed operator P^k_{V+tW} for $t \in \mathbb{R}$ and $W \in \mathbb{W}$, does not depend, neither on the manifold X nor on the explicit expression of V. We may just say that having a compact support for W was important so that the x^2 coefficient wouldn't cause any problem when integrating over $\mathbb{R}$. Here however, as we work on $\mathbb{S}^1$, this is not longer an issue, and so, for any $t \in \mathbb{R}, V \in \tilde{\mathbb{V}}$ and

$$W \in \tilde{\mathbb{W}} := \{W \in \mathcal{C}^0(\mathbb{S}^1); W \geqslant 0\},$$

the eigenpair of the operator $\tilde{P}^k_{V+t\sin^2 xW}$ are analytic in t. As a consequence, we get the validity of the Hellmann–Feynman theorem.

2.4.4 Generic Simplicity Result

Now, denote by $(\tilde{\mathrm{P}})$ the property:

$$\forall k, l \in \mathbb{Z}^*; k^2 \neq l^2 \Rightarrow \operatorname{spec}(\tilde{P}^k_V) \cap \operatorname{spec}(\tilde{P}^l_V) = \phi \qquad (\tilde{\mathrm{P}}).$$

Again this property is equivalent to the validity of the spectral simplicity condition.

As in the previous section, the general scheme is to prove that the set of the good $V's$ satisfying $(\tilde{\mathrm{P}})$ is the intersection of open dense sets, $\tilde{\mathcal{O}}_{k,l,n}$ (which is defined in an obvious way like $\mathcal{O}_{k,l,n}$), in the Baire space $(\tilde{\mathbb{V}}, \|.\|_{\tilde{\mathbb{V}}})$.

The proof of the openness in the previous section depends on proposition 2.6, which is a general lemma on the continuity of eigenvalues. This proposition holds smoothly in this case and thus the openness is okay using the exact same argument.

For the density, we used several lemmas; lemma 2.14, and lemmas 2.15 and 2.16 that depend only on analyticity, Hellmann–Feynman theorem and lemma 2.14. So, what we really want to check is the validity of a version of lemma 2.14 here.

Proposition 2.21. *Let $g \in L^1(\mathbb{S}^1)$. If for all $W \in \tilde{\mathbb{W}}$,*

$$\int_{-\pi}^{\pi} W(x)g(x)dx = 0,$$

then $g = 0$ almost everywhere in $[-\pi, \pi]$.

Proof. For any $[a, b] \subset [-\pi, \pi]$, denote by $\mathbb{1}_{[a,b]}$ the indicator function of the interval $[a, b]$, which is integrable. Let φ be any mollifies and denote by $\varphi_\epsilon(x) = \frac{1}{\epsilon}\varphi(\frac{x}{\epsilon})$. Then, the sequence $W_\epsilon := \varphi_\epsilon * \mathbb{1}_{[a,b]}$ is in $\tilde{\mathbb{W}}$. Moreover, it converges as $\epsilon \to 0$ pointwise to $\mathbb{1}_{[a,b]}$. Now, since g is integrable, we apply Lebesgue dominated convergence theorem to deduce that

$$\lim_{\epsilon \to 0} \int_{-\pi}^{\pi} W_\epsilon(x)g(x)dx = \int_a^b g(x)dx.$$

The left-hand side of the preceding equation is 0 by assumption, so $\int_a^b g(x)dx = 0$. This is true for arbitrary $-\pi \leqslant a, b \leqslant \pi$ which implies that $g = 0$ a.e. in $\mathbb{R}$. $\square$

Finally, as explained, we need the fact that $\tilde{\mathbb{V}}$ equipped with $\|.\|_{\tilde{\mathbb{V}}}$ is a Baire space to conclude that the complement of $\tilde{\mathbb{V}}_b$ is residual in $\tilde{\mathbb{V}}$.

Proposition 2.22. *On $\tilde{\mathbb{V}}$, define the norm $\|V\|_{\tilde{\mathbb{V}}} := \|W\|_{L^\infty(\mathbb{S}^1)}$ for $V = \sin^2(x)W$. Then the space $(\tilde{\mathbb{V}}, \|.\|_{\tilde{\mathbb{V}}})$ is a Baire space.*

Proof. By lemma 2.19, it is sufficient to prove that the metrizable space $(\tilde{\mathbb{V}}, \|.\|_{\tilde{\mathbb{V}}})$ is a complete space.
Let $(V_n)_{n\geqslant 1} = (\sin^2(x)W_n)_{n\geqslant 1}$ be a Cauchy sequence in $(\tilde{\mathbb{V}}, \|.\|_{\tilde{\mathbb{V}}})$. Then the sequence $(W_n)_{n\geqslant 1}$ is Cauchy in $(\mathcal{C}^0(\mathbb{S}^1), \|.\|_{L^\infty(\mathbb{S}^1)})$, which is a complete space. Thus, $(W_n)_{n\geqslant 1}$ is convergent to some W in $(\mathcal{C}^0(\mathbb{S}^1), \|.\|_{L^\infty(\mathbb{S}^1)})$. This implies that $(V_n)_{n\geqslant 1}$ is convergent in $(\tilde{\mathbb{V}}, \|.\|_{\tilde{\mathbb{V}}})$ to $\sin^2(x)W$. Therefore, $(\tilde{\mathbb{V}}, \|.\|_{\tilde{\mathbb{V}}})$ is a complete metrizable space and thus a Baire space. $\square$

Therefore, mimicking the same arguments in the proof of theorem 2.20, if we denote by

$$\tilde{\mathbb{V}}_g := \{V \in \tilde{\mathbb{V}}; (\tilde{P}) \textit{ holds true } \},$$

we prove that

Theorem 2.23. *The set* $\tilde{\mathbb{V}}_g$ *is residual in* $(\tilde{\mathbb{V}}, \|.\|_{\tilde{\mathbb{V}}})$.

Chapter 3

Riemannian Approximation Of Subriemannian Structures

We present a general construction allowing us to see a subriemannian structure as a singular limit of Riemannian metrics. We then study how different geometric or spectral quantities behave along this process. We prove that this approximation scheme induces a volume form, that we compare to Popp's volume.

3.1 Introduction

Let M be a smooth connected manifold of dimension d and consider p smooth vector fields $X^1, ..., X^p$, not necessarily independent, such that they satisfy the Hörmander condition: the vector fields $X^1, ..., X^p$ and their iterated brackets $[X^i; X^j], [X^i; [X^j; X^k]]$,... span the tangent space $T_m M$ at every point $m \in M$ (see [50]).
Let g be the subriemannian metric associated to $\mathscr{C} = \{X^1, ..., X^p\}$ (defined by (1.9)). The structure $(M, \text{span}(\mathscr{C}), g)$ is called a subriemannian structure. The metric g induces a length on the set of horizontal paths, and thus a distance on M called the subriemannian distance.
To analyze functions on subriemannian manifolds, it is necessary to define an appropriate analog of the Laplace operator. We define the sublaplacian

associated to a smooth volume ω on M following (1.7):

$$\Delta = \sum_{i=1}^{p} X_i^* X_i = \sum_{i=1}^{p} -X_i^2 + \mathrm{div}_\omega(X_i) X_i, \tag{3.1}$$

where $\mathrm{div}_\omega(X_i)$ is the divergence of X_i with respect to ω. As said earlier, constructing an intrinsic Laplacian in this context is nontrivial due to the lack of a globally defined metric (we can see from the expression of Δ that its definition depends on the volume form ω). Some natural volume forms were defined since the question was asked in 1982 such as Popp's volume form [13][66] and Hausdorff volume form [2][42][65].
Simultaneously at that time, approximation schemes of subriemannian structures made an appearance as an alternate approach to study properties on subriemannian manifolds.

We study in this chapter the possibility that an approximation scheme induces a volume form. Moreover, we compare the induced volume form with Popp's volume. We first recall some definitions.

3.1.1 Definitions

We recall in this section some definitions in Riemannian and subriemannian geometry.

3.1.1.1 Riemannian Definitions

The Riemannian definitions can be found in any book that handles Riemannian geometry (see [61][71] for instance).
A Riemannian metric $\mathcal{G}$ on a smooth manifold M is a smoothly chosen inner product

$$\mathcal{G}_m := \mathcal{G}(m) : T_mM \times T_mM \to \mathbb{R},$$

on each of the tangent spaces T_mM of M. The smoothness of $\mathcal{G}$ is in the sense that the map $m \mapsto \mathcal{G}_m$, from M to the space of all symmetric positive definite bilinear forms on $TM \times TM$, is smooth. Whenever convenient, we will consider, without loss of generality, $\mathcal{G}$ as a quadratic form (we use the same notation for the quadratic form and its corresponding bilinear form).
For a smooth manifold M equipped with a Riemannian metric $\mathcal{G}$, we define:

- The **length** associated to $\mathcal{G}$ of an A.C. (absolutely continuous) curve on M, $\gamma : [0,1] \to M$ as

$$l_{\mathcal{G}}(\gamma) = \int_0^1 \sqrt{\mathcal{G}_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))}dt.$$

- The **Riemannian distance** between two points x and y in M as:

$$d^R(x,y) = \inf\{l_{\mathcal{G}}(\gamma); \gamma \text{ is A.C. connecting } x \text{ to } y\}.$$

- The **energy functional** associated to $\mathcal{G}$ of an A.C. curve on M, $\gamma : [0,1] \to M$ as

$$\mathcal{E}_{\mathcal{G}}(\gamma) = \int_0^1 \mathcal{G}_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))dt.$$

- A **geodesic** as a curve which is everywhere locally a length minimizer.

3.1.1.2 Subriemannian Definitions

We follow standard subriemannian references (see [3][14][52][66]).
Let M be a smooth connected manifold.

- A **distribution** D on M assigns to any point $m \in M$ a vector subspace $D(m) \subset T_m M$, such that $D(m)$ is the span of a set of smooth vector fields evaluated at m.

- A distribution D is said to satisfy **Hörmander's condition** at a point $m \in M$ if there exists $r = r(m)$ ($r(m) + 1$ is called the step) such that $D_r(m) = T_m M$, where for each $0 \leqslant i \leqslant r - 1$,

$$D_{i+1}(m) = D_i(m) + [D(m), D_i(m)], \tag{3.2}$$

  where we have set $D_0 = D$ and

$$[D(m), D_i(m)] = \text{span}\left\{[X,Y] : X \in D(m), Y \in D_i(m)\right\}.$$

- We say that a set of smooth vector fields on M, $\{X_1, ..., X_p\}$, satisfies Hörmander's condition if $D = \text{span}\{X_1, ..., X_p\}$ satisfies the Hörmander's condition.

- We say that two vector fields X, Y on M commute if $[X, Y] = 0$.
- The vector
$$(\dim(D_0(m)), \dim(D_1(m)), ..., \dim(D_r(m)))$$
is called the **growth vector** at the point m.
- We say that a point m_0 is **regular** if the growth vector is constant in a neighborhood of m_0. Otherwise, m_0 is called **singular**. (M, D) is **equiregular** if every point is regular.
- The **equiregular region** is the largest open set on which the subriemannian structure is equiregular. The **singular region** $\mathcal{Z}$ of the subriemannian structure is the complement of the equiregular region.
- We say that a vector $X \in TM$ is **horizontal** if $X(m) \in D(m)$ for any m. We say that a curve γ is horizontal if for every t, $\dot{\gamma}(t) \in D(\gamma(t))$.

Let $X^1, ..., X^p$ be smooth vector fields and suppose that $D = \text{span}\{X^1, ..., X^p\}$ satisfies the Hörmander condition. Then

- The **subriemannian metric** associated to the vector fields $\{X^1, ..., X^p\}$ is the function $g^0 : D(m) \to \mathbb{R} \cup \{+\infty\}$ given by
$$g^0_m(\mathcal{X}(m)) := g^0(m, \mathcal{X}(m)) = \inf \left\{ \sum_{i=1}^{p} u_i^2 : u \in \mathbb{R}^p, \sum_{i=1}^{p} u_i X^i(m) = \mathcal{X}(m) \right\},$$
with the convention that $\inf\{\varnothing\} = +\infty$.
- The length associated to g^0 of an A.C. path $\gamma(t), t \in [0, 1]$ is given by
$$l_{g^0}(\gamma) = \int_0^1 \sqrt{g^0_{\gamma(t)}(\dot{\gamma}(t))} dt.$$
- The subriemannian distance between x and y is defined as
$$d^{sR}(x, y) = \inf\{l_{g^0}(\gamma) : \gamma \text{ is A.C. connecting } x \text{ to } y \text{ }, \gamma \text{ is horizontal}\}. \tag{3.3}$$

- The subriemannian structure is said to be **almost Riemannian** if $\dim(D_0) = \dim(M)$ at almost every point[1].

The Chow–Rashevskii theorem, known as Chow's theorem, asserts that any two points of a connected subriemannian manifold, endowed with a bracket generating distribution, are connected by a horizontal path in the manifold (see [3][52]). So, d^{sR} is a distance function on M. We set now our framework.

3.1.2 Framework

We recall that for any $n \in \mathbb{N}$, for any $u = (u_1, ..., u_n) \in \mathbb{R}^n$, $|u|_n$ denotes the Euclidean norm. Denote by $\langle ., . \rangle_n$ the usual inner product on $\mathbb{R}^n$. Hereafter, m will always denote a point in the manifold M and for any vector field X, X_m will denote $X(m)$.

Let M be a smooth connected manifold of dimension d. Consider p smooth vector fields $X^{01}, ..., X^{0p}$ (there will not be any use for an exponent of a vector field, so this notation won't cause any confusion), and suppose that $D_0 = \text{span}\{X^{01}, ..., X^{0p}\}$ satisfies Hörmander condition at any point $m \in M$ of step $r(m) + 1$. We assume[2] that $r(m)$ admits a maximum on M and we denote this maximum by r; i.e

$$r = \max\{r(m); m \in M\}.$$

We denote by g^0 the subriemannian metric associated to the vector fields $\{X^{01}, ..., X^{0p}\}$; for any $m \in M$ and $X_m \in D_0(m)$, g^0 is given by

$$g^0_m(X_m) := g^0(m, X_m) = \inf \left\{ |u|_p^2; u \in \mathbb{R}^p, \sum_{i=1}^{p} u_i X^{0i}_m = X_m \right\}. \tag{3.4}$$

Denote by d^0, given by (3.3), the subriemannian distance associated to g^0. For $0 \leqslant i \leqslant r$, Define D_i as in (3.2), and denote by

$$n_i(m) := \dim(D_i(m)).$$

We simply write n_i for $n_i(m)$ when m is fixed.
For $0 \leqslant i \leqslant r$, let $J_i = (j_1, ..., j_{i+1})$, where for any $1 \leqslant k \leqslant i$, $j_k \in \{1, ..., i+1\}$.
Let

$$X_{J_i} = [X^{0j_1}, [X^{0j_2}, [...[X^{0j_i}, X^{0j_{i+1}}]...]].$$

[1] But not at every point, otherwise the structure is Riemannian.
[2] This is true if M is compact

For $0 \leqslant i \leqslant r$, we have

$$\operatorname{card}\{X_{J_i}; J_i \subset \{1, ..., p\}^{i+1}\} = p^{i+1}.$$

Hörmander condition ensures that these vector fields (for all i) span the tangent space at every point.

Now, we make a selection of these vector fields that still span the tangent space. This choice of vectors will make no significant or important difference in the context but will make the examples we give throughout the chapter easier (simpler) to write.
Among these vector fields, there might be a lot of zeros (either by bracketing a vector field with itself, or by bracketing two commuting vector fields), but whether we consider these zero vector fields or not will not make any difference in the context (see the first point of remark 3.12). So, without loss of generality, we exclude the zero vectors.
Now, many of the vector fields X_{J_i} may be repeated the same, and many may be colinear. This may happen in several cases. For instance, if X, Y, Z are smooth vector fields such that $Z - Y$ commutes with X, then $[X, Z] = [X, Y] + [X, Z - Y] = [X, Y]$.
Another case is the following. It is clear that for any i and any $J_i = (j_1, \cdots, j_i, j_{i+1})$, if we define $\hat{J}_i = (j_1, \cdots, j_{i+1}, j_i)$ then $X_{\hat{J}_i} = -X_{J_i}$. In this work, of these two vectors, we only keep one. Observe there may still be the same vector several times but we will keep these pairs in any other circumstances.

Since by Hörmander condition, the vectors in $\bigcup_i \bigcup_{J_i} X_{J_i}$ span the tangent space, then the vectors we choose also span the tangent space. This is because we only excluded zero vectors and colinear vectors.
Although Hörmander condition is still true, this selection will make some differences in the context below, and when it does, we will point it out (see for instance the second point of remark 3.12, and remark 3.25).

So, for $1 \leqslant i \leqslant r$, we denote by N_i the cardinal of the vectors obtained by iterative brackets of length i, of the initial vector fields, among the vectors we chose. We enumerate them as

$$\{X^{i1}, ..., X^{iN_i}\}.$$

Set $N_0 = p$. As we explained, by Hörmander's condition, we have, for every $m \in M$ that

$$T_m M = \operatorname{span}\{X_m^{ij}\}_{0 \leqslant i \leqslant r, 1 \leqslant j \leqslant N_i}. \tag{3.5}$$

3.2 Approximation Scheme

3.2.1 Definition Of The Scheme

Let $N = N_0 + ... + N_r$. For $u \in \mathbb{R}^N$, we write $u = (u_0, u_1, ..., u_r)$, where each u_i is of length N_i. For all $h \in \mathbb{R}\backslash\{0\}$ and all $u \in \mathbb{R}^N$, define the dilation δ_h as

$$\delta_h(u) = (u_0, h^{-1}u_1, h^{-2}u_2, ..., h^{-r}u_r).$$

For $m \in M$, denote by σ_m the map $\sigma_m : \mathbb{R}^N \to T_mM$, such that for $u = (u_i)_{1 \leqslant i \leqslant N} \in \mathbb{R}^N$,

$$\sigma_m(u) = \sum_{i=0}^{r} \sum_{j=1}^{N_i} u_{ij} X_m^{ij}. \tag{3.6}$$

For $m \in M$, and $X_m \in T_mM$, we define the function g_m^h as

$$g_m^h(X_m) := g^h(m, X_m) = \inf \left\{ |\delta_h u|_N^2 ; u \in \mathbb{R}^N, \sigma_m(u) = X_m \right\}. \tag{3.7}$$

The function $h \mapsto g^h$ is monotonically decreasing for any $h > 0$. Indeed, let $(m, X_m) \in TM$ and fix $u = (u_0, ..., u_r) \in \mathbb{R}^N$ such that $\sigma_m(u) = X_m$. Then, as the function $h \in \mathbb{R}^{+*} \mapsto (1/h^2)$ is strictly decreasing, we get that for all $0 < h_1 < h_2$,

$$g_m^{h_2}(X_m) \leqslant |\delta_{h_2} u|_N^2 = \sum_{i=0}^{r} h_2^{-2i} |u_i|^2 < \sum_{i=0}^{r} h_1^{-2i} |u_i|^2 = |\delta_{h_1} u|_N^2.$$

Taking the infimum over all $u \in \mathbb{R}^N$ satisfying $\sigma_m(u) = X_m$, we get that, for all $0 < h_1 < h_2$,

$$g_m^{h_2}(X_m) \leqslant g_m^{h_1}(X_m).$$

We first prove that g^h is a Riemannian metric. We need the following lemmas. The first one is basic linear algebra.

Lemma 3.1. *Let $T : V \to W$ be a linear map and let V' be a subspace of V such that V' is a complement of $K = ker(T)$ in V. Then*

$$T' := T|_{V'} : V' \to T(V)$$

is an isomorphism.

Proof. Obviously, T' is linear and well-defined. We prove now that T' is bijective.
Let $u, v \in V'$ such that $T'(u) = T'(v)$. Then, $T(u-v) = T'(u-v) = 0$. This implies that

$$u - v \in V' \cap K = \{0\}.$$

Thus $u = v$ and so T' is injective.
Now, let $w \in T(V)$. Then there exist $u \in V$ such that $T(u) = w$. As $u \in V$, then $u = u_1 + u_2$ with $u_1 \in V', u_2 \in K$. So,

$$w = T(u) = T(u_1) + T(u_2) = T(u_1) = T'(u_1).$$

Then T' is surjective and therefore bijective. So T' is an isomorphism. $\square$

Lemma 3.2. *Let $\mathcal{M}$ be a smooth manifold. For $m \in \mathcal{M}$, let V be a n dimensional vector space equipped with a smooth inner product denoted by $\langle .,. \rangle_m$. Let $\{e_1(m), ..., e_k(m)\}$ be linearly independent vectors in V, smooth in m, such that, for any $m \in \mathcal{M}$,*

$$\dim\big(\mathrm{span}\{e_1(m), ..., e_k(m)\}\big) = \text{constant}.$$

Let $W(m) = \mathrm{span}\{e_1(m), ..., e_k(m)\}$. Then, there exist $e^1(m), ..., e^{n-k}(m)$ smooth in m, such that for all $m \in \mathcal{M}$, they form a basis of $W(m)^\perp$, where $W(m)^\perp$ is the orthogonal complement of $W(m)$ with respect to the inner product $\langle .,. \rangle_m$.

Proof. Fix m_0, and let $\{w_1, ..., w_{n-k}\}$ be a basis of the fixed subspace $W(m_0)^\perp$. As m moves in a neighborhood of m_0, we have

$$\mathrm{span}\{e_1(m), ..., e_k(m), w_1, ..., w_{n-k}\} = V.$$

Indeed, if we denote by $R(m)$ the representation matrix of $\{e_1(m), ..., e_k(m), w_1, ..., w_{n-k}\}$ in the basis $\{e_1(m_0), ..., e_k(m_0), w_1, ..., w_{n-k}\}$, then $R(m)$ can be written as a block matrix

$$R(m) := \left(\begin{array}{c|c} I_k + \Phi(m-m_0) & \Theta \\ \hline 0 & I_{n-k} \end{array}\right) = I_n + \left(\begin{array}{c|c} \Phi(m-m_0) & \Theta \\ \hline 0 & 0 \end{array}\right),$$

where I_{n-k} is the $n-k$ identity matrix, Θ is a $k \times n-k$ matrix and $\Phi(m-m_0)$ is a $k \times k$ matrix that, by the smoothness and the linear independence of

e_i, converges to 0 as m goes to m_0. So, for m close enough to m_0, and by continuity of the determinant function, we have $\det(R(m)) \neq 0$.
Define now, π_m as the projection on $W(m)$, and let, for $i = 1, ..., n-k$,

$$e^i(m) = w_i - \pi_m w_i.$$

The family $\{e^i(m)\}_{1 \leqslant i \leqslant n-k}$ is a basis of $W(m)^\perp$. Indeed, let $i - \pi_m$ be the orthogonal projection on $W(m)^\perp$, where i represents the identity. Restrict $i - \pi_m$ to $W(m_0)^\perp$. So,

$$(i - \pi_m)\big|_{W(m_0)^\perp} : W(m_0)^\perp \to W(m)^\perp,$$

where both $W(m_0)^\perp$ and $W(m)^\perp$ are of dimension $n-k$. If $u \in \ker(i - \pi_m)$ then $u \in W(m) \cap W(m_0)^\perp = \{0\}$. This is due to the fact that
$\{e_1(m), ..., e_k(m), w_1, ..., w_{n-k}\}$ span V, which gives that

$$V = \text{span}\{e_1(m), ..., e_k(m)\} \oplus \text{span}\{w_1, ..., w_{n-k}\} = W(m) \oplus W(m_0)^\perp.$$

Thus, $(i - \pi_m)\big|_{W(m_0)^\perp}$ is an injective linear finite-dimensional operator whose domain and range have the same dimension and so an isomorphism. So, the image of the set $\{w_1, ..., w_{n-k}\}$ by $i - \pi_m$, which is the set $\{e^i(m)\}_{i=1,...,n-k}$, is a basis of $W(m)^\perp$.
It remains to prove that $m \mapsto \pi_m$ is smooth. Let ξ_0 be a fixed vector in V. We can write

$$\pi_m(\xi_0) = \sum_{i=1}^{k} b_i(m) e_i(m).$$

We need to prove that for any $i \in \{1, ..., k\}$, $m \mapsto b_i(m)$ is smooth. Since $\pi_m(\xi_0) - \xi_0 \in W(m)^\perp$, then, for any $j = 1, ..., k$, we have

$$\langle \pi_m(\xi_0), e_j(m) \rangle_m = \langle \xi_0, e_j(m) \rangle_m.$$

This gives that

$$\sum_{i=1}^{k} b_i(m) \langle e_i(m), e_j(m) \rangle_m = \langle \xi_0, e_j(m) \rangle_m, \tag{3.8}$$

for any $j = 1, ..., k$. The matrix corresponding to this linear system (which we denote Υ_m) is the gram matrix of $\langle ., . \rangle_m$ on $W(m)$ which is invertible. Now, since for all m, Υ_m is non-singular, the determinant of Υ_m is not zero and is a polynomial of the entries of Υ_m. Then by the smoothness of (the entries of) Υ_m, $(1/\det(\Upsilon_m))$ is smooth in m. Using a similar argument, we have that the adjugate matrix of Υ_m is smooth and so its inverse is. Then

$$b(m) := (b_i(m))_{i=1,\dots,k} = (\langle \xi_0, e_j(m)\rangle_m)_{j=1,\dots,k}\Upsilon_m^{-1}$$

is the unique smooth solution of the linear system (3.8). Therefore, we conclude that π_m is smooth in m and therefore $\{e^1(m), ..., e^{n-k}(m)\}$ is a smooth basis of $W(m)^{\perp}$. □

Lemma 3.3. *Let $\mathcal{M}$ be a smooth connected manifold of dimension d, and let $\{X^1, ..., X^N\}$ be N smooth vector fields that span the tangent space at every point. Recall the definition of σ_m given by (3.6).*
The function $\mathcal{G}$ defined on TM by

$$\mathcal{G}_m(X_m) = \inf\{|u|_N^2; u \in \mathbb{R}^N, \sigma_m(u) = X_m\}$$

is a Riemannian metric on M.

Proof. Let $K_m = \ker(\sigma_m)$, and denote by $K_m^{\perp}$ the orthogonal compliment of K_m with respect to $\langle ., .\rangle_N$.
Since $T_mM = \text{span}\{X_m^1, ..., X_m^N\}$ for every $m \in M$, $K_m^{\perp}$ is a linear subspace of $\mathbb{R}^N$ of dimension d. Write $\mathbb{R}^N$ as orthogonal decomposition as follows:

$$\mathbb{R}^N = K_m \bigoplus K_m^{\perp}.$$

Using lemma 3.1, the map

$$\sigma_m' := \sigma_m|_{K_m^{\perp}} : K_m^{\perp} \to T_mM$$

is an isomorphism, and so, there exist a unique $u_{X_m} \in K_m^{\perp}$ such that $\sigma_m(u_{X_m}) = X_m$. So, we can write

$$\sigma^{-1}(\{X_m\}) = \{u_{X_m} + v; v \in K_m\}.$$

This implies that for $u \in \sigma^{-1}(\{X_m\})$, we have

$$\begin{aligned}
|u|_N &= |u_{X_m} + v|_N \\
&= |u_{X_m}|_N + 2\langle u_{X_m}, v\rangle_N + |v|_N \\
&= |u_{X_m}|_N + |v|_N \\
&\geqslant |u_{X_m}|_N,
\end{aligned}$$

Taking infimum over all u in $\mathbb{R}^N$ satisfying $\sigma_m(u) = X_m$ gives that $\mathcal{G}_m(X_m) = |u_{X_m}|_N^2$.
We define now the bilinear form. Let $X_m, Y_m \in T_mM$. Define now,

$$\mathcal{G}_m(X_m, Y_m) = \langle u_{X_m}, u_{Y_m} \rangle_N.$$

We have:

- Linearity of the map $X_m \to u_{X_m}$. Since σ'_m is an isomorphism, then $(\sigma'_m)^{-1} : X_m \mapsto u_{X_m}$ is also an isomorphism and in particular linear.

- Positive definiteness of the map G_m: This map is obviously positive. Moreover, as $\sigma'_m(u_{X_m}) = X_m$, then by the injectivity of σ'_m, we have $X_m = 0$ if and only if $u_{X_m} = 0$. Since $\mathcal{G}_m(X_m) = |u_{X_m}|_N^2$, then $\mathcal{G}_m(X_m) = 0$ if and only if $u_{X_m} = 0$ if and only if $X_m = 0$.

- Smoothness of the map $m \mapsto \mathcal{G}_m$: Choose some coordinates $(x_1, ..., x_d)$ on M. For any $i = 1, ..., N$, write

  $$X_m^i = \sum_{j=1}^{d} a_{ij}(m) \frac{\partial}{\partial x_j}.$$

  Since the vectors $X_m^1, ..., X_m^N$ span T_mM at every m, the matrix $A(m) := (a_{i,j}(m))_{i,j}$ is of rank d. So there is an invertible square submatrix of $A(m)$, say $A_1(m)$, of dimension d, and $A(m)$ can be written, up to reordering (multiplication by a permutation matrix), as a row of matrices

  $$A(m) = \begin{pmatrix} A_1(m) & B(m) \end{pmatrix},$$

  for some $d \times (N-d)$ matrix $B(m)$.
  The subspace K_m has a smooth basis. Indeed, let $\xi_0 \in K_m$, and write $\xi_0 = \begin{pmatrix} \xi_1^t & \xi_2^t \end{pmatrix}$, where ξ_1 and ξ_2 are of dimensions $d \times 1$ and $(N-d) \times 1$ respectively. Since $\xi_0 \in K_m = \ker(\sigma_m)$, we have

  $$\begin{pmatrix} A_1(m) & B(m) \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0.$$

  Then, $\xi_1 = -A_1(m)^{-1}B(m)\xi_2$. For $i = 1, ..., N-d$, define the vectors

  $$e_i(m) = \begin{pmatrix} -A_1(m)^{-1}B(m)\tilde{e}_i \\ \tilde{e}_i \end{pmatrix},$$

where $\tilde{e}_i$ is the $(N-d)\times 1$ column vector

$$\tilde{e}_i := (0, ..., \underset{\text{i-th}}{\underset{\uparrow}{1}}, ..., 0)^t.$$

The vectors $\{e_i(m)\}_{i=1,...,N-d}$ are $N-d$ linearly independent vectors (by the independence of the vectors $\{\tilde{e}\}_{1\leqslant i\leqslant N-d}$) in the $N-d$ dimensional subspace K_m, and so they form a basis for K_m that is smooth by the smoothness of A (that implies the smoothness of $A_1(m)^{-1}$ and $B(m)$). By lemma 3.2, there exist smooth basis $e^1(m), ..., e^d(m)$ for $K_m^{\perp}$.
If we write u_{X_m} as

$$u_{X_m} = \sum_{i=1}^{d} u^i(m)e^i(m),$$

we get, since σ' is an isomorphism, that

$$X_m = \sigma_m(u_{X_m}) = \sum_{i=1}^{d} u^i(m)\sigma_m(e^i(m)).$$

By smoothness of X_m, we get the smoothness of $u^i(m)$, and thus the smoothness of u_{X_m}, which together, with the smoothness of $\langle .,.\rangle_N$, concludes the proof.

□

Remark 3.4. *An important feature of this proof is that it shows that the infimum in the expression of $g_m^h(X_m)$ is attained at a unique $u \in (\ker(\sigma_m))^{\perp} \subset \mathbb{R}^N$.*

As the vector fields in our settings satisfy the Hörmander condition (3.5), we get

Corollary 3.4.1. *For any $h \in \mathbb{R}\backslash\{0\}$, the function g^h defined by (3.7) is a Riemannian metric on M.*

Proposition 3.5. *The following assertions hold true:*

1. *Fix $m \in M$. Then, $g_m^h\big|_{D_0} \leqslant g_m^0\big|_{D_0}$*
2. *Fix $m \in M$. Then we have $\lim_{h\to 0} g_m^h\big|_{D_0} = g_m^0\big|_{D_0}$.*

Proof. 1. We can see this by observing that, for a fixed $m \in M$ and $X_m \in D_0$, we have

$$g^0_m(X_m) = \inf\left\{|\delta_h u|^2_N; u = (u_0, 0, ..., 0) \in \mathbb{R}^N, u_0 \in \mathbb{R}^p, \sigma_m(u) = X_m\right\},$$

and that

$$\left\{|\delta_h u|^2_N; u = (u_0, 0, ..., 0) \in \mathbb{R}^N, u_0 \in \mathbb{R}^p, \sigma_m(u) = X_m\right\} \subset \left\{|\delta_h u|^2_N; u \in \mathbb{R}^N, \sigma_m(u) = X_m\right\}.$$

2. Fix $m \in M$ and $X_m \in D_0$. For any $h > 0$, there exists $u(h) = (u_0(h), ..., u_r(h)) \in \mathbb{R}^N$ such that $\sigma_m(u(h)) = X_m$ and

$$g^h_m(X_m) = |u(h)|^2_N = |u_0(h)|^2_p + \sum_{i=1}^r h^{2i}|u_i(h)|^2_{N_i} \leqslant g^0_m(X_m),$$

where the last inequality is by part one of proposition 3.5. So, we get that $\lim_{h\to 0} u_i(h) = 0$ for any $i = 1, ..., r$. Thus,

$$X_m = \sigma_m(u(h)) = \lim_{h\to 0} \sigma_m(u(h)) = \sigma_m((u_0(0), 0, ..., 0)).$$

Finally, we have that

$$|u_0(h)|^2_p \leqslant |u_0(h)|^2_p + \sum_{i=1}^r h^{2i}|u_i(h)|^2_{N_i} = g^h_m(X_m) \leqslant g^0_m(X_m).$$

Therefore, as $h \to 0$, and by definition of g^0, we get that

$$g^0_m(X_m) \leqslant \lim_{h\to 0} g^h_m(X_m) \leqslant g^0_m(X_m).$$

We conclude.

□

3.2.2 Convergence Of Distances

Denote by d^h the Riemannian distance corresponding to g^h; for any $x, y \in M$,

$$d^h(x, y) = \inf\{l_{g^h}(\gamma); \gamma \text{ is A.C. connecting } x \text{ to } y\}. \tag{3.9}$$

Sometimes the distance function can be defined equivalently using different notations. For instance, the authors in [68] introduced the metric on M as follows. Let $C(\delta)$ denote the class of absolutely continuous curves $\gamma : [0,1] \to M$ such that γ satisfies

$$\dot{\gamma}(t) = \sum_{i=0}^{r}\sum_{j=1}^{N_i} u_{ij}(t) X^{ij}(\gamma(t)), \tag{3.10}$$

with

$$|u_{ij}(t)| \leqslant \delta^i \tag{3.11}$$

Then, they defined the distance as

$$\rho(x,y) = \inf\{\delta > 0; \exists \gamma \in C(\delta) \text{ that connects } x \text{ to } y\}.$$

In fact, the distance we defined by (3.9) can be defined in the same way, by changing (3.11) to a suitable condition. Denote by $C^h(\delta)$ the class of absolutely continuous curves $\gamma : [0,1] \to M$ that satisfies

$$\dot{\gamma}(t) = \sum_{i=0}^{r}\sum_{j=1}^{N_i} u_{ij}(t) X^{ij}(\gamma(t)), \tag{3.12}$$

with

$$\sum_{i=0}^{r}\sum_{j=1}^{N_i} h^{-2i} |u_{ij}(t)|^2 \leqslant \delta^2 \tag{3.13}$$

and define the distance

$$\rho^h(x,y) = \inf\{\delta > 0, \exists \gamma \in C^h(\delta) \text{ that connects } x \text{ to } y\}.$$

Then, $d^h(x,y) = \rho^h(x,y)$. Indeed, let $\delta > 0$ be such that there is $\gamma \in C^h(\delta)$. This means γ satisfies (3.12) and (3.13). Then, by the definition of $d^h(x,y)$ we get

$$d^h(x,y) \leqslant l_{g^h}(\gamma) = \int_0^1 \sqrt{g^h_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))} \leqslant \delta.$$

Take the infimum over all δ to get that $d^h(x,y) \leqslant \rho^h(x,y)$.
Now, observe that

$$d^h(x,y) = \inf\{l_{g^h}(\gamma); \gamma \text{ is a geodesics from } x \text{ to } y\}.$$

Let γ be a geodesic connecting x to y. Then, there exists $\varphi = \gamma \in C^h(\delta)$ for $\delta = l_{g^h}(\gamma)$. Indeed, γ is a geodesic between x and y, and so, it has a constant speed. Thus, we have that

$$g_{\gamma(t)}(\dot{\gamma}(t)) = \int_0^1 \sqrt{g_{\gamma(t)}(\dot{\gamma}(t))}dt. \tag{3.14}$$

Also, we know that there exists $u = (u_{ij})_{0 \leqslant i \leqslant r, 1 \leqslant j \leqslant N_i} \in \mathbb{R}^N$ such that

$$g_{\gamma(t)}(\dot{\gamma}(t)) = \sum_{i=0}^{r} \sum_{j=1}^{N_i} h^{-2i} |u_{ij}(t)|^2. \tag{3.15}$$

Equations (3.14) and (3.15) implies that there exists $\varphi \in C^h(l_{g^h}(\gamma))$ connecting x to y. Thus, $\rho^h(x, y) \leqslant d^h(x, y)$. We conclude.

So, a few differences are made with the authors in [68]. For instance, our condition (3.13) considers the euclidean norm of u while their condition (3.11) considers the L^∞ norm. Also, the scaling by δ is not exactly the same.

Recall that d^0 is the subriemannian distance associated to g^0. As said earlier, many authors proved that the subriemannian distance is the limit of a family of Riemannian distances in the Gromov-Haussdorf sense (see [4][36][40][43][67][83]). We adapt the proof of [40] in our case: we prove that d^h converges to d^0 uniformly on every compact set of M.

Definition 3.6. *For a general Riemannian metric g on M, the Sobolev space $H^1(([0, 1], M), g)$ is defined as*

$$H^1(([0, 1], M), g) = \{\gamma : [0, 1] \to M; \mathcal{E}_g(\gamma) < \infty\}.$$

This definition is equivalent to say that for any chart ϕ on M, $\phi \circ \gamma \in H^1([0, 1], \mathbb{R}^n)$.

Lemma 3.7. *For any two Riemannian metrics g_1 and g_2, the Sobolev spaces $H^1(([0, 1], M), g_1)$ and $H^1(([0, 1], M), g_2)$ are equivalent; that is, for any absolutely continuous curve $\gamma : [0, 1] \to M$, we have*

$$\mathcal{E}_{g_1}(\gamma) < \infty \text{ if and only if } \mathcal{E}_{g_2}(\gamma) < \infty.$$

Proof. Denote by G_1 and G_2 the representation matrices of g_1 and g_2 respectively. Also, for $i = 1, 2$, denote by $\lambda^i_{\min}(m) > 0$ and $\lambda^i_{\max}(m) > 0$ the minimum eigenvalue and the maximum eigenvalue respectively of $G_i(m)$. We compute

$$\begin{aligned}
\mathcal{E}_{g_1}(\gamma) < \infty &= \int_0^1 g_{1\gamma(t)}(\dot\gamma(t), \dot\gamma(t))dt \\
&= \int_0^1 (\dot\gamma(t))^t G_1(\gamma(t))\dot\gamma(t)dt \\
&= \int_0^1 \frac{(\dot\gamma(t))^t G_1(\gamma(t))\dot\gamma(t)}{(\dot\gamma(t))^t G_2(\gamma(t))\dot\gamma(t)} (\dot\gamma(t))^t G_2(\gamma(t))\dot\gamma(t)dt.
\end{aligned}$$

Thus, we get

$$\int_0^1 \frac{\lambda^1_{\min}(t)}{\lambda^2_{\max}(t)} g_{2\gamma(t)}(\dot\gamma(t), \dot\gamma(t)) \leqslant \mathcal{E}_{g_1}(\gamma)dt \leqslant \int_0^1 \frac{\lambda^1_{\max}(t)}{\lambda^2_{\min}(t)} g_{2\gamma(t)}(\dot\gamma(t), \dot\gamma(t))dt. \tag{3.16}$$

Now, for $i = 1, 2$, the eigenvalues of $G_i(m)$ are the roots of the characteristic polynomial of $G_i(m)$, whose coefficients are smooth in m (as they consist of the entries of G_m which is smooth). Now since the roots of a polynomial vary continuously on the coefficients, then they are continuous on M (see [45]). Therefore, (3.16) implies that

$$\begin{aligned}
\inf_{t\in[0,1]} \left(\frac{\lambda^1_{\min}(t)}{\lambda^2_{\max}(t)}\right) \int_0^1 g_{2\gamma(t)}(\dot\gamma(t), \dot\gamma(t)) \leqslant \mathcal{E}_{g_1}(\gamma)dt \\
\leqslant \inf_{t\in[0,1]} \left(\frac{\lambda^1_{\max}(t)}{\lambda^2_{\min}(t)}\right) \int_0^1 g_{2\gamma(t)}(\dot\gamma(t), \dot\gamma(t))dt.
\end{aligned}$$

Since M is compact, there exists $c_1, c_2 > 0$ such that

$$c_1\mathcal{E}_{g_2}(\gamma) \leqslant \mathcal{E}_{g_1}(\gamma) \leqslant c_2\mathcal{E}_{g_2}(\gamma).$$

□

Here, we fix a reference Riemannian metric g^1, on M, and we denote by $H^1([0,1], M) := (H^1([0,1], M), g^1)$. We need the following lemma:

Lemma 3.8. *Let $\mathcal{M}$ be a smooth Riemannian manifold, and let $\aleph^1, \aleph^2$ be two Riemannian metrics on $\mathcal{M}$. Fix some compact subset of $\mathcal{M}$, say K. Then, there exist $c > 1$ such that for any $m \in K$ and $v \in T_m\mathcal{M}$, we have*

$$\frac{1}{c}\aleph^1(v) \leqslant \aleph^2(v) \leqslant c\aleph^1(v).$$

Proof. Denote by A the unit sphere bundle of $\mathcal{M}$ restricted on K,

$$A = \{(m, v) \in T\mathcal{M}; m \in K, v \in T_m\mathcal{M} \text{ and } \aleph^1_m(v) = 1\}.$$

Since K is compact, A is also compact. Moreover, $\aleph^2$ is continuous on $T\mathcal{M}$, and so $\aleph^2\big|_A$ is continuous. A continuous map on a compact set is bounded and so there is some $c_1, c_2 > 0$, such that

$$c_1 \leqslant \aleph^2\big|_A \leqslant c_2.$$

Choose some c such that $\frac{1}{c} \leqslant c_1 \leqslant c_2 \leqslant c$. By definition of A, we get that for any $m \in K$ and $v \in T_m\mathcal{M}$,

$$\frac{1}{c}\aleph^1(v) \leqslant \aleph^2(v) \leqslant c\aleph^1(v). \tag{3.17}$$

Now let $m \in K$ and $v \in T_m\mathcal{M}$. Then, $\tilde{v} := (1/\aleph^1_m(v))v \in A$, so, applying (3.17) to $\tilde{v}$ and by homogeneity of quadratic forms, we get

$$\frac{1}{c[\aleph^1_m(v)]^2}\aleph^1(v) \leqslant \frac{1}{[\aleph^1_m(v)]^2}\aleph^2(v) \leqslant \frac{c}{[\aleph^1_m(v)]^2}\aleph^1(v),$$

which implies that (3.17) is true for all $m \in K$ and $v \in T_mM$. □

Theorem 3.9. *The distance d^h converges to d^0 uniformly on every compact set of M.*

Proof. Suppose to contrary, there exist a compact K, $\epsilon_0 > 0$, such that for every h, there exist $x_h, y_h \in K$ such that

$$|d^h(x_h, y_h) - d^0(x_h, y_h)| > \epsilon_0. \tag{3.18}$$

First, observe that, since K is compact, then up to a subsequence, the sequences x_h and y_h are convergent to some x_0 and y_0 respectively (with respect to d^1).
Now, For $x, y \in M$, for all $h \geqslant 0$, we have that $d^0(x, y) \geqslant d^h(x, y)$. Indeed, let γ be a horizontal curve connecting x to y. We have,

$$l_{g^0}(\gamma) = \int_0^1 \sqrt{g^0_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))}dt \geqslant \int_0^1 \sqrt{g^h_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))}dt = l_{g^h}(\gamma) \geqslant d^h(x, y).$$

The second inequality holds because γ is horizontal and $g^h\big|_{D_0} \leqslant g^0\big|_{D_0}$. Taking infimum over all horizontal curves γ connecting x to y, we get

$$d^h(x,y) \leqslant d^0(x,y). \tag{3.19}$$

Thus, (3.18) implies that for all $h \geqslant 0$,

$$d^h(x_h,y_h) - d^0(x_h,y_h) < -\epsilon_0. \tag{3.20}$$

Moreover, (3.19) implies that there exists a constant $c = \sup_{x,y\in K}\{d^0(x,y)\}$ such that all $h \geqslant 0$, we have

$$d^h(x_h,y_h) < c. \tag{3.21}$$

For $h > 0$, let $\gamma_h : [0,1] \to M$ be a minimizing geodesic connecting x_h and y_h parametrized such that

$$g^h_{\gamma_h}(\dot{\gamma_h},\dot{\gamma_h}) = cst.$$

Now, denote by $B(x,c)$, the ball of center x and radius c with respect to the reference distance d^1; that is

$$B(x,c) = \{y \in M, d^1(x,y) < c\}.$$

Then, there exist $h_0 > 0$ such that for all $0 < h \leqslant h_0$ and all $t \in [0,1]$, $\gamma_h(t) \in B(x_0, 1+c)$. Indeed, for all $0 < h < 1$ and $t \in [0,1]$, we compute

$$d^1(x_h,\gamma_h(t)) \leqslant d^h(x_h,\gamma_h(t)) \leqslant d^h(x_h,y_h) \leqslant c,$$

and so, $\gamma_h(t) \in B(x_h,c)$. Moreover, we compute

$$d^1(x_0,\gamma_h(t)) \leqslant d^1(x_0,x_h) + d^1(x_h,\gamma_h(t)) \leqslant d^1(x_0,x_h) + c.$$

Since x_h converges to x_0 as $h \to 0$, then there exist $h_0 > 0$ such that $d^1(x_0,x_h) \leqslant 1$. Thus, for all $0 < h \leqslant h_0$,

$$\gamma_h \subset B(x_0, 1+c) \subset \overline{B(x_0,1+c)}.$$

Then lemma 3.8, with the fact that g^h is strictly decreasing for $h > 0$, implies that there exist a constant $c_1 > 1$ such that for all $0 < h < \min\{1, h_0\}$, we have

$$\int_0^1 g^1_{\gamma_h}(\dot\gamma_h(t), \dot\gamma_h(t))dt \leqslant c_1 \int_0^1 g^1_{\gamma_h}(\dot\gamma_h(t), \dot\gamma_h(t))dt \leqslant c_1 \int_0^1 g^h_{\gamma_h}(\dot\gamma_h(t), \dot\gamma_h(t))dt \leqslant c_1 c^2. \tag{3.22}$$

Thus, γ_h is uniformly bounded in $H^1([0,1], M) \xhookrightarrow{compact} C^0([0,1], M)$. This implies that γ_h converges uniformly in $C^0([0,1], M)$ and weakly in $H^1([0,1], M)$. By uniqueness of limit, the weak limit equals the uniform limit, which we denote by γ_0. The curve γ_0 is an absolutely continuous (as it is in $H^1([0,1], M)$) curve that connects x_0 to y_0.
Moreover, γ_0 is horizontal. Indeed, for any $h > 0$, we have

$$\int_0^1 g^h_{\gamma_0(t)}(\dot\gamma_0(t), \dot\gamma_0(t))^{\frac{1}{2}} dt \leqslant d^h(x_0, y_0) \leqslant d^0(x_0, y_0). \tag{3.23}$$

From (the proof of) lemma 3.3, we know that there exist unique $u_{\dot\gamma_0(t)} = u(t) = (u_0(t), ..., u_r(t))$, $u_i \in \ker(\sigma'_{\gamma(t)})^\perp \subset R^{N_i}$ such that

$$g^h_{\gamma_0(t)}(\dot\gamma_0(t), \dot\gamma_0(t)) = |\delta_h u(t)|^2_N = \sum_{i=0}^r h^{-2i} |u_i(t)|^2_{N_i}. \tag{3.24}$$

Now, since γ_0 is an absolutely continuous function, then its derivative is continuous almost everywhere in $[0,1]$. Also, as $\sigma'_{\gamma(t)}$ is an isomorphism, then it has a continuous inverse, and so

$$u(t) = (\sigma'_{\gamma(t)})^{-1}(\dot\gamma(t))$$

is continuous almost everywhere in $[0,1]$. So, $u_i(t)$ is continuous almost everywhere in $[0,1]$ and $u_i(t)^2$ is integrable on $[0,1]$.
Thus, (3.24) and (3.23) implies that for all $h > 0$ and any $1 \leqslant i \leqslant r$,

$$h^{-2i} \int_0^1 |u_i(t)|^2_{N_i} dt \leqslant d^0(x_0, y_0).$$

This gives that for any $1 \leqslant i \leqslant r$,

$$\int_0^1 |u_i(t)|^2_{N_i} dt = 0,$$

which in turn gives that $u_i(t) = 0$ a.e. This implies that $g^h_{\gamma_0(t)}(\dot\gamma_0(t), \dot\gamma_0(t)) = |u_0(t)|^2_{N_0}$ a.e. which implies that γ_0 is horizontal.
Now, we have that

$$\mathcal{E}_{g^0}(\gamma_0)^{1/2} \leqslant \liminf_{h\to 0}[\mathcal{E}_{g^h}(\gamma_h)^{1/2}]. \tag{3.25}$$

Indeed, using the Cauchy Schwartz inequality, with the fact that γ_0 is horizontal, we have that for all $h > 0$,

$$\int_0^1 g_{\gamma_h(t)}(\dot\gamma_h(t), \dot\gamma_0(t))dt \leqslant \mathcal{E}_{g^h}(\gamma_0)^{1/2}\mathcal{E}_{g^h}(\gamma_h)^{1/2} = \mathcal{E}_{g^0}(\gamma_0)^{1/2}\mathcal{E}_{g^h}(\gamma_h)^{1/2}.$$

Weak convergence in $H^1([0,1], M)$ implies (3.25). Finally, we get

$$d^0(x_0, y_0) \leqslant \mathcal{E}_{g^0}(\gamma_0)^{1/2} \leqslant \liminf_{h\to 0}[\mathcal{E}_{g^h}(\gamma_h)^{1/2}] = \liminf_{h\to 0} d^h(x_h, y_h),$$

which contradicts (3.20). □

Throughout this chapter, we give some examples, which are standard in the context of subriemannian geometry.

Example 3.1 (The Grushin Case On $\mathbb{R}^2$). *On $\mathbb{R}^2$, consider the smooth vector fields*

$$\mathcal{X}_1 = \partial_x \text{ and } \mathcal{X}_2 = x\partial_y.$$

On $\mathbb{R}^2\backslash\{x = 0\}$, $\mathcal{X}_1$ and $\mathcal{X}_2$ span $\mathbb{R}^2$ (step 1). Denote by $\mathcal{X}_3 = [\mathcal{X}_1, \mathcal{X}_2] = \partial_y$. On the singular line $\{x = 0\}$, $\mathcal{X}_1, \mathcal{X}_2$ and $\mathcal{X}_3$ span $\mathbb{R}^2$ (step 2). Following definition (3.4), define the subriemannian metric g^0 on $\mathbb{R}^2$ as

$$g^0(\mathcal{X}) = \inf\{u_1^2 + u_2^2; u = (u_i)_{i=1,2} \in \mathbb{R}^2, \sum_{i=1}^{2} u_i\mathcal{X}_i = \mathcal{X}\}.$$

Following definition (3.7), define the Riemannian metric g^h on $\mathbb{R}^2$ as

$$g^h(\mathcal{X}) = \inf\{u_1^2 + u_2^2 + h^{-2}u_3^2; u = (u_i)_{i=1,2,3} \in \mathbb{R}^3, \sum_{i=1}^{3} u_i\mathcal{X}_i = \mathcal{X}\}.$$

For $\mathcal{X} = (a, b) \in \mathbb{R}^2$, direct computation implies that

$$g^h((a, b)) = a^2 + \frac{b^2}{x^2 + h^2},$$

clearly a smoothly defined inner product on $\mathbb{R}^2$*. Moreover, if* $x \neq 0$*, then*

$$g^0((a,b)) = a^2 + \frac{b^2}{x^2},$$

and so, as $h \to 0$*,* $g^h \to g^0$*. On* $\{x = 0\}$*, for* g^0 *to be finite,* b *must be* 0*, which means that* $\mathcal{X} = a\mathcal{X}_1$*, that is* $\mathcal{X}$ *is horizontal.*

Example 3.2 (The Heisenberg Case On $\mathbb{R}^3$)**.** *On* $\mathbb{R}^3$*, consider the smooth vector fields*

$$\mathcal{X}_1 = \partial_x \text{ and } \mathcal{X}_2 = \partial_y + x\partial_z.$$

For any $x \in \mathbb{R}^3$*,* $\mathcal{X}_1, \mathcal{X}_2$ *and* $\mathcal{X}_3 = [\mathcal{X}_1, \mathcal{X}_2] = \partial_z$ *span* $\mathbb{R}^3$ *(equiregular case of step 2). Following definition (3.4), define the subriemannian metric* g^0 *on* $\mathbb{R}^3$ *as*

$$g^0(\mathcal{X}) = \inf\{u_1^2 + u_2^2\,;\, u = (u_i)_{i=1,2} \in \mathbb{R}^2,\, \sum_{i=1}^{2} u_i\mathcal{X}_i = \mathcal{X}\}.$$

Following definition (3.7), define the Riemannian metric g^h *on* $\mathbb{R}^3$ *as*

$$g^h(\mathcal{X}) = \inf\{u_1^2 + u_2^2 + h^{-2}u_3^2\,;\, u = (u_i)_{i=1,2,3} \in \mathbb{R}^3,\, \sum_{i=1}^{3} u_i\mathcal{X}_i = \mathcal{X}\}.$$

For $\mathcal{X} = (a,b,c) \in \mathbb{R}^3$*, direct computation implies that*

$$g^h((a,b,c)) = a^2 + b^2 + \frac{(c - xb)^2}{h^2},$$

which is clearly a smoothly defined inner product on $\mathbb{R}^3$*. Moreover, for any* $(a,b,c) \in \mathbb{R}^3$*,*

$$g^0((a,b,c)) = a^2 + b^2.$$

For g^0*, which is the limit of* g^h *as* $h \to 0$*, to be finite,* $c - xb$ *must be* 0*, which means that* $\mathcal{X} = a\mathcal{X}_1 + b\mathcal{X}_2$*, that is* $\mathcal{X}$ *is horizontal.*

Example 3.3 (The Martinet Case On $\mathbb{R}^3$)**.** *On* $\mathbb{R}^3$*, consider the smooth vector fields*

$$\mathcal{X}_1 = \partial_x \text{ and } \mathcal{X}_2 = \partial_y + \frac{x^2}{2}\partial_z.$$

On $\mathbb{R}^3\backslash\{x = 0\}$, $\mathcal{X}_1, \mathcal{X}_2$ and $\mathcal{X}_3 = [\mathcal{X}_1, \mathcal{X}_2] = x\partial_z$ span $\mathbb{R}^3$ (step 2). On $\{x = 0\}$, $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$ and $\mathcal{X}_4 = [\mathcal{X}_1, \mathcal{X}_3] = \partial_z$ span $\mathbb{R}^3$ (step 3). Following definition (3.4), define the subriemannian metric g^0 on $\mathbb{R}^3$ as

$$g^0(\mathcal{X}) = \inf\{u_1^2 + u_2^2\,;\, u = (u_i)_{i=1,2} \in \mathbb{R}^3,\, \sum_{i=1}^{2} u_i\mathcal{X}_i = \mathcal{X}\}.$$

Following definition (3.7), define the Riemannian metric g^h on $\mathbb{R}^3$ as

$$g^h(\mathcal{X}) = \inf\{u_1^2 + u_2^2 + h^{-2}u_3^2 + h^{-4}u_4^2\,;\, u = (u_i)_{i=1,2,3,4} \in \mathbb{R}^4,\, \sum_{i=1}^{4} u_i\mathcal{X}_i = \mathcal{X}\}.$$

For $\mathcal{X} = (a, b, c) \in \mathbb{R}^3$, direct computation implies that

$$g^h((a, b, c)) = a^2 + b^2 + \frac{(2c - bx^2)^2}{4h^2(h^2 + x^2)},$$

which is clearly a smoothly defined inner product on $\mathbb{R}^3$. Moreover, for any $(a, b, c) \in \mathbb{R}^3$,

$$g^0(a, b, c) = a^2 + b^2.$$

For g^0, which is the limit of g^h as $h \to 0$, to be finite, $c - \frac{x^2}{2}b$ must be 0, which means that $\mathcal{X} = a\mathcal{X}_1 + b\mathcal{X}_2$, that is $\mathcal{X}$ is horizontal.

3.3 The Volume Form dvolg^h

As our goal is to find a volume form associated to the subriemannian structure, induced from the preceding approximation scheme, we consider the volume form dvolg^h and study its properties and behavior as $h \to 0$. For a general local frame $(Z_1, ..., Z_d)$ of T_mM, the volume form dvolg^h is given by

$$\mathrm{dvol}g^h = \sqrt{|\det(G_h)|}|d\nu_1, ..., d\nu_d|, \tag{3.26}$$

where G_h is the representation matrix of g^h in $(Z_1, ..., Z_d)$ and $(\nu_1, ..., \nu_d)$ is the dual basis to $(Z_1, ..., Z_d)$. From this expression, our study will aim at analyzing the properties of the determinant of the matrix G_h.

3.3.1 Expression Of G_h^{-1}

We first give a general expression for G_h^{-1} in the frame $(Z_1, ..., Z_d)$. For $0 \leqslant i \leqslant r$, denote by A_i the $N_i \times d$ matrices defined such that the j-th row of A_i are the coefficients of X^{ij} in this frame (so that A_i has N_i rows and d columns). Denote by G_h the representation Gram matrix of g^h in this frame. The $A_i's$ along side with G_h depend on the point m. The dependence will be explicit when needed.

Theorem 3.10. *For all $h \in \mathbb{R}\backslash\{0\}$, we have*

$$G_h{}^{-1} = \sum_{i=0}^{r} h^{2i} A_i^t A_i, \tag{3.27}$$

where the matrices G_h^{-1} and A_i for $i = 0, ..., r$ have just been defined above.

Proof. Let Σ be the representation matrix of the map σ_m defined by (3.6), and observe that

$$\Sigma = \begin{pmatrix} A_0^t & A_1^t & ... & A_r^t \end{pmatrix}. \tag{3.28}$$

Denote by I^h the block matrix

$$I^h = \left(\begin{array}{c|c|c|c} I_{N_0} & 0 & 0 & 0 \\ \hline 0 & h^{-2} I_{N_1} & 0 & 0 \\ \hline 0 & 0 & \ddots & 0 \\ \hline 0 & 0 & 0 & h^{-2r} I_{N_r} \end{array}\right),$$

where for $i = 0, ..., r$, I_{N_i} is the $N_i \times N_i$ identity matrix. Note that $(I^h)^{-1} = I^{1/h}$.

Let $\mathcal{X} \in T_m M$. We define the functions:

$$\begin{aligned} \mathcal{F} : \mathbb{M}_{N\times 1} &\to \mathbb{R} \\ U &\mapsto U^t I^h U, \end{aligned}$$

and

$$\begin{aligned} \tilde{\mathcal{F}} : \mathbb{M}_{N\times 1} &\to \mathbb{M}_{d\times 1} \\ U &\mapsto \Sigma U - \mathcal{X}. \end{aligned}$$

We write

$$\begin{aligned} g^h(\mathcal{X}) &= \inf\{|\delta_h u|_N^2, u \in \mathbb{R}^N, \sigma_m(u) = \mathcal{X}\} \\ &= \inf\{U^t I^h U; \Sigma U = \mathcal{X}\} \\ &= \inf\{\mathcal{F}(U); \tilde{\mathcal{F}}(U) = 0\}. \end{aligned}$$

From lemma 3.3, there exists a unique $U_0 \in (\ker(\Sigma))^\perp$ such that $g^h(\mathcal{X}) = \mathcal{F}(U_0)$. Let

$$\mathbb{S} := \{U \in \mathbb{M}_{N\times 1}; \tilde{\mathcal{F}}(U) = 0 = \tilde{\mathcal{F}}(U_0)\}.$$

For any $U \in \mathbb{S}$, we have $0 = \tilde{\mathcal{F}}(U) - \tilde{\mathcal{F}}(U_0) = \Sigma(U - U_0)$, and so, $\mathbb{S} = \{U_0 + \tilde{U}; \tilde{U} \in \ker(\Sigma)\}$, which means that $\mathbb{S}$ is an affine space. This implies that it is a submanifold of dimension $N - d$. Moreover, we have

$$\ker(\Sigma) = T_{U_0}\mathbb{S}. \tag{3.29}$$

Indeed, let $\mathcal{Y} \in T_{U_0}\mathbb{S}$. By definition, there exist a curve $\gamma(t)$ on $\mathbb{S}$ such that $\gamma(0) = U_0$ and $\gamma'(0) = \mathcal{Y}$. Since $\gamma(t) \in \mathbb{S}$, then $\Sigma(\gamma(t)) = \mathcal{X}$. Differentiate with respect to t to get that $\mathcal{Y} \in \ker(\Sigma)$, and so, $T_{U_0}\mathbb{S} \subset \ker(\Sigma)$. Now, Hörmander condition implies that σ_m is surjective and so $\dim(\ker(\Sigma)) = N - d$. Moreover, for $\tilde{\mathcal{X}} \in \mathbb{R}^d$, $\mathcal{X} + \tilde{\mathcal{X}} \in \mathbb{R}^d$. So since σ_m is surjective, there exists a unique U such that $\Sigma U = \mathcal{X} + \tilde{\mathcal{X}}$, which implies that there exists U such that $\tilde{\mathcal{F}}(U) = \tilde{\mathcal{X}}$. Therefore $\tilde{\mathcal{F}}$ is also surjective and

$$\dim(T_{U_0}\mathbb{S}) = \dim(\mathbb{S}) = N - d = \dim(\ker(\Sigma)),$$

and we get (3.29).
Since the infimum of $\mathcal{F}$ restricted to the constraint $\tilde{\mathcal{F}} = 0$ is attained at U_0, then the Lagrange Multiplier method implies that U_0 is a critical point of the function $\mathcal{F} - \lambda\tilde{\mathcal{F}}$. In particular $d\mathcal{F}|_{U_0} = \lambda\, d\tilde{\mathcal{F}}\Big|_{U_0}$. So, for any critical point U_0 and any $W \in T_{U_0}\mathbb{S}$, we have that $d\mathcal{F}|_{U_0}(W) = 0$. On the other hand, we have

$$d\mathcal{F}|_{U_0}(W) = U_0^t I^h W + W^t I^h U_0 = 2W^t I^h U_0.$$

Thus, for any $W \in \ker(\Sigma)$, one gets that $W^t I^h U_0 = 0$. This implies that

$$I^h U_0 \in (\ker(\Sigma))^\perp = \operatorname{Im}(\Sigma^t).$$

Then there exists $V \in \mathbb{M}_{d\times 1}$ such that $I^h U_0 = \Sigma^t V$. Since $\tilde{\mathcal{F}} U_0 = 0$, we get that $\mathcal{X} = \Sigma I^{1/h} \Sigma^t V$.

We now observe that the square matrix $\mathbb{T} := \Sigma I^{1/h}\Sigma^t$ is invertible. Indeed, let $\mathbb{U}$ be such that $\mathbb{T}\mathbb{U} = 0$. We set $W = I^{1/h}\Sigma^t\mathbb{U}$. We have $W \in \ker(\Sigma)$ and $\Sigma^t\mathbb{U} = I^h W$. So, $I^h W \in \mathrm{Im}(\Sigma^t) = (\ker(\Sigma))^\perp$. Then, $W^t I^h W = 0$ which implies that $W = 0$ which implies that $\Sigma^t\mathbb{U} = 0$. Since Σ^t is of rank d, we get that $\mathbb{U} = 0$. This implies that the square matrix $\mathbb{T}$ has a trivial kernel and thus it is invertible.
Thus, we get

$$U_0 = I^{1/h}\Sigma^t(\Sigma I^{1/h}\Sigma^t)^{-1}\mathcal{X}.$$

We finally calculate:

$$\begin{aligned}
\mathcal{X}^t G_h \mathcal{X} = g(\mathcal{X}) &= U_0^t I^h U_0 \\
&= \mathcal{X}^t(\Sigma I^{1/h}\Sigma^t)^{-1}\Sigma^t {I^h}^{-1} I^h I^{1/h}\Sigma^t(\Sigma I^{1/h}\Sigma^t)^{-1}\mathcal{X} \\
&= \mathcal{X}^t(\Sigma I^{1/h}\Sigma^t)^{-1}\mathcal{X}.
\end{aligned}$$

Using (3.28), we conclude that

$$G_h{}^{-1} = \sum_{i=0}^{r} h^{2i} A_i^t A_i.$$

□

Remark 3.11. *This theorem is true for a general frame on M. In particular, it is true for a local coordinate frame or an adapted frame (definition 3.20).*

Remark 3.12. *We give two remarks concerning the choice of vectors made in subsection 3.1.2.*

1. *Our work will all be based on the matrices $A_i^t A_i$ and so we could include the zero vectors obtained from the iterative brackets of the initial vector fields in the definition of A_i, as the matrix $A_i^t A_i$ will not change by adding a row of zeros for A_i.*

2. *We recall that in setting our framework, we just included one of the two vectors X_{J_i} and $X_{\hat{J}_i} = -X_{J_i}$ for any $i > 0$. If we define $\hat{A}_i$ as we defined A_i but including all the vectors with their opposite, we will have that $\forall i > 0$,*

$$\begin{aligned}
\hat{A}_i^t\hat{A}_i &= (X^{i1})^t X^{i1} + ... + {X^{iN_i}}^t X^{iN_i} + (-X^{i1})^t(-X^{i1}) + ... + (-X^{iN_i})^t(-X^{iN_i}) \\
&= 2\left((X^{i1})^t X^{i1} + ... + (X^{iN_i})^t X^{iN_i}\right) = 2A_i^t A_i.
\end{aligned}$$

Example 3.4. *Following example 3.1, we denote by G_h the representation matrix of g^h. We compute*

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}, A_1 = \begin{pmatrix} 0 & 1 \end{pmatrix} \text{ and } G_h^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & h^2 + x^2 \end{pmatrix}$$

and observe that, $A_0^t A_0 + h^2 A_1^t A_1 = G_h^{-1}$.

Example 3.5. *Following example 3.2, we denote by G_h the representation matrix of g^h. We compute*

$$A_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x \end{pmatrix}, A_1 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \text{ and } G_h^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x \\ 0 & x & h^2 + x^2 \end{pmatrix},$$

and observe that, $A_0^t A_0 + h^2 A^t{}_1 A_1 = G_h^{-1}$.

Example 3.6. *Following example 3.3, we denote by G_h the representation matrix of g^h. We compute*

$$A_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{x^2}{2} \end{pmatrix}, A_1 = \begin{pmatrix} 0 & 0 & x \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$$

and

$$G_h^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{x^2}{2} \\ 0 & \frac{x^2}{2} & h^4 + h^2 x^2 + \frac{x^4}{4} \end{pmatrix},$$

and observe that, $A_0^t A_0 + h^2 A^t{}_1 A_1 + h^4 A_2^t A_2 = G_h^{-1}$.

3.3.2 Point-wise Limiting behavior Of det(G_h^{-1})

In this subsection, we aim at obtaining some information about $\det(G_h)$, and its behavior as $h \to 0$. Since we have the nice expression (3.27) of G_h^{-1}, we study the determinant of G_h^{-1} instead, and deduce things for $\det(G_h)$.

As a consequence of (3.27), the determinant of G_h^{-1} is a polynomial in h with non-negative coefficients. This is because the determinant of a matrix polynomial with Hermitian positive-semidefinite coefficients is a polynomial with non-negative coefficients (see [38]). However, this expression provides no more information on what happens near 0. For instance, we can't know what is the first non-zero coefficient (the leading term in the expansion) and so we can't know the power of h in the expansion near 0. This is important because we aim to prove that the determinant of G_h^{-1} is asymptotic near 0 to something of order $h^{\varsigma(m)}$ to be determined, with a coefficient that does not vanish on the equiregular region.

Thus, we need another approach that allows us to obtain some knowledge about this coefficient. We will present two approaches, one using a detailed spectral description of the eigenvalues of G_h^{-1} written in a local coordinates frame, and one using the special properties of adapted coordinates (subsection 3.3.2.3). The spectral approach is more important than the second one that depends on the type of the coordinates chosen. In this approach, we study the spectrum of the metric which, besides our main interest, may provide valuable insights into the geometry and curvature of the underlying manifold. For that, we describe the spectrum of G_h^{-1}.

3.3.2.1 Spectral Approach

Choose some coordinates $x = (x_1, ..., x_d)$ that are defined on an open set $U = U_x$. We define A_i in the same way as in the previous subsection, with respect to the coordinates x. Denote by G_h the representation Gram matrix of g^h in the coordinate x.

Again, the A_i's, G_h and $\mathrm{dvol}g^h$ depend on the point in the coordinates chosen, so an appropriate notation will be introduced when convenient, but for now, when no dependence is shown, this means that we work at a fixed point in the fixed coordinate frame. Let $\mathcal{V}_0 = \mathrm{Im}(A_0^t A_0)$ and let, for $1 \leqslant j \leqslant r$,

$$\mathcal{V}_j = \mathcal{V}_{j-1} + \mathrm{Im}(\mathcal{A}_j),$$

where

$$\mathcal{A}_j = A_j^t A_j\Big|_{(\mathcal{V}_{j-1})^\perp}, \tag{3.30}$$

where $\mathcal{V}_j^\perp$ represents the orthogonal compliment of $\mathcal{V}_j$ with respect to the canonical inner product in $\mathbb{R}^d$.

Observe that, as G_h^{-1} is a polynomial in h (expression (3.10)), it is an analytic function in h and thus the analytic perturbation theory (Appendix B) is applicable; the eigenbranches of the self-adjoint matrix G_h^{-1} are analytic

functions in h.
Recall that for $0 \leqslant i \leqslant r$, we denote by $n_i(m) := \dim(D_i(m))$. When we fix a point m, we simply write n_i.

Theorem 3.13. *Fix some point $m \in U$. For any $0 \leqslant j \leqslant r$, there are $n_j - n_{j-1}$ eigenbranches $\{\lambda_i^j(h)\}_{1 \leqslant i \leqslant n_j - n_{j-1}}$ of G_h^{-1} such that*

$$\lambda_i^j(h) = h^{2j} \eta_i^j(h),$$

where for any j, the analytic functions $\{\eta_i^j(h)\}_{1 \leqslant i \leqslant n_j - n_{j-1}}$ converges, as $h \to 0$, to the $n_j - n_{j-1}$ non-zero eigenvalues of $\mathcal{A}_j$.

Proof. For $1 \leqslant k \leqslant r$, let (H_k) be the following hypothesis: if we set

$$S_k(h) = A_0^t A_0 + h^2 A_1^t A_1 + ... + h^{2k} A_k^t A_k, \tag{3.31}$$

then, we have that

$$\operatorname{spec}(S_k(h)) = \{\{h^{2j}\{\rho_i^j(h)\}\}_{0 \leqslant j \leqslant k, 1 \leqslant i \leqslant n_j - n_{j-1}}, 0, ..., 0\}, \tag{3.32}$$

where, for any $0 \leqslant j \leqslant k$, the analytic functions $\{\rho_i^j(h)\}_{1 \leqslant i \leqslant n_j - n_{j-1}}$ converge to the non-zero eigenvalues of $\mathcal{A}_j$.
Moreover, if $\{\zeta_i^j(h)\}_{0 \leqslant j \leqslant k, 1 \leqslant i \leqslant n_j - n_{j-1}}$ is a set of orthonormal eigenfunctions corresponding to the non-zero eigenvalues of $S_k(h)$, then

$$\operatorname{span}\{\zeta_i^j(0), 0 \leqslant j \leqslant k, 1 \leqslant i \leqslant n_j - n_{j-1}\} = \mathcal{V}_k.$$

First, observe that for any k, $\operatorname{rank}(S_k(h))$, is independent of h, for $h \neq 0$, and is equal to n_k. Indeed,

$$S_k(h) = \begin{pmatrix} A_0^t & hA_1^t & ... & h^k A_k^t \end{pmatrix} \begin{pmatrix} A_0^t & hA_1^t & ... & h^k A_k^t \end{pmatrix}^t$$

and so for $h \neq 0$, we have

$$\operatorname{rank}(S_k(h)) = \operatorname{rank} \begin{pmatrix} A_0^t & hA_1^t & ... & h^k A_k^t \end{pmatrix}.$$

Now, multiplying two linearly dependent (respectively independent) vectors by a non-zero constant does not change the fact that they are linearly dependent (resp. independent). So, the rank of $S_k(h)$ is independent of h for

$h \neq 0$. For $h = 1$, the matrix $S_k(1)$ is the representation matrix of the vectors spanning D_k in local coordinates, which means that for any $h > 0$,

$$\operatorname{rank}(S_k(h)) = \operatorname{rank}(S_k(1)) = \operatorname{rank}(D_k) = n_k.$$

Also, for any k, the family of $d \times d$ matrices $S_k(h)$ is analytic (polynomials) in h, and so applying analytic perturbation theory in the finite-dimensional case, it implies that all eigenvalues of $S_k(h)$ for any $1 \leqslant k \leqslant r$ are analytic in h.

We prove (H_k) by induction on k. We first prove (H_1). By continuity of the spectrum (see corollary A.1.1), the difference between the $i-th$ eigenvalue of $A_0^t A_0$ and the $i-th$ eigenvalue of $S_1(h) = A_0^t A_0 + h^2 A_1^t A_1$ for any $1 \leqslant i \leqslant d$ is of order h^2 ($O(h^2)$). So, by analyticity of the spectrum, we can write

$$\operatorname{spec}(S_1(h)) = \left\{ \{\epsilon_i^0(h)\}_{1 \leqslant i \leqslant n_0}, h^2 \{\epsilon_i^1(h)\}_{1 \leqslant i \leqslant d-n_0} \right\}, \tag{3.33}$$

where $\{\epsilon_i^j(h)\}$ are analytic for any $j = 1, 2$ and any i, and for $j = 0$, it converges to the n_0 non-zero eigenvalues of $A_0^t A_0$. Moreover, let

$$\left\{ \{\xi_i^0(h)\}_{1 \leqslant i \leqslant n_0}, \{\xi_i^1(h)\}_{1 \leqslant i \leqslant d-n_0} \right\} \tag{3.34}$$

be a set of orthonormal eigenfunctions corresponding to the eigenvalues in (3.33).
Fix $i_0 \in \{1, ..., d - n_0\}$, and write

$$\epsilon_{i_0}^1(h) = \nu_{0,i_0} + h^2 \nu_{1,i_0} + ... \text{ and } \xi_{i_0}^1(h) = \xi_{0,i_0} + h^2 \xi_{1,i_0} + ...$$

Writing the eigenvalue equation for any $h > 0$ and comparing coefficients with respect to the powers of h, we get the following:

$$A_0^t A_0 \xi_{0,i_0} = 0, \tag{3.35}$$

and

$$A_1^t A_1 \xi_{0,i_0} + A_0^t A_0 \xi_{1,i_0} = \nu_{0,i_0} \xi_{0,i_0}. \tag{3.36}$$

Equation (3.35) implies that $\xi_{0,i_0} \in \ker(A_0^t A_0)$.
Denote by $\mathcal{Q}^0$ the orthogonal projection on $\ker(A_0^t A_0)$. Equation (3.35) implies that $\mathcal{Q}^0 \xi_{0,i_0} = \xi_{0,i_0}$. Multiplying equation (3.36) on the left by $\mathcal{Q}^0$, we get

$$(\mathcal{Q}^0 A_1^t A_1 \mathcal{Q}^0 - \nu_{0,i_0}) \mathcal{Q}^0 \xi_{0,i_0} = 0.$$

Then, $(\nu_{0,i_0}, \xi_{0,i_0})$ is an eigenpair of the matrix $\mathcal{A}_1$.
Now, by orthogonality of the set of eigenvectors in (3.34), we get that, for any $1 \leqslant i, \tilde{i} \leqslant d - n_0$,

$$\langle \xi_{0,i}, \xi_{0,\tilde{i}} \rangle_{\mathbb{R}^d} = \lim_{h \to 0} \langle \xi_i^1(h), \xi_{\tilde{i}}^1(h) \rangle_{\mathbb{R}^d} = 0$$

which implies that $\{\xi_{0,i}\}_{1 \leqslant i \leqslant d-n_0}$, are orthogonal thus linearly independent. Then, the $d - n_0$ linearly independent vectors $\{\xi_{0,i}\}_{1 \leqslant i \leqslant d-n_0}$ form an eigenbasis for $\mathcal{V}_0^\perp$ (which is of dimension $d - n_0$) and thus, $\{\nu_{0,i}\}_{1 \leqslant i \leqslant d-n_0}$ cover all eigenvalues of $\mathcal{Q}^0 A_1^t A_1 \mathcal{Q}^0$. Therefore,

$$\operatorname{spec}(S_1(h)) = \left\{ \{\epsilon_i^0(h)\}_{1 \leqslant i \leqslant n_0}, h^2 \{\epsilon_i^1(h)\}_{1 \leqslant i \leqslant n_1 - n_0}, 0...0 \right\},$$

with $\{\epsilon_i^1(h)\}_{1 \leqslant i \leqslant n_1 - n_0}$ converging to the non-zero eigenvalues of $\mathcal{Q}^0 A_1^t A_1 \mathcal{Q}^0$. Moreover, let

$$\left\{ \{\xi_i^j(h)\}_{0 \leqslant j \leqslant 1, 1 \leqslant i \leqslant n_j - n_{j-1}}, \{\xi_i^z\}_{1 \leqslant i \leqslant d - n_1} \right\}$$

be a set of orthonormal eigenfunctions, where the first part corresponds to the non-zero eigenvalues of $\operatorname{spec}(S_k(h))$ and the second one corresponds to the zero eigenvalues. By analyticity, $\{\xi_i^0(0)\}_{1 \leqslant i \leqslant n_0}$ span $\mathcal{V}_0$. Now, we prove that $\{\xi_i^1(0)\}_{1 \leqslant i \leqslant n_1 - n_0}$ span $\operatorname{Im}(\mathcal{A}_1)$. First, since $\{\xi_i^1(0)\}_{1 \leqslant i \leqslant n_1 - n_0}$ are eigenfunctions of $\mathcal{A}_1$, then we clearly have

$$\operatorname{span}\{\xi_i^1(0), 1 \leqslant i \leqslant n_1 - n_0\} \subset \operatorname{Im}(\mathcal{A}_1).$$

Now let $u \in \operatorname{Im}(\mathcal{A}_1)$. Then there exists $v \in \mathcal{V}_0^\perp$ such that $u = \mathcal{A}_1(v)$. In the first part, we proved that

$$\operatorname{span} \left\{ \{\xi_i^1(0)\}_{1 \leqslant i \leqslant n_1 - n_0}, \{\xi_i^z\}_{1 \leqslant i \leqslant d - n_1} \right\} = \mathcal{V}_0^\perp.$$

So, we can write v as a linear combination:

$$v = \sum_{i=1}^{n_1 - n_0} a_i \xi_i^1(0) + \sum_{i=1}^{d - n_0} b_i \xi_i^z.$$

Then, because $\{\xi_i^z\}_{1 \leqslant i \leqslant d - n_1}$ corresponds to the zero eigenvalues, we get

$$\begin{aligned} u &= \mathcal{A}_1(v) \\ &= \sum_{i=1}^{n_1 - n_0} a_i \mathcal{A}_1(\xi_i^1(0)) + \sum_{i=1}^{d - n_0} b_i \mathcal{A}_1(\xi_i^z) \\ &= \sum_{i=1}^{n_1 - n_0} a_i \epsilon_i^1(0) \xi_i^1(0) \in \operatorname{span}\{\xi_i^1(0), 1 \leqslant i \leqslant n_1 - n_0\}. \end{aligned}$$

Therefore, we get

$$\operatorname{span}\{\xi_i^j(h), 0 \leqslant j \leqslant 1, 1 \leqslant i \leqslant n_j - n_{j-1}\} = \mathcal{V}_1.$$

This finishes the proof that (H_1) is true.

Suppose now that (H_k) holds true. Write the spectrum of $S_k(h)$ as in (3.32):

$$\operatorname{spec}(S_k(h)) = \{\{h^{2j}\{\rho_i^j(h)\}\}_{0 \leqslant j \leqslant k, 1 \leqslant i \leqslant n_j - n_{j-1}}, 0, ..., 0\},$$

and let $\{\zeta_i^j(h)\}_{0 \leqslant j \leqslant k, 1 \leqslant i \leqslant n_j - n_{j-1}}$ be set of orthonormal eigenfunctions corresponding to the non-zero eigenvalues of $S_k(h)$. Denote by $\mathcal{J}_k(h)$ the subspace

$$\mathcal{J}_k(h) := \operatorname{span}\{\zeta_i^j(h), 0 \leqslant j \leqslant k, 1 \leqslant i \leqslant n_j - n_{j-1}\}.$$

Then, by induction hypothesis, we have:

$$\mathcal{J}_k(0) = \mathcal{V}_k. \tag{3.37}$$

We prove (H_{k+1}). Again, by continuity and analyticity of the spectrum, we can write

$$\operatorname{spec}(S_{k+1}(h)) = \{\{h^{2j}\tilde{\rho}_i^j(h)\}_{0 \leqslant j \leqslant k, 1 \leqslant i \leqslant n_j - n_{j-1}}, \{h^{2k+2}\tilde{\rho}_i^{k+1}(h)\}_{1 \leqslant i \leqslant d - n_k}\}, \tag{3.38}$$

with

$$|h^{2j}(\tilde{\rho}_i^j(h) - \rho_i^j(h))| = O(h^{2k+2}), \tag{3.39}$$

and

$$\tilde{\zeta}_i^j(0) = \zeta_i^j(0), \tag{3.40}$$

for all $0 \leqslant j \leqslant k$ and $1 \leqslant i \leqslant n_k - n_{k-1}$, where

$$\left\{\{\tilde{\zeta}_i^j(h)\}_{0 \leqslant j \leqslant k, 1 \leqslant i \leqslant n_j - n_{j-1}}, \{\tilde{\zeta}_i^{k+1}(h)\}_{1 \leqslant i \leqslant d - n_k}\right\} \tag{3.41}$$

is a set of orthonormal eigenfunctions corresponding to the eigenvalues in (3.38). This implies, by the induction hypothesis, that $(\tilde{\rho}_i^j(0), \tilde{\zeta}_i^j(0))$ is an eigenpair of $\mathcal{A}_j$ for any $0 \leqslant j \leqslant k, 1 \leqslant i \leqslant n_j - n_{j-1}$.

Fix $i_1 \in \{1, ..., d - n_k\}$ and consider $\tilde{\rho}_{i_1}^{k+1}(h)$ with corresponding eigenfunction $\tilde{\zeta}_{i_1}^{k+1}(h)$. Now, (3.37) and (3.40) implies that $\{\tilde{\zeta}_i^j(0)\}_{0 \leqslant j \leqslant k, 1 \leqslant i \leqslant n_j}$ is an orthonormal basis of $\mathcal{V}_k$. So, we get

$$\tilde{\rho}_{i_1}^{k+1}(h) = O(h^{2k+2}) \ \textit{and} \ \tilde{\zeta}_{i_1}^{k+1}(0) \in \mathcal{V}_k^{\perp} \quad \text{as } h \to 0. \tag{3.42}$$

Let $\mathcal{P}_h^k$ and $\mathcal{Q}_h^k$ be the orthogonal projections on $\mathcal{J}_k(h)$ and $(\mathcal{J}_k(h))^\perp$ respectively. Write

$$\tilde{\zeta}_{i_1}^{k+1}(h) = \mathcal{P}_h^k\tilde{\zeta}_{i_1}^{k+1}(h) + \mathcal{Q}_h^k\tilde{\zeta}_{i_1}^{k+1}(h).$$

As $\mathcal{J}_k(h)$ is the span of analytic functions, it is analytic, and by lemma 3.2, $(\mathcal{J}_k(h))^\perp$ is. Thus, $\mathcal{P}_h^k$ and $\mathcal{Q}_h^k$ are also analytic, and by (3.42), we have that

$$\mathcal{P}_h^k\tilde{\zeta}_{i_1}^{k+1}(h) = o(1) \text{ and } \mathcal{Q}_h^k\tilde{\zeta}_{i_1}^{k+1}(h) = \tilde{\zeta}_{i_1}^{k+1}(0) + o(1), \tag{3.43}$$

as $h \to 0$. Therefore, the eigenvalue equation implies that

$$h^{2k+2}\tilde{C}_h\tilde{\zeta}_{i_1}^{k+1}(h) + h^{2k+2}\tilde{B}_h\tilde{\zeta}_{i_1}^{k+1}(h) = h^{2k+2}\tilde{\rho}_{i_1}^{k+1}(h)\mathcal{Q}_h^k\tilde{\zeta}_{i_1}^{k+1}(h), \tag{3.44}$$

where

$$\tilde{C}_h = \mathcal{Q}_h^k A_{k+1}^t A_{k+1}\mathcal{P}_h^k$$

and

$$\tilde{B}_h = \mathcal{Q}_h^k A_{k+1}^t A_{k+1}\mathcal{Q}_h^k.$$

Equation (3.44) gives, after dividing by h^{2k+2} and writing $\tilde{C}_h\tilde{\zeta}_{i_1}^{k+1}(h) = o(1)$ (by (3.43), the following equation

$$o(1) + \tilde{B}_h\tilde{\zeta}_{i_1}^{k+1}(h) = \tilde{\rho}_{i_1}^{k+1}(h)\mathcal{Q}_h^k\tilde{\zeta}_{i_1}^{k+1}(h).$$

The subspace $\mathcal{J}(h)$ is analytic in h, and so $\tilde{B}_h$ is analytic, so at $h = 0$ we get

$$(\tilde{Q}_0 A_{k+1}^t A_{k+1}\tilde{Q}_0 - \tilde{\rho}_{i_1}^{k+1}(0))\tilde{Q}_0\tilde{\zeta}_{i_1}^{k+1}(0) = 0. \tag{3.45}$$

Therefore, (3.37) and (3.45) imply that $(\tilde{\rho}_{i_1}^{k+1}(0), \tilde{\zeta}_{i_1}^{k+1}(0))$ is an eigenpair of $\mathcal{A}_{k+1}$.

Finally, the orthonormality of the eigenfunctions in (3.41) implies the orthonormality of $\{\tilde{\zeta}_i^{k+1}(0)\}_{1\leqslant i\leqslant d-n_k}$. Therefore, the $d - n_k$ orthogonormal (thus linearly independent) vectors $\{\tilde{\zeta}_i^{k+1}(0)\}_{1\leqslant i\leqslant d-n_k}$ form an eigenbasis of $\mathcal{V}_k$ (which is of dimension $d-n_k$ by (3.37)). So, they cover all the eigenvalues of $\mathcal{A}_{k+1}$.

It remains to prove that the span of the eigenvectors that correspond to the non-zero eigenvalues of $S_{k+1}(h)$ span $\mathcal{V}_{k+1}$ at $h = 0$. We write the eigenvectors in (3.41) as follows

$$\left\{\{\tilde{\zeta}_i^j(h)\}_{0\leqslant j\leqslant k, 1\leqslant i\leqslant n_j-n_{j-1}}, \{\tilde{\zeta}_i^{k+1}(h)\}_{1\leqslant i\leqslant n_k-n_{k-1}}, \{\tilde{\zeta}_i^z\}_{1\leqslant i\leqslant d-n_{k+1}}\right\}, \tag{3.46}$$

where $\{\tilde{\zeta}_i^z\}_{1\leqslant i\leqslant d-n_{k+1}}$ corresponds to the 0 eigenvalues. Again, (3.40) and (3.37) implies that $\{\tilde{\zeta}_i^j(0)\}_{0\leqslant j\leqslant k, 1\leqslant i\leqslant n_j-n_{j-1}}$ span $\mathcal{V}_k$. Now, we prove that $\{\tilde{\zeta}_i^{k+1}(0)\}_{1\leqslant i\leqslant n_k-n_{k-1}}$ span $\mathrm{Im}(\mathcal{A}_k)$. Since $\{\tilde{\zeta}_i^{k+1}(0)\}_{1\leqslant i\leqslant n_k-n_{k-1}}$ are eigenvectors of $\mathcal{A}_k$ it is clear that

$$\mathrm{span}\{\tilde{\zeta}_i^{k+1}(0), 1\leqslant i\leqslant n_k-n_{k-1}\}\subset \mathrm{Im}(\mathcal{A}_k).$$

Now, let $u\in\mathrm{Im}(\mathcal{A}_k)$. Then, there is $v\in\mathcal{V}_k^\perp$ such that $u=\mathcal{A}_k(v)$. We proved that

$$\mathrm{span}\left\{\{\tilde{\zeta}_i^{k+1}(0)\}_{1\leqslant i\leqslant n_k-n_{k-1}}, \{\tilde{\zeta}_i^z\}_{1\leqslant i\leqslant d-n_{k+1}}\right\}=\mathcal{V}_k^\perp.$$

So, we can write v as a linear combination

$$v=\sum_{i=1}^{n_{k+1}-n_k} a_i\tilde{\zeta}_i^{k+1}(0)+\sum_{i=1}^{d-n_{k+1}} b_i\tilde{\zeta}_i^z.$$

Then, because $\{\tilde{\zeta}_i^z\}_{i=1,\dots,d-n_{k+1}}$ correspond to the zero eigenvalues, we get

$$\begin{aligned} u &= \mathcal{A}_k(v)\\ &= \sum_{i=1}^{n_{k+1}-n_k} a_i\mathcal{A}_k(\tilde{\zeta}_i^{k+1}(0))+\sum_{i=1}^{d-n_{k+1}} b_i\mathcal{A}_k(\tilde{\zeta}_i^z)\\ &= \sum_{i=1}^{n_{k+1}-n_k} a_i\tilde{\rho}_i^{k+1}(0)(\tilde{\zeta}_i^{k+1}(0))\in\mathrm{span}\{\tilde{\zeta}_i^{k+1}(0), 1\leqslant i\leqslant n_k-n_{k-1}\}.\end{aligned}$$

Therefore, we get

$$\mathrm{span}\left\{\{\tilde{\zeta}_i^j(h)\}_{0\leqslant j\leqslant k, 1\leqslant i\leqslant n_j-n_{j-1}}, \{\tilde{\zeta}_i^{k+1}(h)\}_{1\leqslant i\leqslant n_k-n_{k-1}}\right\}=\mathcal{V}_{k+1}.$$

We conclude (H_{k+1}).
By Hörmander condition, the induction stops at $k=r$ and we get the final result. □

Remark 3.14. *The relevant part of the induction in the proof of theorem 3.13 is for $0\leqslant j\leqslant r(m)$. For $j>r(m)$, $n_j(m)=d$ and so the statement of theorem indicates that there are 0 eigenbranches of order h^{2j}. This is because at m, $S_{r(m)}(h)$ defined by (3.31) generates all the d non-zero eigenbranches of $G_h^{-1}(m)$, and with any further perturbation of higher order, the d new eigenbranches will only be perturbed by a big O of this higher order, but the behavior near $h=0$ remains the same.*

As said before, expression (3.27) doesn't give much information about the behavior of $\det(G_h^{-1})$ as h approach 0. However, theorem 3.13 implies that the spectrum of G_h^{-1} can be written as follows:

$$\mathrm{spec}(G_h^{-1}(m)) = \{\underbrace{\text{Order 0 terms}}_{n_0(m)}, \underbrace{\text{Order 2 terms}}_{n_1(m)-n_0(m)} \ldots \underbrace{\text{Order 2r terms}}_{d-n_{r-1}(m)}\},$$

As the determinant of a matrix is the product of the eigenvalues we get the following corollary.

Corollary 3.14.1. *For a fixed $m \in U$, the determinant of G_h^{-1} has the following expansion*

$$\det(G_h^{-1}(m)) = f_h(m)h^{2\varsigma(m)},$$

where

$$\varsigma(m) = \sum_1^r i[n_i(m) - n_{i-1}(m)], \tag{3.47}$$

and $f_h(m)$ is given by

$$f_h(m) = \prod_{j=0}^{r} \prod_{i=1}^{n_j - n_{j-1}} \eta_i^j(h)(m),$$

with $\eta_i^j(h)(m)$ being introduced in theorem 3.13, where we set $\prod_{i=1}^0 = 1$.

Moreover, $f_h(m)$ converges, as $h \to 0$, to $f(m) \neq 0$, where $f(m)$ is the product of the non-zero eigenvalues of $\mathcal{A}_i(m)$ (given by (3.30)) for $i = 0, ..., r$.

In other words, if we write

$$\det(G_h^{-1}(m)) = \sum_{i \geqslant 0} a_i(m)h^{2i},$$

then, $a_i(m) = 0 \, \forall i < \varsigma(m)$ and $f(m) = a_{\varsigma(m)} > 0$.

3.3.2.2 Dependence With Respect To The Point

In the previous section, we obtained the expansion near $h = 0$ for $\det(G_h^{-1})$ at a fixed point $m \in U$; that is, theorem 3.13 was point-wise and the behavior in corollary 3.14.1 is a point-wise behavior. Now, we allow m to move in U and we study the characteristics of the function $f(m)$, that was defined in corollary 3.14.1. We recall the definition of the singular set in subsection 3.1.1.2 and define the set $\mathcal{Z}_U = U \cap \mathcal{Z}$.

Proposition 3.15. *On every connected component of the equiregular region $M\backslash\mathcal{Z}$, $\varsigma(m)$ is independent of m and is equal to $Q-d$, where Q denotes the Hausdorff dimension on this component.*

Proof. Let $\mathscr{K}$ be a connected component of $M\backslash\mathcal{Z}$. The growth vector

$$(n_0(\mathscr{K}), ..., n_{r-1}(\mathscr{K}), d)$$

is constant (independent of m). Thus, using the expression of $\varsigma(m)$ in (3.47), $\varsigma(m)$ is a constant which we denote $\varsigma(\mathscr{K})$.

The Hausdorff dimension on $\mathscr{K}$ equals

$$Q(\mathscr{K}) = \sum_{i=0}^{r}(i+1)[n_i(\mathscr{K}) - n_{i-1}(\mathscr{K})],$$

where we set $n_{-1}(\mathscr{K}) = 0$ (for information about the Hausdorff dimension, see [65]). So we have

$$Q(\mathscr{K}) - \varsigma(\mathscr{K}) = \sum_{i=0}^{r}(i+1)[n_i(\mathscr{K}) - n_{i-1}(\mathscr{K})] - \sum_{i=1}^{r} i[n_i(\mathscr{K}) - n_{i-1}(\mathscr{K})] = d.$$

□

Proposition 3.16. *The function f is a smooth function on each component of $U\backslash\mathcal{Z}_U$.*

Proof. The matrix G_h^{-1} is the inverse of the representation matrix of the Riemannian metric on M, so its determinant is smooth in m and is independent on the coordinates chosen. This implies that for each k, the coefficients a_k in (3.48) are smooth in m. Now, corollary 3.14.1 gives the asymptotic expansion of $\det(G_h^{-1})$; in particular, it gives the first non-zero coefficient. So, on a connected component $\mathscr{K}$ of $M\backslash\mathcal{Z}$, this non-zero coefficient happens for a power $\varsigma(\mathscr{K})$ of h independent of m, and so in this case $f = a_{\varsigma(\mathscr{K})}$ which is smooth and non-zero. □

Proposition 3.17. *The function f vanishes on the singular region $\mathcal{Z}_U$.*

Proof. Recall that $(n_0(m), n_1(m), ..., n_{r(m)-1}(m), d)$ denotes the growth vector at a point in M. For any $m \in M$, we can write the spectrum of $G_h^{-1}(m)$ as

$$\begin{aligned}
\operatorname{spec}(G_h^{-1}) = \big\{ & \{\lambda_i(h,m), 1 \leqslant i \leqslant n_0(m)\}, \\
& \{h^2\lambda_i(h,m), n_0(m)+1 \leqslant i \leqslant n_1(m)\}, \\
& \quad \dots \\
& \{h^{2r(m)}, n_{r(m)-1}(m)+1 \leqslant i \leqslant d\} \qquad \big\}
\end{aligned}$$

with $\lim_{h\to 0}\lambda_i(h,m) \neq 0$ for $1 \leqslant i \leqslant d$. Therefore, on any connected component $\mathscr{K}$ of $M\backslash\mathcal{Z}$, $\det(G_h^{-1}(m))$ is of order $h^{2\varsigma(\mathscr{K})}$, and if the determinant of $G_h^{-1}(m)$ has the expression

$$\det(G_h^{-1}(m)) = \sum_{k\geqslant 0} a_k(m)h^{2k}, \tag{3.48}$$

then $a_k(m) = 0$ for all $k < \varsigma(\mathscr{K})$ and $a_{\varsigma(\mathscr{K})}(m) = f(m) > 0$ on $\mathscr{K}$.

Now, let $\tilde{m} \in \mathcal{Z}$. Then, there exist $i_0 \in \{0, ..., r(m)\}$ such that

$$n_{i_0}(\tilde{m}) < n_{i_0}(\mathscr{K}).$$

Then, there exist $i_1 \in \{i_0+1, ..., r\}$ such that

$$n_{i_1}(\tilde{m}) - n_{i_1-1}(\tilde{m}) > n_{i_1}(\mathscr{K}) - n_{i_1-1}(\mathscr{K}).$$

This is because for any $m \in M$, $n_i(m) \leqslant n_{i+1}(m)$ for any $0 \leqslant i \leqslant r(m)$, and eventually, by Hörmander condition, $n_{r(m)} = d$.
Therefore, $\det(G_h^{-1}(\tilde{m}))$ is of order $h^{\varsigma(\tilde{m})}$ with $\varsigma(\tilde{m}) > \varsigma(\mathscr{K})$. Thus, $a_k(\tilde{m}) = 0$ for all $k < \varsigma(\tilde{m})$, in particular, $a_\varsigma(\tilde{m}) = f(\tilde{m}) = 0$.

The other direction follows the same reasoning; suppose $m \in \mathscr{K}$ such that $f(m) = 0$. Then $\det(G_h^{-1}(m))$ is of order greater than $h^{\varsigma(\mathscr{K})}$ which is a contradiction. □

Proposition 3.18. *The volume form induced is independent of the choice of coordinates.*

Proof. First, observe that the volume form induced from the approximation scheme written in a local coordinate chart x is given by

$$(1/\sqrt{|f(x)|})|dx_1 \wedge ... \wedge dx_d|.$$

Fix two coordinates $x = (x_1, ..., x_d)$ and $y = (y_1, ..., y_d)$ and denote by Φ the canonical diffeomorphism between x and y (map of change of coordinates). Let $\mathcal{Y}$ be a vector field and let $\mathcal{Y}_x$ and $\mathcal{Y}_y$ be such that

$$\mathcal{Y} = \mathcal{Y}_x.\partial_x = \mathcal{Y}_y.\partial_y.$$

Then, $\mathcal{Y}_x = (Jac(\Phi))^t\mathcal{Y}_y$, where $Jac(\Phi)$ denotes the Jacobian matrix of Φ. This implies that

$$A_j(x) = A_j(y)Jac(\Phi),$$

where $A_j(x), A_j(y)$, are the matrices defined in the introduction of this section, in x and y respectively. By the expression given in (3.27), this implies that

$$G_h^{-1}(x) = (Jac(\Phi))^t(G_h^{-1}(y))Jac(\Phi),$$

where $G_h^{-1}(x), G_h^{-1}(y)$ are the representation matrices of the metric g^h, represented in x and y respectively. Thus,

$$f_h(x) = |\det(Jac(\Phi))|^2 f_h(y). \tag{3.49}$$

Letting $h \to 0$ in (3.49), we get

$$f(x) = |\det(Jac(\Phi))|^2 f(y).$$

Since

$$|dx_1 \wedge ... \wedge dx_d| = |\det(Jac(\Phi))||dy_1 \wedge ... \wedge dy_d|,$$

we finally deduce that

$$(1/\sqrt{|f(x)|})|dx_1 \wedge ... \wedge dx_d| = (1/\sqrt{|f(y)|})|dy_1 \wedge ... \wedge dy_d|.$$

$\square$

Remark 3.19. *In the preceding proposition, we re-proved the fact that the determinant of a Riemannian metric is independent of the choice of coordinates. Using this fact, we can say directly that f is the leading order of* $\det(G_h^{-1})$ *as $h \to 0$, and so it is independent of the coordinated chosen.*

As a corollary, we deduce the following.

Corollary 3.19.1. *In an equiregular subriemannian setting, where $\mathcal{Z} = \emptyset$, f is a smooth strictly positive function on M. Thus, it defines a smooth volume form on M written in local coordinate x as*

$$d\mathcal{P}_o = \frac{1}{\sqrt{f(x)}}|dx_1 \wedge ... \wedge dx_d|. \tag{3.50}$$

.

Example 3.7. *Following example 3.4, we compute*

$$\mathrm{spec}(A_0^t A_0) = \{1, x^2\} \text{ and } \mathrm{spec}(G_h^{-1}) = \{1, x^2 + h^2\}.$$

Away from the singular line $\{x = 0\}$, the two eigenvalues are of order 0*. So*

$$\det(G_h^{-1}) = x^2 + h^2 \sim x^2.$$

In this case, $\varsigma(x) = 0$ and $f_h(x) = x^2 + h^2 \to f(x) = x^2$. At $\{x = 0\}$, one eigenvalue, 1, is of order 0 *and one eigenvalue, $x^2 + h^2$, is of order h^2. So*

$$\det(G_h^{-1}) = h^2 \sim h^2.$$

This is expected as the growth vector equals (2) *for $x \neq 0$, and* $(1, 2)$ *at* $\{x = 0\}$.

Example 3.8. *Following example 3.5, we compute*

$$\mathrm{spec}(A_0^t A_0) = \{0, 1, x^2 + 1\}, \text{ and } \mathrm{spec}(G_h^{-1}) = \{\lambda_1(h, x), \lambda_2(h, x), \lambda_3(h, x)\},$$

where $\lambda_1(h, x) = 1$,

$$\lambda_2(h, x) = \frac{h^{-2}x^2 + \sqrt{(h^{-2}x^2 + h^{-2} + 1)^2 - 4h^{-2}} + h^{-2} + 1}{2h^{-2}} \sim 2(x^2 + 1),$$

and

$$\lambda_3(h, x) = \frac{h^{-2}x^2 - \sqrt{(h^{-2}x^2 + h^{-2} + 1)^2 - 4h^{-2}} + h^{-2} + 1}{2h^{-2}} \sim h^2(1 + o(h^{-2})).$$

For any x, two eigenvalues, $\lambda_1(h, x)$ and $\lambda_2(h, x)$ are of order 0 and one eigenvalue, $\lambda_3(h, x)$, is of order h^2. This is expected, as this is the equiregular case, and the growth vector equals $(2, 3)$ *is constant everywhere. So, at any x,*

$$\det(G_h^{-1}) = h^2 \sim h^2.$$

In this case, $\varsigma(x) = 1$ and $f_h(x) = 1 \to f(x) = 1$.

Example 3.9. *Following example 3.6, we compute*

$$\operatorname{spec}(A_0^t A_0) = \{0, 1, x^2 + 1\}, \text{ and } \operatorname{spec}(G_h^{-1}) = \{\lambda_1(h,x), \lambda_2(h,x), \lambda_3(h,x)\},$$

where $\lambda_1(h,x) = 1$,

$$\lambda_3(h,x) = \frac{h^{-4}x^4 + 4h^{-4} + 4h^{-2}x + \sqrt{H(h,x)} + 4}{8h^{-4}},$$

and

$$\lambda_2(h,x) = \frac{h^{-4}x^4 + 4h^{-4} + 4h^{-2}x - \sqrt{H(h,x)} + 4}{8h^{-4}},$$

with

$$H(h,x) = (h^{-4}x^4 + 4h^{-4} + 4h^{-2}x + 4)^2 - 16h^{-4}(4h^{-2}x + 4).$$

Away from the singular plane $\{x = 0\}$*, two eigenvalues,* $\lambda_1(h,x)$ *and* $\lambda_2(h,x)$ *are of order* 0 *and one eigenvalue,* $\lambda_3(h,x)$ *is of order* h^2 *as* $\lambda_3(h,x) = h^2(4x + o(h^{-2}) + o(h^{-4}))$*. So*

$$\det(G_h^{-1}) = h^2x^2 + h^4 \sim h^2x^2.$$

In this case, $\varsigma(x) = 1$ *and* $f_h(x) = x^2 + h^2 \to f(x) = x^2$*. At* $\{x = 0\}$*, two eigenvalues,* $\lambda_1(h,x)$ *and* $\lambda_2(h,x)$*, are of order* 0 *and one eigenvalue,* $\lambda_3(h,x)$ *is of order* h^4*, as* $\lambda_3(h,x) = h^4(4 + o(h^{-4}))$*. So*

$$det(G_h^{-1}) = h^4 \sim h^4.$$

This is expected as the growth vector equals $(2,3)$ *for* $x \neq 0$*, and* $(2,2,3)$ *at* $\{x = 0\}$*.*

3.3.2.3 Second Approach

In this subsection, we recover corollary 3.14.1 using an adapted frame of the tangent space. We first define an adapted basis.

Definition 3.20. *Let* D *be a distribution on* M *such that* D *satisfies the Hörmander condition with step* $r + 1$*. Define* D_i *following (3.2). We say a local frame* $Z_1, ..., Z_d$ *is adapted if* $Z_1, ..., Z_{n_i}$ *is a local frame for* D_i*, for any* $0 \leqslant i \leqslant r$*, where* $n_i = dim(D_i(m))$*, and* $Z_1, ..., Z_{n_0}$ *are orthonormal.*

In this subsection, we recover corollary 3.14.1 using the special characteristics of an adapted frame

So, fix some adapted frame $Z_1, ..., Z_d$. Consider our framework. Let $A_{(i)}$ be the $N_i \times d$ matrix defined such that the j-th row of $A_{(i)}$ is the coefficients of X^{ij} in the adapted frame (so that $A_{(i)}$ has N_i rows and d columns). We first describe the matrix $A_{(i)}$.

For $0 \leqslant i \leqslant r$. The rows of the matrix $A_{(i)}$ are the vectors X^{ij} for $1 \leqslant j \leqslant N_i$ written in the adapted frame $Z_1, ..., Z_d$. For each $1 \leqslant j \leqslant N_i$, the vector X^{ij} is in D_i which has $\{Z_1, ..., Z_{n_i}\}$ as a basis. Thus, the last $d - n_i$ columns of the matrix $A_{(i)}$ are zero columns. More precisely, if we denote by a_{ij}^k the coefficient of Z_k in the expression of X^{ij};

$$X^{ij} = \sum_{k=1}^{d} a_{ij}^k Z_k,$$

then $a_{ij}^k = 0$ for all $k > n_i$ and $A_{(i)}$ has the following expression

$$A_{(i)} = \begin{pmatrix} a_{i1}^1 & \dots & a_{i1}^{n_i} & 0 & \dots & 0 \\ a_{i2}^1 & \dots & a_{i2}^{n_i} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & 0 & \dots & 0 \\ a_{iN_i}^1 & \dots & a_{iN_i}^{n_i} & 0 & \dots & 0 \end{pmatrix}. \tag{3.51}$$

Now, for any $0 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant N_i$ define the coefficients $\bar{a}_{ij}^k$ as follows:

$$\bar{a}_{ij}^k = \begin{cases} 0 & 1 \leqslant k \leqslant n_{i-1} \\ a_{ij}^k & n_{i-1} + 1 \leqslant k \leqslant n_i \end{cases},$$

and define the matrix $\bar{A}_{(i)} := (\bar{a}_{ij}^k)_{1 \leqslant j \leqslant N_i, 1 \leqslant k \leqslant n_i}$ with N_i rows and n_i columns ($\bar{A}_{(i)}$ denotes the representation matrix of X^{ij} in D_i mod D_{i-1}, letting all the coefficients that correspond to $Z_1, ..., Z_{n_{i-1}}$ equal 0).

For $0 \leqslant i \leqslant r$, we introduce the matrix M_i as the non-zero block of $\bar{A}_{(i)}^t \bar{A}_{(i)}$:

$$\bar{A}_{(i)}^t \bar{A}_{(i)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ * & * & * & * \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & * & 0 \\ 0 & * & 0 \\ 0 & * & 0 \\ 0 & * & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The red-shaded region in $\bar{A}_{(i)}$ corresponds to the vectors $Z_{n_{i-1}+1}, ..., Z_{n_i}$, and M_i is the blue-shaded sub-matrix.

Proposition 3.21. *Fix $m \in M$. If we denote by $\tilde{G}_h$ the representation Gram matrix of g^h with respect to the adapted frame $Z_1, ..., Z_d$, then, near $h = 0$, we have*

$$\det(\tilde{G}_h^{-1}) \sim h^{2\varsigma} \prod_{i=1}^{r} \det(M_i). \tag{3.52}$$

Proof. By theorem 3.10, We have that

$$\tilde{G}_h^{-1} = \sum_{i=0}^{r} h^{2i} A_{(i)}^t A_{(i)}. \tag{3.53}$$

Using the characteristics of the matrix $A_{(i)}$ (description (3.51), formula (3.53) implies that

$$\tilde{G}_h^{-1} = \left(\begin{array}{c|c|c|c|c} M_0 + O(h^2) & O(h^2) & ... & O(h^{2(r-1)}) & O(h^{2r}) \\ \hline O(h^2) & h^2 M_1 + O(h^4) & O(h^4) & ... & \vdots \\ \hline O(h^4) & O(h^4) & h^4 M_2 + O(h^6) & ... & \vdots \\ \hline \vdots & ... & ... & \ddots & O(h^{2r}) \\ \hline O(h^{2r}) & ... & ... & O(h^{2r}) & h^{2r} M_r \end{array}\right). \tag{3.54}$$

Let

$$G_{(1)}(h) = \left(\begin{array}{c|c|c|c|c} M_0 & 0 & ... & 0 & 0 \\ \hline 0 & h^2 M_1 & 0 & ... & \vdots \\ \hline 0 & 0 & h^4 M_2 & ... & \vdots \\ \hline \vdots & ... & ... & \ddots & 0 \\ \hline 0 & ... & ... & 0 & h^{2r} M_r \end{array}\right),$$

and

$$G_{(2)}(h) = \tilde{G}_h^{-1} - G_{(1)}(h) = \left(\begin{array}{c|c|c|c|c} O(h^2) & O(h^2) & ... & O(h^{2(r-1)}) & O(h^{2r}) \\ \hline O(h^2) & O(h^4) & O(h^4) & ... & \vdots \\ \hline O(h^4) & O(h^4) & O(h^6) & ... & \vdots \\ \hline \vdots & ... & ... & \ddots & O(h^{2r}) \\ \hline O(h^{2r}) & ... & ... & O(h^{2r}) & 0 \end{array}\right).$$

Moreover, we have that M_i is invertible, and so $G_{(1)}(h)$ is. Indeed, M_i is an $n_i - n_{i-1}$ matrix with

$$\mathrm{rank}(M_i) = \mathrm{rank}\left(\bar{A}^t_{(i)}\bar{A}_{(i)}\right) = \mathrm{rank}\left(\bar{A}_{(i)}\right).$$

As $n_i \leqslant N_i$, then

$$\mathrm{rank}\left(\bar{A}_{(i)}\right) \leqslant n_i - n_{i-1}.$$

As $\{Z_{n_{i-1}+1}, ..., Z_{n_i}\}$ are basis for D_i, then $\mathrm{rank}(\bar{A}_{(i)}) = n_i - n_{i-1}$. Thus, $\mathrm{rank}(M_i) = n_i - n_{i-1}$, and so it has full rank (in particular, invertible). Therefore, we have

$$\begin{aligned}\det(\tilde{G}_h^{-1}) &= \det(G_{(1)}(h)(I_d + (G_{(1)}(h))^{-1}G_{(2)}(h))) \\ &= \det(G_{(1)}(h))\det(I_d + (G_{(1)}(h))^{-1}G_{(2)}(h)).\end{aligned} \tag{3.55}$$

Now, as $(G_{(1)}(h))^{-1}G_{(2)}(h)$ is simply multiplying the i^{th} row of $G_{(2)}(h)$ by $h^{-2i}M_i^{-1}$ for any $0 \leqslant i \leqslant r$, then, the limit of $(G_{(1)}(h))^{-1}G_{(2)}(h)$, which we denote by $\bar{G}$, is a strictly lower triangular matrix. Then the spectrum of $I_d + \bar{G}$ is given by

$$\mathrm{spec}(I_d + \bar{G}) = \{\underbrace{1, ..., 1}_{d \text{ times}}\},$$

and so its determinant is 1. Since the determinant function is a continuous function from $\mathbb{M}_{d\times d}$ to $\mathbb{R}$, then $\det(I_d+(G_{(1)}(h))^{-1}G_{(2)}(h)) \sim 1$ and therefore, near $h = 0$,

$$\det(G_h^{-1}) \sim \det(G_{(1)}(h)) = h^{2\varsigma(m)} \prod_{i=1}^{r} \det(M_i).$$

□

Finally, the same arguments as in subsection 3.3.2.2 give the characteristics of the function $\prod_{i=1}^r \det(M_i(m))$. Consequently, in an equiregular setting, it defines a smooth volume form on M given in the local adapted frame $(Z_1, ..., Z_d)$ by

$$d\mathcal{P}_o = \frac{1}{\sqrt{\prod_{i=1}^r \det(M_i)}} |\nu_1 \wedge ... \wedge \nu_d|, \tag{3.56}$$

where we recall that $(\nu_1 \wedge ... \wedge \nu_d)$ is the dual frame to $(Z_1, ..., Z_d)$.

The thing that made this approach much easier than that used in the first approach is the fact that the frame $Z_1, ..., Z_d$ is an adapted frame that gives the nice structure of the matrix in 3.54.

3.4 Relation With Popp's Volume

In this subsection, we suppose that we are working in an equiregular setting. We define Popp's volume following [13], where the formula of Popp's volume is given in terms of any adapted frame of the tangent bundle. We then compare our volume form obtained by the approximation scheme to Popp's volume.

3.4.1 Definition: Popp's Volume

Popp's volume is defined by inducing a canonical inner product on $gr_m(D)$ via the brackets, and then using a non-canonical isomorphism between the graded vector space $gr_m(D)$ and the tangent space T_mM, to define an inner product on the whole tangent space. We follow the definition given in [13] of Popp's volume with respect to an adapted basis.

Fix an adapted local frame $Z_1, ..., Z_d$. Denote by $\nu_1, ..., \nu_d$ the dual frame to $Z_1, ..., Z_d$. For $j = 1, ..., r$, we define the adapted structure constants $b^l_{i_1 \dots i_j} \in \mathcal{C}^\infty(M)$ as follows:

$$[Z_{i_1}, [Z_{i_2}, ..., [Z_{i_j}, Z_{i_{j+1}}]...]] = \sum_{l=n_{j-1}+1}^{n_j} b^l_{i_1 i_2 \dots i_j} Z_l \text{ mod } D^{j-1}, \tag{3.57}$$

where $1 \leqslant i_1, ..., i_j \leqslant n_0$. We define the $n_j - n_{j-1}$ dimensional square positive definite matrix B_j as follows

$$[B_j]^{hl} = \sum_{i_1,...,i_j=1}^{n_0} b^h_{i_1 i_2 \dots i_j} b^l_{i_1 i_2 \dots i_j}, \quad j = 0, ..., m, \tag{3.58}$$

where B_0 is the $n_0 \times n_0$ identity matrix.

Definition 3.22. *The Popp's volume is given by*

$$d\mathcal{P} = \frac{1}{\sqrt{\prod_{j=1}^r \det(B_j)}} |\nu_1 \wedge ... \wedge \nu_d|. \tag{3.59}$$

3.4.2 Comparison With Popp's volume

Now, we prove that the volume associated to the approximation scheme coincides with Popp's volume up to a constant to be determined. We need the following proposition.

Proposition 3.23. *The matrix M_0 defined with respect to the adapted frame $Z_1, ..., Z_d$ is the identity matrix; that is $M_0 = I_{n_0}$.*

Proof. Again, the fact that we are dealing with an adapted frame gives the nice expression of $\tilde{G}_h^{-1}$ in (3.54) that implies, with the invertibility of M_i that the matrix $\bar{G}$, which is the limit of matrix $(G_{(1)}(h))^{-1}G_{(2)}(h)$ is strictly lower triangular and that

$$\lim_{h\to 0} \det(I_d + (G_{(1)}(h))^{-1}G_{(2)}(h)) = 1.$$

Thus, for h small enough, $I_d + (G_{(1)}(h))^{-1}G_{(2)}(h)$ is invertible. Since

$$G_h^{-1} = G_{(1)}(h)(I_d + (G_{(1)}(h))^{-1}G_{(2)}(h)), \tag{3.60}$$

we get, taking the inverse of both sides in equation (3.60) and multiplying by $G_{(1)}(h)$ on the right, that

$$G_h G_{(1)}(h) = (I_d + (G_{(1)}(h))^{-1}G_{(2)}(h))^{-1}.$$

Thus,

$$M_0 = G_h G_{(1)}(h)\big|_{D_0} = (I_d + (G_{(1)}(h))^{-1}G_{(2)}(h))^{-1}\big|_{D_0}.$$

Since the limit of $I_d + (G_{(1)}(h))^{-1}G_{(2)}(h)$ (which is $I_d + \bar{G}$) exists and is invertible, we get

$$M_0 = (I_d + \bar{G})^{-1}\big|_{D_0}.$$

Now, as $\bar{G}$ is a strictly lower triangular matrix, it is nilpotent, say of index n. So,

$$(I_d + \bar{G})^{-1} = I_d + \sum_{k=1}^{n-1}(-1)^k\bar{G}^k.$$

Therefore, since for any k, $\bar{G}^k\big|_{D_0} = 0$ we get that $M_0 = I_{n_0}$. □

Recall the definition of M_i in subsection 3.3.2.3. Recall that our volume form is given by

$$d\mathcal{P}_o = \frac{1}{\sqrt{\prod_{i=1}^r \det(M_i)}}|\nu_1 \wedge ... \wedge \nu_d|.$$

Then, using (3.59), it is enough to compare the matrix M_i to the matrix B_i defined by 3.58 for any $1 \leqslant i \leqslant r$ ($M_0 = B_0 = I_{n_0}$), to deduce the relation between $d\mathcal{P}_o$ and $d\mathcal{P}$. In the following theorem, we establish the relation between the entries of M_i and B_i.

Theorem 3.24. *Denote by* $(\mu_{(i),\kappa_1\kappa_2})_{1\leqslant\kappa_1,\kappa_2\leqslant n_i-n_{i-1}}$ *the entries of the matrix* M_i*. Then, for every* $1\leqslant i\leqslant r$*, and any* $1\leqslant\kappa_1,\kappa_2\leqslant n_i-n_{i-1}$*, we have*

$$\mu_{(i),\kappa_1\kappa_2}=\frac{1}{2}[B_i]^{\kappa_1\kappa_2},$$

where $[B_i]^{\kappa_1\kappa_2}$ *are the entries of the matrix* B_i *defined by (3.58).*

Proof. Denote by $J:=\{1,...,p\}$. For any $j\in J$, write $X^{0j}=\sum_{l=1}^{n_0}a^l_{(0)j}Z_l$. Then, for $j_1,...,j_{i+1}\in J$, we have

$$[X^{0j_1},[X^{0j_2}...[X^{0j_i},X^{0j_{i+1}}]...]=\sum_{l_1,...,l_{i+1}=1}^{n_0}a^{l_1}_{(0)j_1}...a^{l_{i+1}}_{(0)j_{i+1}}[Z_{l_1},[Z_{l_2},...[Z_{l_i},Z_{l_{i+1}}]...]. \tag{3.61}$$

Then, for any $1\leqslant i\leqslant r$, any $n_{i-1}<n\leqslant n_i$, and any $(j_1,...,j_{i+1})\in J^{i+1}$, we get that the coefficient of $[X^{0j_1},[X^{0j_2}...[X^{0j_i},X^{0j_{i+1}}]...]$ in $Z_n\in D_i$ mod D_{i-1} is given by

$$a^n_{(i),(j_1,...,j_{i+1})}=\sum_{l_1,...,l_{i+1}=1}^{n_0}a^{l_1}_{(0)j_1}...a^{l_{i+1}}_{(0)j_{i+1}}b^n_{l_1,...,l_{i+1}}, \tag{3.62}$$

where the $b's$ are the adapted structure constants defined in (3.57). Denote by $\tilde{J}:=\{1,...,n_0\}$ and by $\mathcal{J}_i$ a subset of $\tilde{J}^{i+1}$ that enumerates the vectors X^{ij}. We recall that that the difference between $\tilde{J}^{i+1}$ and $\mathcal{J}_i$ consists of getting rid of the 0 and keeping one of the two vectors corresponding to $(j_0,\cdots,j_i,j_{i+1})$ and $(j_0,\cdots,j_{i+1},j_i)$.

Therefore, for any $1 \leqslant i \leqslant r$ and any $1 \leqslant \kappa_1, \kappa_2 \leqslant n_i - n_{i-1}$, we compute

$$
\begin{aligned}
\mu_{(i),\kappa_1\kappa_2} &= \sum_{(j_1,\dots,j_{i+1})\in\mathcal{J}_i} a^{\kappa_1}_{(i),(j_1,\dots,j_{i+1})} a^{\kappa_2}_{(i),(j_1,\dots,j_{i+1})} \\
&= \frac{1}{2} \sum_{(j_1,\cdots,j_{i+1})\in\tilde{J}^{i+1}} a^{\kappa_1}_{(i),(j_1,\dots,j_{i+1})} a^{\kappa_2}_{(i),(j_1,\dots,j_{i+1})} \\
&= \frac{1}{2} \sum_{\substack{l_1,\dots,l_{i+1}\in\tilde{J} \\ m_1,\dots,m_{i+1}\in\tilde{J}}} \sum_{(j_1,\dots,j_{i+1})\in\tilde{J}^{i+1}} a^{l_1}_{(0)j_1} \dots a^{l_{i+1}}_{(0)j_{i+1}} b^{\kappa_1}_{l_1,\dots,l_{i+1}} a^{m_1}_{(0)j_1} \dots a^{m_{i+1}}_{(0)j_{i+1}} b^{\kappa_2}_{m_1,\dots,m_{i+1}} \\
&= \frac{1}{2} \sum_{\substack{l_1,\dots,l_{i+1}\in\tilde{J} \\ m_1,\dots,m_{i+1}\in\tilde{J}}} \left(\sum_{j_1\in\tilde{J}} a^{l_1}_{(0)j_1} a^{m_1}_{(0)j_1} \right) \dots \left(\sum_{j_{i+1}\in\tilde{J}} a^{l_{i+1}}_{(0)j_{i+1}} a^{m_{i+1}}_{(0)j_{i+1}} \right) b^{\kappa_1}_{l_1,\dots,l_{i+1}} b^{\kappa_2}_{m_1,\dots,m_{i+1}} \\
&= \frac{1}{2} \sum_{\substack{l_1,\dots,l_{i+1}\in\tilde{J} \\ m_1,\dots,m_{i+1}\in\tilde{J}}} \delta^{m_1}_{l_1} \dots \delta^{m_{i+1}}_{l_{i+1}} b^{\kappa_1}_{l_1,\dots,l_{i+1}} b^{\kappa_2}_{m_1,\dots,m_{i+1}} \\
&= \frac{1}{2} \sum_{l_1,\dots,l_{i+1}\in\tilde{J}} b^{\kappa_1}_{l_1,\dots,l_{i+1}} b^{\kappa_2}_{l_1,\dots,l_{i+1}} = \frac{1}{2} [B_i]^{\kappa_1\kappa_2},
\end{aligned}
\tag{3.63}
$$

where the Kronecker deltas are due to proposition 3.23. The factor (1/2) popped out due to the choice made about choosing the vector fields the matrices $A_{(i)}$ are representing (and that we considered in our framework). Taking the sum over $(j_1, \dots, j_{i+1}) \in \mathcal{J}_i$ means that we are considering the enumeration of the vector fields we chose. When moving to $(j_1, \dots, j_{i+1}) \in J^{i+1}$, it means that we now included all the vector fields that are formed by the iterative brackets, and thus doubling the quantity on the right-hand side of (3.63). □

Recall that $d\mathcal{P}$ denotes the Popp's volume.

Corollary 3.24.1. *Recall that $d\mathcal{P}_o$ denotes the volume induced from our approximation scheme. We have*

$$
d\mathcal{P} = \frac{1}{\sqrt{2^r}} d\mathcal{P}_o. \tag{3.64}
$$

Remark 3.25. *If we define the A_i's including all these vectors (including the ones we excluded in our framework), we get that $d\mathcal{P} = d\mathcal{P}_o$. This is following remark 3.12 which is the reason why the constant* $1/2$ *popped out in our computation in (3.63).*

Remark 3.26. *We can deduce from corollary 3.24.1 that in the non-equiangular case, the function f defined in corollary 3.14.1, is defined on $M \backslash \mathcal{Z}$.*

Example 3.10. *Following example 3.7, the volume form obtained from the approximation scheme is given by*

$$d\mathcal{P}_o = (1/\sqrt{f_0(x)})|dx \wedge dy| = (1/|x|)|dx \wedge dy|.$$

Now, $\mathbb{R}^2 \backslash \{x = 0\}$ is the equiregular region with $r = 0$. We have

$$G_h = \begin{pmatrix} 1 & 0 \\ 0 & x^2 \end{pmatrix}.$$

Direct computation shows that $g^h(\mathcal{X}_i, \mathcal{X}_j) = \delta_i^j$ for $i, j = 1, 2$. So, $\mathcal{X}_1, \mathcal{X}_2$ is an adapted frame. Following (3.59), and as $r = 0$ on the equiregular region of $\mathbb{R}^2$, we have

$$d\mathcal{P} = \nu_1 \wedge \nu_2 = |dx \wedge \frac{1}{|x|} dy| = \frac{1}{|x|}|dx \wedge dy| = d\mathcal{P}_o,$$

where ν_1, ν_2 are the dual frame to $\mathcal{X}_1, \mathcal{X}_2$.

Example 3.11. *Following example 3.8, the volume form obtained from the approximation scheme is given by*

$$d\mathcal{P}_o = (1/\sqrt{f_0(x)})|dx \wedge dy \wedge dz| = |dx \wedge dy \wedge dz|.$$

Now, this is an equiregular setting with $r = 1$. Using the expression of G_h^{-1}, we get

$$G_h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 + x^2 h^{-2} & -xh^{-2} \\ 0 & -xh^{-2} & h^{-2} \end{pmatrix}.$$

Direct computation shows that $g^h(\mathcal{X}_i, \mathcal{X}_j) = \delta_i^j$ for $i, j = 1, 2$. So, $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$ is an adapted frame. Following (3.59), and as $r = 1$ on $\mathbb{R}^3$, we have

$$d\mathcal{P} = \nu_1 \wedge \nu_2 \wedge \nu_3 = \frac{1}{\sqrt{2}}|dx \wedge dy \wedge (dz - xdy)| = \frac{1}{\sqrt{2}}|dx \wedge dy \wedge dz| = \frac{1}{\sqrt{2}} d\mathcal{P}_o,$$

where ν_1, ν_2, ν_3 are the dual frame to $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$.

Example 3.12. *Following example 3.9, the volume form obtained from the approximation scheme is given by*

$$d\mathcal{P}_o = (1/\sqrt{f_0(x)})|dx \wedge dy \wedge dz| = (1/|x|)|dx \wedge dy \wedge dz|.$$

Now, $\mathbb{R}^2\backslash\{x = 0\}$ *is the equiregular region with* $r = 1$*. We have*

$$G_h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 + \frac{x^2}{2} & -1 \\ 0 & -1 & \frac{2}{x^2} \end{pmatrix}.$$

Direct computation shows that $g^h(\mathcal{X}_i, \mathcal{X}_j) = \delta_i^j$ *for* $i, j = 1, 2..$ *So,* $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$ *is an adapted frame. Following (3.59), and as* $r = 1$ *on the equiregular region of* $\mathbb{R}^3$*, we have*

$$d\mathcal{P} = \nu_1 \wedge \nu_2 \wedge \nu_3 = \frac{1}{\sqrt{2}}|dx \wedge dy \wedge (\frac{1}{|x|}dz - \frac{|x|}{2}dy)| = \frac{1}{\sqrt{2}|x|}|dx \wedge dy \wedge dz| = \frac{1}{\sqrt{2}}d\mathcal{P}_o,$$

where ν_1, ν_2, ν_3 *are the dual frame to* $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$*.*

In each of the preceding standard examples, the initial vector fields are orthonormal. Also, in all these examples, the coefficient of $\nu_1 \wedge ... \wedge \nu_d$ is equal to a constant $(1/\sqrt{2^r})$, and the function is obtained by calculating the dual frame. We give an example where this is not the case.

Example 3.13. *On* $\mathbb{R}^4$*, consider the following smooth vector fields:* $\mathcal{X}_1 = \partial_x, \mathcal{X}_2 = \partial_y, \mathcal{X}_3 = x\partial_z + y\partial t$*. On* $\mathbb{R}^4\backslash\{x = 0\}$*,* $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$ *and* $\mathcal{X}_4 = [\mathcal{X}_2, \mathcal{X}_3] = \partial_t$ *span* $\mathbb{R}^4$*. Then, this is an equiregular setting with* $r = 1$*. Following (3.27), we have*

$$G_h^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & x^2 + h^2 & xy \\ 0 & 0 & xy & y^2 + h^2 \end{pmatrix}.$$

So, $det(G_h^{-1}) = h^4 + h^2(x^2 + y^2) = h^2(h^2 + x^2 + y^2)$*. As* $\varsigma = 1$*, then* $f_h(x) = h^2 + x^2 + y^2$ *and so* $f_0(x) = x^2 + y^2$*. Then, the volume given by the approximation is given by*

$$d\mathcal{P}_o = (1/\sqrt{x^2 + y^2})|dx \wedge dy \wedge dz \wedge dt|.$$

Now, we have

$$\Upsilon_s = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{y^2+h^2}{h^2+h(x^2+y^2)} & \frac{-xy}{h^2+h(x^2+y^2)} \\ 0 & 0 & \frac{-xy}{h^2+h(x^2+y^2)} & \frac{y^2+x^2}{h^2+h(x^2+h^2)} \end{pmatrix}.$$

Then, direct computation shows that

$$g^h(\mathcal{X}_3) = \frac{x^2+y^2}{1+\frac{x^2+y^2}{h}} := a(x,y)^2.$$

Now, setting

$$\{\tilde{\mathcal{X}}_1, \tilde{\mathcal{X}}_2, \tilde{\mathcal{X}}_3, \tilde{\mathcal{X}}_4\} = \{\mathcal{X}_1, \mathcal{X}_2, (1/a(x,y))\mathcal{X}_3, \mathcal{X}_4\},$$

we get that $g^h(\tilde{\mathcal{X}}_i, \tilde{\mathcal{X}}_j) = \delta_i^j$ *for* $i,j = 1,...,4$. *Thus* $\{\tilde{\mathcal{X}}_1, \tilde{\mathcal{X}}_2, \tilde{\mathcal{X}}_3, \tilde{\mathcal{X}}_4\}$ *is an adapted frame. Then following (3.57), and as*

$$[\mathcal{X}_1, \mathcal{X}_3] = \partial_z = \frac{1}{xa(x,y)}\tilde{\mathcal{X}}_3 + \frac{-y}{x}\tilde{\mathcal{X}}_4,$$

we get that $b_{23}^4 = 1$ *and* $b_{13}^4 = (y/x)$. *Then, following (3.58), we get*

$$B_2 = \left(\frac{2}{x^2}(x^2+y^2)\right).$$

Thus, (3.59) implies that

$$d\mathcal{P} = \frac{|x|}{\sqrt{2}\sqrt{x^2+y^2}}\nu_1\wedge\nu_2\wedge\nu_3\wedge\nu_4 = \frac{|x|}{\sqrt{2}\sqrt{x^2+y^2}}|dx\wedge dy\wedge\frac{1}{|x|}dz\wedge dt| = \frac{1}{\sqrt{2}}d\mathcal{P}_o.$$

Chapter 4

Convergence Of Spectrum

We study the sublaplacian on a compact manifold and prove the convergence of the spectrum in the previous approximation scheme.

4.1 Introduction

In this chapter, we suppose that M is a compact orientable manifold.

If we define the sublaplacian Δ_0 with respect to a fixed volume $d\omega$, a natural question arises, whether one can construct a family of Riemannian metrics such that the spectrum of the Riemannian Laplace operator defined with respect to $d\omega$ converges to the spectrum of $(\Delta_0, d\omega)$.

Another question that seems more challenging is whether we can construct a family of Riemannian metrics such that the spectrum of the Riemannian Laplace operator defined with respect to the volume form $\mathrm{dvol}g^h$ of the metric converges to the spectrum of $(\Delta_0, d\omega)$.

In fact, in these settings, only little is known about the convergence of the spectrum of the Laplacians (see [39][40][76]). In some specific settings (which implies that $\mathrm{dvol}g^h = h^2 d\omega$), it was shown that the family Δ_h converges to a hypoelliptic operator Δ, and that the eigenvalues of Δ_h converge to those of Δ. This was first observed by Fukaya in [39] and then proved by Ge in [40] (See also [76] for contact manifold case).

Here, we prove the convergence theorem in our framework, which is much more general. Suppose that the subriemannian structure is equiregular and recall that $d\mathcal{P}$ denotes the Popp volume. We construct a family of Laplace operators Δ_h such that the spectrum of $(\Delta_h, \mathrm{dvol}g^h)$ converges to that of $(\Delta_0, d\mathcal{P})$. It is logical to consider the Riemannian family associated to the approximation scheme of chapter 3 as our candidate. This is because we know from the previous chapter that the family $h^{-\varsigma}\mathrm{dvol}g^h$ converges to $d\mathcal{P}$ (the main result of chapter 3).

Using some uniform subelliptic estimates, and standard theorems of spectral theory, we first prove the convergence in the fixed volume form case, where the family of elliptic operators approximating $(\Delta_0, d\omega)$ is defined with respect to $d\omega$. Then, by proving the validity of the uniform estimates, we prove that the results remain true in the case where the family of elliptic operators is defined with respect to the volume form of the family of Riemannian metrics. We show that these estimates follow from the previous case by proving nice properties on the function f_h, and the adjoint taken with respect to $\mathrm{dvol}g^h$ of X^{ij}.

Approaches using subelliptic estimates are usually popular in the context of studying a subriemannian manifold, especially since uniform versions of such estimates, when coupled with similar approximation schemes, allow to extend known Riemannian results to the subriemannian setting (see [27][32] for instance).

We first recall the definition of a hypoelliptic and subelliptic operator.

Definition 4.1.
- *An operator P is hypoelliptic on M if $Pu \in \mathcal{C}^\infty(M)$ implies that $u \in \mathcal{C}^\infty(M)$.*
- *An operator P is subelliptic of order p with ϵ loss of derivatives if $Pu \in H^s(M)$ implies that $u \in H^{s+p-\epsilon}(M)$ for some $\epsilon < 1$.*

Let's define the divergence operator.

Definition 4.2. *Fix a volume form $d\omega$ on M and let X be a smooth vector field. The divergence* $\mathrm{div}_\omega(X)$ *is defined as the function satisfying: For any $u, v \in \mathcal{C}^\infty(M)$,*

$$\int_M uXv d\omega = \int_M \Big[\big(-X - \mathrm{div}_\omega(X)\big)\, u\Big]\, v d\omega. \tag{4.1}$$

We recall the chain rule for the divergence operator.

Corollary 4.2.1. *Let $d\omega$ be a Riemannian volume form on M. For any vector field X on M and any function f, we have*

$$\mathrm{div}_\omega(fX) = X(f) + f\mathrm{div}_\omega(X). \tag{4.2}$$

Proof. For any smooth functions u and v, we have

$$\begin{aligned}\int \big(-fXu - \mathrm{div}_\omega(fX)u\big)\, v\, d\omega &= \int u(fX)v\, d\omega \\ &= \int \big(-X(fu) - (\mathrm{div}_\omega X)fu\big)\, v\, d\omega \\ &= \int \big(-fXu - (Xf)u - (\mathrm{div}_\omega X)fu\big)\, v\, d\omega.\end{aligned}$$

We deduce our claim by identification. □

The divergence operator satisfies the following property.

Proposition 4.3. *Let $d\omega_1$ and $d\omega_2$ be two (non-vanishing) volume forms on M and let α be such that $d\omega_1 = \alpha d\omega_2$. Then, the following assertions hold true:*

1. *For any vector field X on M, we have*

$$\alpha \mathrm{div}_{\omega_1}(X) = \mathrm{div}_{\omega_2}(\alpha X). \tag{4.3}$$

2. *For any vector field X on M, we have*

$$\mathrm{div}_{\omega_2}(X) = \alpha X\left(\frac{1}{\alpha}\right) + \mathrm{div}_{\omega_1}(X). \tag{4.4}$$

Proof. 1. By definition, $\mathrm{div}_{\omega_2}(\alpha X)$ is the function satisfying

$$\int_M u(\alpha X) v d\omega_2 = \int_M \Big[\big(-\alpha X - \mathrm{div}_{\omega_2}(\alpha X)\big)\, u\Big]\, v d\omega_2, \tag{4.5}$$

for all $u, v \in \mathcal{C}^\infty(M)$. By the definition of α in proposition (4.3), equation (4.5) implies that

$$\int_M uXv d\omega_1 = \int_M \left[\left(-X - \frac{1}{\alpha}\mathrm{div}_{\omega_2}(\alpha X)\right) u\right] v d\omega_1. \tag{4.6}$$

Finally, by definition, (4.6) implies that

$$\mathrm{div}_{\omega_1}(X) = \frac{1}{\alpha}\mathrm{div}_{\omega_2}(\alpha X).$$

2. The second property follows by using the chain rule and (4.3).

□

We introduce now the (sub)Laplace operators.

4.1.1 Sublaplace Operator

We associate on the compact manifold M a fixed volume form that we denote $d\omega$ and we denote by ϕ the function such that locally, $d\omega = \phi(x)dx$. We denote by $L^2_\omega(M) := L^2_\omega((M,\mathbb{R}), d\omega)$ the Hilbert space associated to $d\omega$, given by

$$L^2_\omega(M) := \left\{ u : M \to \mathbb{R};\ \|u\|^2_{L^2_\omega} := \int_M |u|^2 d\omega < \infty \right\}.$$

On $L^2_\omega(M)$, we define the sublaplace operator as

$$\Delta_0 = \sum_{j=0}^{p} (X^{0j})^{*_\omega} X^{0j} = \sum_{j=0}^{p} \left(-(X^{0j})^2 - \operatorname{div}_\omega(X^{0j}) X^{0j} \right), \tag{4.7}$$

where the star denotes the adjoint with respect to $d\omega$. Hörmander [50] first proved, that under Hörmander's condition, the operator Δ_0, which is called a type 1 Hörmander operator, is hypoelliptic. He proved this by proving that Δ_0 satisfies a subelliptic estimate (which implies that Δ_0 is subelliptic and thus hypoelliptic).

4.1.2 Family Of Riemannian Laplacians

For any $h > 0$, we define on $L^2_\omega(M)$, the family of elliptic operators:

$$\Delta_h = \sum_{i=0}^{r} \sum_{j=1}^{N_i} h^{2i} (X^{ij})^{*_\omega} X^{ij} = \sum_{i=0}^{r} \sum_{j=1}^{N_i} h^{2i} \left(-(X^{ij})^2 - \operatorname{div}_\omega(X^{ij}) X^{ij} \right),$$

where the star denotes the adjoint with respect to $d\omega$.
Recall that $\varsigma(m)$ was given in corollary 3.14.1 by

$$\varsigma(m) = \sum_{1}^{r} i[n_i(m) - n_{i-1}(m)]. \tag{4.8}$$

If the subRiemannian manifold is equiregular, then $\varsigma(m)$ is constant on M, and is denoted here after by ς.

Now, denote by $L^2_h(M) := L^2_h((M,\mathbb{R}), \mathrm{dvol} g^h)$ the Hilbert space associated to $h^{2\varsigma}\mathrm{dvol} g^h$, given by

$$L^2_h(M) := \left\{ u : M \to \mathbb{R};\ \|u\|^2_{L^2_h(M)} := \int_M |u|^2 h^{2\varsigma} \mathrm{dvol} g^h < \infty \right\}.$$

For any $h > 0$, we define on $L^2_h(M)$, the family of elliptic operators:

$$\tilde{\Delta}_h = \sum_{i=0}^{r}\sum_{j=1}^{N_i} h^{2i}(X^{ij})^{*_h}X^{ij} = \sum_{i=0}^{r}\sum_{j=1}^{N_i} h^{2i}\left(-(X^{ij})^2 - \mathrm{div}_h(X^{ij})X^{ij}\right),$$

where the star denotes the adjoint with respect to $\mathrm{dvol}g^h$ and div_h denote the divergence with respect to $\mathrm{dvol}g^h$.
For $h > 0$, the operators Δ_h and $\tilde{\Delta}_h$ are elliptic. Since M is compact, then Δ_h and $\tilde{\Delta}_h$ with domain $\mathcal{C}^\infty(M)$ are essentially self-adjoint on $L^2_\omega(M)$ and $L^2_h(M)$ respectively and their self-adjoint extensions have compact resolvents. Thus, their spectrum consists of an increasing sequence of positive eigenvalues of finite multiplicities that converge to $+\infty$.
Hereafter, when we write X^*, the $*$ is taken with respect to the fixed volume form; that is $X^* = X^{*_\omega}$.

4.2 Uniform Estimates

At some point in the proof of the convergence, we will need the boundedness in any Sobolev space, of the eigenfunctions of Δ_h and $\tilde{\Delta}_h$. One way of doing this is to prove an estimate showing that the norm of the smooth functions on M is controlled by the norm of Δ_h and $\tilde{\Delta}_h$. A useful tool will be the so-called subelliptic estimate.
As said earlier, Hörmander [50] first proved that the operator Δ_0 (or any type one Hörmander operator) is hypoelliptic. Kohn [58] then proved this result by proving a subelliptic estimate on Δ_0 using pseudo-differential calculus. Since then, many authors who worked with approximation schemes investigated the validity of such uniform estimates applied to an approximating family of operators (see for instance [30][48]).
Here, we prove a uniform parameter-dependent version of the famous local subelliptic estimate, which allows us to obtain a uniform subelliptic estimate and deduce some information about the operators defined above. In section 4.2.2 we deal with Δ_h; we adapt Kohn's proof of subellipticity to prove that a uniform estimate is true for Δ_h and then, in section 4.2.3, weuse the nice properties of the functions f_h obtained in chapter 3 to prove a uniform estimate for $\tilde{\Delta}_h$.

It is worth mentioning that although in this proof we lose having an optimal gain of regularity (optimal order of subellipticity, which is $1/r$ as proved in [75]), this is not important to us, as having any order will imply the self-adjointness and the compactness of the resolvent of Δ_0, and the boundedness of the eigenfunctions of Δ_h and $\tilde{\Delta}_h$.

4.2.1 The Sobolev Space $H^s(M)$

For $s \in \mathbb{R}$, denote by Λ^s the following operator:

$$\Lambda^s = (Id + \Delta_1)^{\frac{s}{2}}, \tag{4.9}$$

where Id is the identity operator. For $s \in \mathbb{R}$, we define the space $H^s_\omega(M)$ with respect to $d\omega$ as

$$H^s_\omega(M) = \{u \in L^2_\omega(M);\ \|u\|_{H^s_\omega(M)} := \|\Lambda^s u\|_{L^2_\omega(M)} < \infty\}. \tag{4.10}$$

In what follows, the constants may have the same notation though be of different values.

4.2.2 First Case: Fixed Volume Form

As we said earlier, we adapt Kohn's proof and prove a uniform version of the subelliptic estimate. Thus our proof will mainly depend on some very well-known facts in the pseudo-differential calculus, that we state in Appendix A.3.

We first prove a simple proposition.

Proposition 4.4. *For any smooth u on M, for any $1 \leqslant j \leqslant p$ and for all $h \in [0,1]$, we have*

$$\left\| X^{0,j} u \right\|^2_{L^2_\omega} \leqslant \frac{1}{2}\left(\|\Delta_h u\|^2_{L^2_\omega} + \|u\|^2_{L^2_\omega} \right).$$

Proof. We compute

$$\left\| X^{0,j} u \right\|^2_{L^2_\omega} \leqslant \sum_{i=0}^{r} \sum_{j=1}^{N_i} h^{2i} \left\| X^{ij} u \right\|^2_{L^2_\omega} \leqslant \langle \Delta_h u, u \rangle_{L^2_\omega} \leqslant \frac{1}{2}\left(\|\Delta_h u\|^2_{L^2_\omega} + \|u\|^2_{L^2_\omega} \right).$$

□

Our aim is to prove that: $\exists \epsilon > 0, \forall s \in \mathbb{R}, \exists C(s), \forall h \in [0,1], \forall u \in \mathcal{C}^\infty(M)$,

$$\|u\|_{H_\omega^{\epsilon+s}} \leqslant C(s) \left(\|\Delta_h u\|_{H_\omega^s} + \|u\|_{H_\omega^s} \right). \tag{4.11}$$

We first prove (4.11) for $s = 0$ using mainly the results of section A.3.

Theorem 4.5. *There exist $\epsilon > 0$ and a constant $C > 0$ such that for all $h \in [-1, 1]$ and all $u \in \mathcal{C}^\infty(M)$,*

$$\|u\|_{H_\omega^\epsilon} \leqslant C \left(\|\Delta_h u\|_{L_\omega^2} + \|u\|_{L_\omega^2} \right). \tag{4.12}$$

Proof. In this proof, all inner products are with respect to $L_\omega^2(M)$.
Denote by $\mathcal{P}$ the set of all pseudo-differential operators of order zero. For all $\epsilon \in]0, 1]$, let

$$\mathcal{P}_\epsilon = \{P \in \mathcal{P}; \|Pu\|_{H_\omega^\epsilon} \leqslant C(\epsilon) \left(\|\Delta_h u\|_{L_\omega^2} + \|u\|_{L_\omega^2} \right), \forall u \in C^\infty(M), \forall h \in [0,1]\}.$$

We have the following properties:

(a) For all $\epsilon_1 \leqslant \epsilon_2$, $\mathcal{P}_{\epsilon_2} \subseteq \mathcal{P}_{\epsilon_1}$. This is obvious as $H_\omega^{\epsilon_2}$ is continuously embedded in $H_\omega^{\epsilon_1}$ for all $\epsilon_1 \leqslant \epsilon_2$.

(b) If $P \in \mathcal{P}_\epsilon$ then $P^* \in \mathcal{P}_\epsilon$ for any $\epsilon \leqslant \frac{1}{2}$.
Indeed, let $P \in \mathcal{P}_\epsilon$. We compute

$$\begin{aligned}
\|P^* u\|_{H_\omega^\epsilon}^2 = \|\Lambda^\epsilon P^* u\|_{L_\omega^2}^2 &= \langle \Lambda^\epsilon P^* u, \Lambda^\epsilon P^* u \rangle \\
&= \langle P^* u, \Lambda^{2\epsilon} P^* u \rangle \\
&= \langle (P\Lambda^{2\epsilon} P^* - P^* \Lambda^{2\epsilon} P) u, u \rangle + \langle P^* \Lambda^{2\epsilon} P u, u \rangle \\
&= \langle (P\Lambda^{2\epsilon} P^* - P^* \Lambda^{2\epsilon} P) u, u \rangle + \|Pu\|_{H_\omega^\epsilon}^2 .
\end{aligned}$$

Using theorem A.13, $P\Lambda^{2\epsilon} P^* - P^* \Lambda^{2\epsilon} P$ is of order $-1 + 2\epsilon$, which is non-positive for $\epsilon \leqslant \frac{1}{2}$. So, for $\epsilon \leqslant \frac{1}{2}$, it is bounded in $L_\omega^2(M)$ by corollary A.12.1, and we get

$$\|P^* u\|_{H_\omega^\epsilon}^2 \leqslant C_1(\epsilon) \|u\|_{L_\omega^2}^2 + \|Pu\|_{H_\omega^\epsilon}^2 \leqslant C(\epsilon) \left(\|\Delta_h u\|_{L_\omega^2} + \|u\|_{L_\omega^2} \right).$$

(c) For $\epsilon < \frac{1}{2}$, $\mathcal{P}_\epsilon$ is a left and right ideal in $\mathcal{P}$.
Indeed, by theorem A.12, pseudo-differential operators of order 0 are

bounded in any Sobolev space. So we compute, for $P \in \mathcal{P}_\epsilon$ and every $A \in \mathcal{P}$,

$$\|APu\|_{H^\epsilon_\omega} \leqslant C(\epsilon)\|Pu\|_{H^\epsilon_\omega},$$

which implies that $\mathcal{P}_\epsilon$ is left ideal in $\mathcal{P}$. Moreover, by (b), $P^* \in \mathcal{P}_\epsilon$, so as $\mathcal{P}_\epsilon$ is left ideal, $A^*P^* \in \mathcal{P}_\epsilon$. By (b) again, $P^*A^* = (A^*P^*)^* \in \mathcal{P}_\epsilon$. This implies that $\mathcal{P}_\epsilon$ is right ideal in $\mathcal{P}$.

(d) For all $j = 1, ..., n_0$ and all $\epsilon \leqslant \frac{1}{2}$, $X^{0,j}\Lambda^{-1} \in \mathcal{P}_\epsilon$.
Indeed, for $\epsilon \leqslant 1$, the operator $\Lambda^{\epsilon-1}$ is of non-positive order. So it is bounded in $L^2_\omega(M)$ using corollary A.12.1. We compute

$$\begin{aligned}\left\|\Lambda^{-1}X^{0,j}u\right\|_{H^\epsilon_\omega} &= \left\|\Lambda^{\epsilon-1}X^{0,j}u\right\|_{L^2_\omega}\\ &\leqslant c(\epsilon)\left\|X^{0,j}u\right\|_{L^2_\omega}\\ &\leqslant C(\epsilon)\left(\|\Delta_h u\|_{L^2_\omega} + \|u\|_{L^2_\omega}\right),\end{aligned}$$

where the last inequality is due to proposition 4.4. Then $\Lambda^{-1}X^{0,j} \in \mathcal{P}_\epsilon$ for any $j = 1, ..., n_0$.
Now, we observe that

$$X^{0,j}\Lambda^{-1} = -(\Lambda^{-1}X^{0,j})^* + \mathrm{div}_\omega(X^{0,j})\Lambda^{-1}.$$

The first term is in $\mathcal{P}_\epsilon$ using (b). Now, the functions $\mathrm{div}_\omega(X^{0,j})$ are smooth on M, so bounded by some constant. Since Λ^{-1} is of negative order, it is bounded in $L^2_\omega(M)$ by corollary A.12.1 So, the second term is in $\mathcal{P}_\epsilon$. We get that for $\epsilon \leqslant \frac{1}{2}$, $X^{0,j}\Lambda^{-1} \in \mathcal{P}_\epsilon$.

(e) If $P \in \mathcal{P}_\epsilon$, then $[X^{0,j}, P] \in \mathcal{P}_{\frac{\epsilon}{2}}$ for all $j = 1, ..., n_0$ and $\epsilon \leqslant \frac{1}{2}$.
Indeed, let $P \in \mathcal{P}_\epsilon$ and let $\sigma = \frac{\epsilon}{2}$. Let $j = 1, ..., n_0$, and denote by T the pseudo-differential operator of order ϵ defined as $T := \Lambda^{2\sigma}[X^{0,j}, P]$
We compute

$$\begin{aligned}\left\|[X^{0,j}, P]u\right\|^2_{H^\sigma_\omega} &= \langle [X^{0,j}, P]u, Tu\rangle\\ &= \langle X^{0,j}Pu, Tu\rangle - \langle PX^{0,j}u, Tu\rangle\\ &\leqslant |\langle X^{0,j}Pu, Tu\rangle| + |\langle PX^{0,j}u, Tu\rangle|.\end{aligned}$$

Observe that P^*T and TP^* are of order ϵ and have the same principal symbol. Then, by theorem A.13, we have that $P^*T - TP^*$ is of order $\epsilon - 1$ which is negative for $\epsilon < 1$. So, by corollary A.12.1, it is bounded in $L^2_\omega(M)$.
Moreover, T is of order ϵ, so by theorem A.12, it is bounded from $H^\epsilon_\omega(M)$ to $L^2_\omega(M)$.
It follows that

$$\begin{aligned}
|\langle PX^{0,j}u, Tu\rangle| &= |\langle X^{0,j}u, P^*Tu\rangle| \\
&\leqslant \left\|X^{0,j}u\right\|^2_{L^2_\omega} + \|P^*Tu\|^2_{L^2_\omega} \\
&\leqslant \left\|X^{0,j}u\right\|^2_{L^2_\omega} + \|TP^*u\|^2_{L^2_\omega} + C_0(\epsilon)\|u\|^2_{L^2_\omega} \\
&\leqslant \left\|X^{0,j}u\right\|^2_{L^2_\omega} + C_1(\epsilon)\|P^*u\|^2_{H^\epsilon_\omega} + C_0(\epsilon)\|u\|^2_{L^2_\omega} \\
&\leqslant C_1(\epsilon)\|P^*u\|^2_{H^\epsilon_\omega} + C_2(\epsilon)(\|\Delta_h u\|^2_{L^2_\omega} + \|u\|^2_{L^2_\omega}) \\
&\leqslant C_3(\epsilon)(\|\Delta_h u\|_{L^2_\omega} + \|u\|_{L^2_\omega})^2
\end{aligned}$$

since P^* is in $\mathcal{P}_\epsilon$.

For the second term, we write

$$\begin{aligned}
\langle X^{0,j}Pu, Tu\rangle &= \langle Pu, (X^{0,j})^*Tu\rangle \\
&= \langle Pu, T(X^{0,j})^*u\rangle + \langle Pu, [(X^{0,j})^*, T]u\rangle \\
&\leqslant \|T^*Pu\|_{L^2_\omega}\|(X^{0,j})^*u\|_{L^2_\omega} + \|[(X^{0,j})^*, T]^*Pu\|_{L^2_\omega}\|u\|_{L^2_\omega}.
\end{aligned}$$

Since T^* is a pseudo-differential operators of order ϵ we have

$$\|T^*Pu\|_{L^2_\omega} \leqslant C\|Pu\|_{H^\epsilon_\omega}$$

The same estimate applies to $\|[(X^{0,j})^*, T]^*Pu\|_{L^2_\omega}$ since $[(X^{0,j})^*, T]^*$ is also of order ϵ. Finally, we use the fact that $(X^{0,j})^* = -X^{0,j} + c_j$ for some smooth function c_j and that $P \in \mathcal{P}_\epsilon$ to conclude.

(f) For $0 \leqslant i \leqslant r$, for all $1 \leqslant j \leqslant N_i$, we have $X^{ij}\Lambda^{-1} \in \mathcal{P}_{\frac{1}{2^{i+1}}}$.
We will show this for $i = 2$. So, let $1 \leqslant j, k \leqslant n_0$. Note that

$$[X^{0,j}, X^{0,k}]\Lambda^{-1} = [X^{0,j}, X^{0,k}\Lambda^{-1}] - X^{0,k}\Lambda^{-1}\Lambda[X^{0,j}, \Lambda^{-1}].$$

By parts (d) and (e),

$$[X^{0,j}, X^{0,k}\Lambda^{-1}] \in \mathcal{P}_{\frac{1}{4}}.$$

Now, $\Lambda[X^{0,j}, \Lambda^{-1}]$ is of order zero, so using parts (c) and (d),

$$X^{0,k}\Lambda^{-1}\Lambda[X^{0,j}, \Lambda^{-1}] \in \mathcal{P}_{\frac{1}{2}}.$$

Using part (a), we get that $[X^{0,j}, X^{0,k}]\Lambda^{-1} \in \mathcal{P}_{\frac{1}{4}}$. By recurrence, we get that $X^{ij}\Lambda^{-1} \in \mathcal{P}_{\frac{1}{2^{i+1}}}$ for all $0 \leqslant i \leqslant r, 1 \leqslant j \leqslant N_i$.

(g) We have $\mathcal{P}_{\frac{1}{2^{r+1}}} = \mathcal{P}$.
Indeed, using (4.9), the definition of Λ, we have that

$$Id = \Lambda^{-2} + \sum_{i=0}^{r}\sum_{j=1}^{N_i} \Lambda^{-2}(X^{ij})^* X^{ij}.$$

Using Hörmander condition, the vector fields span the tangent space, and so we have that $\Lambda^{-1}(X^{ij})^* \in \mathcal{P}_{\frac{1}{2^{r+1}}}$ for every $j \in \{1, ..., N_i\}$, and so, by parts (b) and (c), $\Lambda^{-1}(X^{ij})^* X^{ij}\Lambda^{-1} \in \mathcal{P}_{\frac{1}{2^{r+1}}}$. Observing that

$$\Lambda^{-2}(X^{ij})^* X^{ij} = \Lambda^{-1}(X^{ij})^* X^{ij}\Lambda^{-1} + \Lambda^{-1}[\Lambda^{-1}(X^{ij})^* X^{ij}\Lambda^{-1}, \Lambda],$$

we deduce that $Id \in \mathcal{P}_{\frac{1}{2^{r+1}}}$.

Therefore, we obtain (4.12). $\square$

Before proving the full uniform subelliptic estimate, we record a useful result.

Proposition 4.6. *The following holds true:* $\forall 0 \leqslant i \leqslant r, \forall 1 \leqslant j \leqslant N_i, \forall \zeta > 0, \forall h > 0, \forall u \in \mathcal{C}^\infty(M)$,

$$\left\| h^i X^{ij} u \right\|_{L^2_\omega} \leqslant \frac{\zeta}{\sqrt{2}} \|\Delta_h u\|_{L^2_\omega} + \frac{1}{\sqrt{2}\zeta} \|u\|_{L^2_\omega}. \tag{4.13}$$

Proof. We compute

$$\begin{aligned}
\left\| h^i X^{ij} u \right\|^2_{L^2_\omega} &\leqslant \sum_{i=0}^{r}\sum_{j=1}^{N_i} \left\| h^i X^{ij} u \right\|^2_{L^2_\omega} = \langle h^i X^{ij} u, h^i X^{ij} u \rangle_{L^2_\omega} \\
&= \left\langle \sum_{i=0}^{r}\sum_{j=1}^{N_i} h^{2i}(X^{ij})^* X^{ij} u, h^i u \right\rangle_{L^2_\omega} = \langle \Delta_h u, h^i u \rangle_{L^2_\omega} \\
&\leqslant \|\Delta_h u\|_{L^2_\omega} \|u\|_{L^2_\omega} \leqslant \frac{\zeta^2}{2} \|\Delta_h u\|^2_{L^2_\omega} + \frac{1}{2\zeta^2} \|u\|^2_{L^2_\omega}.
\end{aligned}$$

Now, since for any $a, b > 0$, $\sqrt{a+b} \leqslant \sqrt{a} + \sqrt{b}$, we deduce (4.13). $\square$

Theorem 4.7 (Uniform Subelliptic Estimate)**.** *The following holds true:* $\exists \epsilon > 0, \forall s \in \mathbb{R}, \exists C(s), \forall h \in [-1,1], \forall u \in \mathcal{C}^\infty(M)$,

$$\|u\|_{H^{\epsilon+s}_\omega} \leqslant C(s) \left(\|\Delta_h u\|_{H^s_\omega} + \|u\|_{H^s_\omega} \right). \tag{4.14}$$

Proof. Let $s \in \mathbb{R}$ and apply (4.12) to $\Lambda^s u$:

$$\|\Lambda^s u\|_{H^\epsilon_\omega} \leqslant c\big(\|\Delta_h \Lambda^s u\|_{L^2_\omega} + \|\Lambda^s u\|_{L^2_\omega} \big). \tag{4.15}$$

Observe that

$$\|\Delta_h \Lambda^s u\|_{L^2_\omega} \leqslant \|[\Delta_h, \Lambda^s]u\|_{L^2_\omega} + \|\Delta_h u\|_{H^s_\omega}. \tag{4.16}$$

We calculate

$$\begin{aligned}
\|[\Delta_h, \Lambda^s]u\|_{L^2_\omega} &\leqslant \sum_{i=0}^{r} \sum_{j=1}^{N_i} h^{2i} \left\| [(X^{ij})^* X^{i,j}, \Lambda^s]u \right\|_{L^2_\omega} \\
&\leqslant \sum_{i=0}^{r} \sum_{j=1}^{N_i} h^{2i} \left\| [(X^{ij})^*, \Lambda^s] X^{i,j} u \right\|_{L^2_\omega} + \sum_{i=0}^{r} \sum_{j=1}^{N_i} h^{2i} \left\| (X^{ij})^* [X^{i,j}, \Lambda^s]u \right\|_{L^2_\omega} \\
&= I_s + J_s.
\end{aligned} \tag{4.17}$$

Combining (4.16) and (4.17), we have

$$\|\Delta_h \Lambda^s u\|_{L^2_\omega} \leqslant I_s + J_s + \|\Delta_h u\|_{H^s_\omega}. \tag{4.18}$$

We first deal with I_s. Fix $s \in \mathbb{R}$. Using that $[(X^{ij})^*, \Lambda^s]\Lambda^{-s}$ and $[X^{i,j}, \Lambda^s]\Lambda^{-s}$

are bounded on L^2_ω, we compute

$$\begin{aligned} I_s &= \sum_{i=0}^{r}\sum_{j=1}^{N_i} h^{2i}\left\|[(X^{ij})^*,\Lambda^s]X^{i,j}u\right\|_{L^2_\omega} \\ &= \sum_{i=0}^{r}\sum_{j=1}^{N_i} h^{2i}\left\|[(X^{ij})^*,\Lambda^s]\Lambda^{-s}\Lambda^s X^{i,j}u\right\|_{L^2_\omega} \\ &\leqslant c(s)\sum_{i=0}^{r}\sum_{j=1}^{N_i} h^{2i}\left\|\Lambda^s X^{i,j}u\right\|_{L^2_\omega} \qquad (4.19) \\ &\leqslant c(s)\sum_{i=0}^{r}\sum_{j=1}^{N_i} h^{2i}\left(\left\|X^{i,j}\Lambda^s u\right\|_{L^2_\omega} + \left\|[X^{i,j},\Lambda^s]u\right\|_{L^2_\omega}\right) \\ &\leqslant c(s)\sum_{i=0}^{r}\sum_{j=1}^{N_i} h^{2i}\left(\left\|X^{i,j}\Lambda^s u\right\|_{L^2_\omega} + C(s)\|u\|_{H^s_\omega}\right). \end{aligned}$$

Using proposition 4.6, $\exists c_1(s)\forall\zeta, \exists C_1(s,\zeta), \forall h \leqslant 1, \forall u \in \mathcal{C}^\infty(M)$,

$$I_s \leqslant c_1(s)\zeta\|\Delta_h\Lambda^s u\|_{L^2_\omega} + C_1(s,\zeta)\|u\|_{L^2_\omega}. \qquad (4.20)$$

We now deal with J_s. Observe that for any $h \leqslant 1$,

$$\begin{aligned} J_s &= \sum_{i=0}^{r}\sum_{j=1}^{N_i} h^{2i}\left\|(X^{ij})^*[X^{i,j},\Lambda^s]u\right\|_{L^2_\omega} \\ &\leqslant \sum_{i=0}^{r}\sum_{j=1}^{N_i} h^{2i}\left\|[X^{i,j},\Lambda^s](X^{ij})^*u\right\|_{L^2_\omega} + \sum_{i=0}^{r}\sum_{j=1}^{N_i} h^{2i}\left\|[(X^{ij})^*,[X^{i,j},\Lambda^s]]u\right\|_{L^2_\omega} \\ &\leqslant \sum_{i=0}^{r}\sum_{j=1}^{N_i} h^{2i}\left\|[X^{i,j},\Lambda^s](X^{ij})^*u\right\|_{L^2_\omega} + c(s)\|u\|_{H^s_\omega} \\ &= K_s + c(s)\|u\|_{H^s_\omega}, \end{aligned} \qquad (4.21)$$

where the second inequality is due to the fact that $[(X^{ij})^*,[X^{i,j},\Lambda^s]]\Lambda^{-s}$ is of order 0.

Using the same steps as in (4.19), we have that

$$\begin{aligned} K_s &\leqslant c(s) \sum_{i=0}^{r} \sum_{j=1}^{N_i} h^{2i} \left(\left\| (X^{i,j})^* \Lambda^s u \right\|_{L^2_\omega} + C(s) \|u\|_{H^s_\omega} \right) \\ &\leqslant c(s) \sum_{i=0}^{r} \sum_{j=1}^{N_i} h^{2i} \left(\left\| X^{i,j} \Lambda^s u \right\|_{L^2_\omega} + \left\| c^{ij} \Lambda^s u \right\|_{L^2_\omega} + C(s) \|u\|_{H^s_\omega} \right), \end{aligned} \tag{4.22}$$

where $c^{ij} = \mathrm{div}_\omega(X^{ij})$ are smooth functions on M. Thus, we get that

$$K_s \leqslant c(s) \sum_{i=0}^{r} \sum_{j=1}^{N_i} h^{2i} \left(\left\| X^{i,j} \Lambda^s u \right\|_{L^2_\omega} + \sup_{m \in M} |c^{ij}(m)| C(s) \|u\|_{H^s_\omega} \right). \tag{4.23}$$

Then, again using proposition 4.6, we get that

$$\begin{gathered} \exists c_2(s), \forall \zeta, \exists C_2(s,\zeta), \forall h \leqslant 1, \forall u \in \mathcal{C}^\infty(M), \\ J_s \leqslant c_2(s) \zeta \|\Delta_h \Lambda^s u\|_{L^2_\omega} + C_2(s,\zeta) \|u\|_{L^2_\omega}. \end{gathered} \tag{4.24}$$

Plugging (4.20) and (4.24) in (4.18, we get that

$$\begin{gathered} \exists c_1(s), c_2(s), \forall \zeta, \exists C_1(s,\zeta), C_2(s,\zeta), \forall h \leqslant 1, \forall u \in \mathcal{C}^\infty(M), \\ \|\Delta_h \Lambda^s u\|_{L^2_\omega} \leqslant (c_1(s)+c_2(s))\zeta \|\Delta_h \Lambda^s u\|_{L^2_\omega} + (C_1(s,\zeta)+C_2(s,\zeta)) \|u\|_{L^2_\omega} + \|\Delta_h u\|_{H^s_\omega}. \end{gathered} \tag{4.25}$$

Choose ζ small enough so that $(c_1(s) + c_2(s))\zeta < 1$. Denote this ζ by ζ_0. Then, (4.25) implies that for any $h \leqslant 1$, for any $u \in \mathcal{C}^\infty(M)$,

$$\|\Delta_h \Lambda^s u\|_{L^2_\omega} \leqslant \frac{C_1(s,\zeta_0) + C_2(s,\zeta_0)}{1 - (c_1(s) + c_2(s))\zeta_0} \|u\|_{L^2_\omega} + \frac{1}{1 - (c_1(s) + c_2(s))\zeta_0} \|\Delta_h u\|_{H^s_\omega}. \tag{4.26}$$

Finally, plugging (4.26) into (4.15), we get that: $\exists c_3(s) > 0 \forall h \leqslant 1, \forall u \in \mathcal{C}^\infty(M)$,

$$\|u\|_{H^{\epsilon+s}_\omega} = \|\Lambda^s u\|_{H^\epsilon_\omega} \leqslant c_3(s) \left(\|\Delta_h u\|_{H^s_\omega} + \|u\|_{L^2_\omega} \right).$$

This concludes the proof. □

We give some corollaries of the subelliptic estimate.

Corollary 4.7.1. *There exist $\epsilon > 0$ such that for any $n \in \mathbb{N}$, there exists $C(n) > 0$, for any u smooth on M, and any $h \in [-1,1]$, we have,*

$$\|u\|_{H^{n\epsilon}_\omega} \leqslant C(n) \left(\|\Delta_h^n u\|_{L^2_\omega} + \left\| \Delta_h^{n-1} u \right\|_{L^2_\omega} + \ldots \|\Delta_h u\|_{L^2_\omega} + \|u\|_{L^2_\omega} \right). \tag{4.27}$$

Proof. Take $s = \epsilon$ in (4.11), we get

$$\|u\|_{H^{2\epsilon}_\omega} \leqslant C\left(\|\Delta_h u\|_{H^\epsilon_\omega} + \|u\|_{H^\epsilon_\omega}\right) \leqslant C\left(\|\Delta_h u\|_{H^\epsilon_\omega} + \|\Delta_h u\|_{L^2_\omega} + \|u\|_{L^2_\omega}\right).$$

Now, apply (4.11) for s=0 and $\Delta_h u$, to get that

$$\|\Delta_h u\|_{H^\epsilon_\omega} \leqslant C\left(\left\|\Delta_h^2 u\right\|_{L^2_\omega} + \|\Delta_h u\|_{L^2_\omega}\right).$$

We get (4.27) for $n = 2$. We go on recursively to get (4.27) for any n. □

Moreover, we recover the well-known characteristics of the sublaplacian:

Corollary 4.7.2. *The operator Δ_0 is hypoelliptic. Moreover, the operator Δ_0 with domain $\mathcal{C}^\infty(M)$ is essentially self-adjoint on $L^2(M)$.*

Proof. As Δ_0 is subelliptic, it is hypoelliptic.
Now, let $u \in L^2_\omega(M)$ such that $-\Delta_0 u = \lambda u$ in the distributional sense, for some $\lambda > 0$. As Δ_0 is subelliptic, $-\Delta_0 u = \lambda u \in L^2_\omega(M)$ implies that $u \in \mathcal{C}^\infty(M)$. For all $v \in \mathcal{C}^\infty(M)$, we have

$$\lambda\langle u, v\rangle_{L^2_\omega} = -\langle \Delta_0 u, v\rangle_{\mathscr{D}(M)'\mathscr{D}(M)} = -\langle u, \Delta_0 v\rangle_{L^2_\omega}. \tag{4.28}$$

Taking $v = u$ in (4.28), we get that

$$\lambda\|u\|_{L^2_\omega} + \langle u, \Delta_0 u\rangle_{L^2_\omega} = 0, \tag{4.29}$$

that implies, as both terms are non-negative, that $u = 0$. We conclude by applying theorem A.5.1. □

Denote by $\mathcal{D}(\Delta_0)$ the domain of the unique self-adjoint extension.

Corollary 4.7.3. *The operator $(\Delta_0, \mathcal{D}(\Delta_0))$ has a compact resolvent.*

Proof. Inequality (4.11) implies that $\mathcal{D}(\Delta_0) \subset H^\epsilon_\omega(M) \xhookrightarrow{compact} L^2(M)$ by Rellich–Kondrachov theorem. We conclude. □

In particular, its spectrum consists of an increasing sequence of positive eigenvalue of finite multiplicities converging to $+\infty$. Moreover, $L^2_\omega(M)$ has a basis of orthonormal eigenfunctions of Δ_0.

4.2.3 Second Case: Equiregular Case

In this subsection, we assume that M is a compact orientable equiregular subriemannian manifold. Moreover, we suppose that $d\omega = d\mathcal{P}_o$. We aim now to prove a version of the estimate (4.11) with $\tilde{\Delta}_h$ instead of Δ_h. The family of Laplace operators $\tilde{\Delta}_h$ are defined with respect to $L^2_h(M)$ and so, some of the above estimates may now depend on h. As the main difference between the two cases is the dependence of $L^2_h(M)$ on h, we will compare it with $L^2_\omega(M)$. It is evident that the two spaces are equivalent for any h, but uniformity is needed.

4.2.3.1 Uniform Equivalence Between L^2_ω And L^2_h

Recall from corollary 3.14.1 in chapter 3 that f_h is the function such that locally, $\mathrm{dvol}g^h = \sqrt{f_h(x)}h^\varsigma|dx|$ (the determinant of the matrix $h^{-2\varsigma}G_h$). Also, in the equiregular case, we have that

$$f_h(m) = f(m) + \sum_{k\geqslant 1} h^{2k} a_{\varsigma+k}(m),$$

for some smooth functions $a_{\varsigma+k}$. We recall that according to [38], the functions $a_{\varsigma+k}$ are non-negative.

Proposition 4.8. *The functions f_h and f satisfy the following assertions.*

1. *The convergence of f_h to f is uniform.*

2. *The function $(1/f_h)$ is uniformly bounded on M.*

Proof. 1. The function f_h is a polynomial in h with smooth coefficients and leading term f, so it is easy to see that

$$\lim_{h\to 0}\left(\sup_{m\in M}|f_h(m) - f(m)|\right) = 0,$$

which implies uniform convergence.

2. Using the expression of f_h and the fact that the coefficients $a_{\varsigma+k}$ is

non-negative for all k, we have that

$$\begin{aligned}\left|\frac{1}{f_h(m)} - \frac{1}{f(m)}\right| &= \left|\frac{1}{f(m) + \sum_{k\geqslant 1} h^{2k} a_{\varsigma+k}(m)} - \frac{1}{f(m)}\right| \\ &= \left|\frac{-\sum_{k\geqslant 1} h^{2k} a_{\varsigma+k}(m)}{f(m)(f(m) + \sum_{k\geqslant 1} h^{2k} a_{\varsigma+k}(m))}\right| \\ &\leqslant \frac{1}{|f(m)|}.\end{aligned}$$

Then the function $(1/f_h)$ is bounded above by $(2/f)$ which is a constant (in h) that blows up if and only if the subriemannian structure is non-equiangular.

□

As a corollary, we have

Corollary 4.8.1. *The exists $c_1, c_2 > 0$, $h_1 > 0$ such that for all $h \in [0, h_1]$,*

$$c_1 \|u\|_{L^2_\omega(M)} \leqslant \|u\|_{L^2_h(M)} \leqslant c_2 \|u\|_{L^2_\omega(M)}\,.$$

Proof. By proposition 4.8, f_h is uniformly convergent to f which implies that there exists $h_0 > 0$ such that for any $h \in [0, h_0]$, we have $|f_h| \leqslant |f| + 1$. So, for any $h \in [0, h_0]$, we compute

$$\begin{aligned}\|u\|^2_{L^2_\omega} &= \int_M |u|^2 d\omega \\ &= \int_U |u(x)|^2 \frac{\sqrt{f_h(x)}}{\sqrt{f_h(x)}} \phi(x)|dx| \\ &\leqslant \sup_{m\in M}\left(\phi(m)\sqrt{f(m)+1}\right)\int_U |u(x)|^2 \frac{|dx|}{\sqrt{f_h(x)}} = c\|u\|^2_{L^2_h}\,.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\|u\|^2_{L^2_h} &= \int_M |u|^2 h^{2\varsigma} \mathrm{dvol} g^h \\ &= \int_U |u(x)|^2 \frac{\phi(x)}{\sqrt{f_h(x)}\phi(x)} |dx| \\ &\leqslant \sup_{m\in M}\left(\frac{1}{\phi(m)}\sqrt{\frac{2}{f(m)}}\right)\int_U |u(x)|^2 \phi(x)|dx| = c\|u\|^2_{L^2_\omega}\,.\end{aligned}$$

Take $h_1 = \min\{1, h_0\}$. We conclude. □

Let us first prove some straightforward properties of $\tilde{\Delta}_h$.

4.2.3.2 Relation Between Δ_h And $\tilde{\Delta}_h$

We investigate now the relation between Δ_h and $\tilde{\Delta}_h$. The two operators Δ_h and $\tilde{\Delta}_h$ have the same 'principle part' but differ in the lower order terms. We will use the following proposition to prove that $\tilde{\Delta}_h$ converges to Δ_0 on one hand and to obtain the subelliptic estimate for $\tilde{\Delta}_h$ on the other hand.

Proposition 4.9. *There exist N functions $\{f_h^{ij}\}_{0\leqslant i\leqslant r, 1\leqslant j\leqslant N_i}$ smooth on M and analytic in h for any i, j such that*

$$\Delta_h = \tilde{\Delta}_h + \sum_{i=0}^{r}\sum_{j=1}^{N_i} h^{2i} f_h^{ij} X^{ij}. \tag{4.30}$$

Proof. By definition of Δ_h and $\tilde{\Delta}_h$, it is clear that

$$f_h^{ij} = \mathrm{div}_\omega(X^{ij}) - \mathrm{div}_h(X^{ij}).$$

Moreover, we know that for any $0 \leqslant i \leqslant r, 1 \leqslant j \leqslant N_i$, f_h^{ij} is smooth on M by the smoothness of X^{ij}. So we prove that f_h^{ij} is analytic in h for any i, j. Let α_h be such that $h^{2\varsigma}\mathrm{dvol}g^h = \alpha_h d\omega$. Locally,

$$\alpha_h = \sqrt{\frac{f}{f_h}}. \tag{4.31}$$

Then, using the second part of proposition 4.3, we have

$$f_h^{ij} = \mathrm{div}_\omega(X^{ij}) - \mathrm{div}_h(X^{ij}) = \alpha_h X^{ij}\left(\frac{1}{\alpha_h}\right). \tag{4.32}$$

Now, f_h is analytic in h and doesn't vanish for any $m \in M$ and for any h, then α_h and $\frac{1}{\alpha_h}$ are analytic functions in h and thus f_h^{ij} is. □

Denote by $\mathcal{L}$ the space of linear bounded functions from $H^2_\omega(M)$ to $L^2_\omega(M)$;

$$\mathcal{L} = \mathcal{L}(H^2_\omega(M), L^2_\omega(M)).$$

Corollary 4.9.1. *The operator $\Delta_h - \tilde{\Delta}_h$ converges to 0 in operator norm, that is,*

$$\left\|\Delta_h - \tilde{\Delta}_h\right\|_{\mathcal{L}} \to 0 \qquad \text{as } h \to 0. \tag{4.33}$$

Proof. Since f_h converges to f as $h \to 0$, then using expression (4.31), $\alpha_h \to 1$ as $h \to 0$, which implies using the expression of f_h^{ij} in (4.32) that $f_h^{ij} \to 0$ as $h \to 0$. Therefore, using (4.30), for any $h > 0$, we have

$$\left\|\Delta_h - \tilde{\Delta}_h\right\|_{\mathcal{L}} \leqslant \sum_{i=0}^{r}\sum_{j=1}^{N_i} h^{2i}\left\|f_h^{ij}X^{ij}\right\|_{\mathcal{L}} \leqslant \sum_{i=0}^{r}\sum_{j=1}^{N_i} h^{2i}\left(\max_{m\in M}|f_h^{ij}(m)|\right)\left\|X^{ij}\right\|_{\mathcal{L}} \to 0.$$

□

Now we prove a version of the uniform subelliptic estimate for $\tilde{\Delta}_h$.

4.2.3.3 Subelliptic Estimate

Recall that h_0 is such that for any $|h| \leqslant h_0$, $L^2_\omega(M)$ and $L^2_h(M)$ are uniformly equivalent. Recall that $h_1 = \min\{h_0, 1\}$.

First, we have the validity of proposition 4.4 with $\tilde{\Delta}_h$:

Proposition 4.10. *The following holds true:* $\exists C > 0, \forall 0 \leqslant i \leqslant r, \forall h \in [-h_1, h_1], \forall u \in \mathcal{C}^\infty(M)$,

$$\left\|X^{0,j}u\right\|_{L^2_\omega} \leqslant C\left(\left\|\tilde{\Delta}_h u\right\|^2_{L^2_\omega} + \|u\|^2_{L^2_\omega}\right).$$

Proof. We compute

$$\begin{aligned}\left\|X^{0,j}u\right\|^2_{L^2_\omega} &\leqslant c\left\|X^{0,j}u\right\|^2_{L^2_h} \leqslant c\sum_{i=0}^{r}\sum_{j=1}^{N_i} h^{2i}\left\|X^{ij}u\right\|^2_{L^2_h} \\ &\leqslant c\langle\tilde{\Delta}_h u, u\rangle_{L^2_h(M)} \leqslant \frac{c}{2}\left(\left\|\tilde{\Delta}_h u\right\|^2_{L^2_h} + \|u\|^2_{L^2_h}\right) \\ &\leqslant C\left(\left\|\tilde{\Delta}_h u\right\|^2_{L^2_\omega} + \|u\|^2_{L^2_\omega}\right).\end{aligned}$$

Where the constants are due to the uniform equivalence between the Sobolev spaces. □

We also have a version of proposition (4.6).

Proposition 4.11. *The following holds true:* $\exists C > 0, \forall 0 \leqslant i \leqslant r, \forall 1 \leqslant j \leqslant N_i, \forall \zeta^2 > 0, \forall |h| \leqslant h_1, \forall u \in \mathcal{C}^\infty(M)$,

$$\left\| h^i X^{ij} u \right\|_{L^2_\omega} \leqslant C \left(\frac{\zeta}{\sqrt{2}} \left\| \tilde{\Delta}_h u \right\|_{L^2_\omega} + \frac{1}{\sqrt{2}\zeta} \|u\|_{L^2_\omega} \right). \tag{4.34}$$

Proof. We compute

$$\begin{aligned} \left\| h^i X^{ij} u \right\|^2_{L^2_\omega} &\leqslant c \sum_{i=0}^{r} \sum_{j=1}^{N_i} \left\| h^i X^{ij} u \right\|^2_{L^2_h} = c \sum_{i=0}^{r} \sum_{j=1}^{N_i} \langle h^{2i} (X^{ij})^{*_h} X^{ij} u, u \rangle_{L^2_h} \\ &= c \langle \tilde{\Delta}_h u, u \rangle_{L^2 - h} \leqslant c \left\| \tilde{\Delta}_h u \right\|_{L^2_h} \|u\|_{L^2_h} \leqslant C \left\| \tilde{\Delta}_h u \right\|_{L^2_\omega} \|u\|_{L^2_\omega} \\ &\leqslant \frac{C\zeta^2}{2} \left\| \tilde{\Delta}_h u \right\|^2_{L^2_\omega} + \frac{C}{2\zeta^2} \|u\|^2_{L^2_\omega}, \end{aligned} \tag{4.35}$$

where the constants are due to the uniform convergence between $L^2_\omega(M)$ and $L^2_h(M)$ for $h \in [-h_1, h_1]$.
Now, since $\sqrt{a+b} \leqslant \sqrt{a} + \sqrt{b}$ for any $a, b > 0$, and since (4.35) is true for any $\zeta > 0$, we deduce (4.34). □

Theorem 4.12. *The following holds true:* $\exists \epsilon > 0, \forall s \in \mathbb{R}, \exists C(s) > 0, \forall h \in [-h_1, h_1], \forall u \in \mathcal{C}^\infty(M)$,

$$\|u\|_{H^{\epsilon+s}_\omega(M)} \leqslant C(s) \left(\left\| \tilde{\Delta}_h u \right\|_{H^s_\omega(M)} + \|u\|_{H^s_\omega(M)} \right). \tag{4.36}$$

Proof. Proposition 4.10 allows us to follow the same steps (a)-(g) in the proof of theorem 4.5 to prove (4.36) for $s = 0$. This is because that proof didn't depend on the explicit expression of Δ_h, and the adjoints and norms are always taken with respect to $d\omega$. So, $\exists \epsilon > 0, \exists c > 0, \forall h \in [-h_1, h_1], \forall u \in \mathcal{C}^\infty(M)$,

$$\|u\|_{H^\epsilon_\omega(M)} \leqslant c \left(\left\| \tilde{\Delta}_h u \right\|_{L^2_\omega(M)} + \|u\|_{L^2_\omega(M)} \right). \tag{4.37}$$

Now, apply (4.37) to $\Lambda^s u$ to get that

$$\|\Lambda^s u\|_{L^2_\omega} \leqslant c \left(\left\| \tilde{\Delta}_h \Lambda^s u \right\|_{L^2_\omega} + \|\Lambda^s u\|_{L^2_\omega} \right). \tag{4.38}$$

As we did before, we now investigate $\left\|\tilde{\Delta}_h \Lambda^s u\right\|_{L^2_\omega}$. We have

$$\begin{aligned}\left\|\tilde{\Delta}_h \Lambda^s u\right\|_{L^2_\omega} &\leqslant \left\|[\tilde{\Delta}_h, \Lambda^s]u\right\|_{L^2_\omega} + \left\|\tilde{\Delta}_h u\right\|_{H^s_\omega} \\ &\leqslant \left\|[\tilde{\Delta}_h - \Delta_h, \Lambda^s]u\right\|_{L^2_\omega} + \|[\Delta_h, \Lambda^s]u\|_{L^2_\omega} + \left\|\tilde{\Delta}_h u\right\|_{H^s_\omega} \\ &= \tilde{I}_s + \tilde{J}_s + \left\|\tilde{\Delta}_h u\right\|_{H^s_\omega}.\end{aligned} \tag{4.39}$$

We deal with $\tilde{I}_s$ and $\tilde{J}_s$ separately. Consider first $\tilde{I}_s$. Using proposition 4.9, we have

$$\left\|[\tilde{\Delta}_h - \Delta_h, \Lambda^s]u\right\|_{L^2_\omega} \leqslant \sum_{i=0}^{r}\sum_{j=1}^{N_i}\left\|[f_h^{ij}X^{ij}, \Lambda^s]\Lambda^{-s}\Lambda^s u\right\|_{L^2_\omega},$$

where f_h^{ij} is a smooth function that is analytic in h by proposition 4.9. The operator

$$[f_h^{ij}X^{ij}, \Lambda^s]\Lambda^{-s}$$

is a 0 order operator, whose symbol depends linearly on f_h^{ij}. Moreover, its norm in $\mathcal{L}(L^2_\omega)$ depends only on a finite number of derivatives of the symbol (see [84, Theorem 4.23]). Then by the smoothness and the analyticity of f_h^{ij}, its norm is uniformly bounded with respect to h, and we get that

$$\tilde{I}_s = \left\|[\tilde{\Delta}_h - \Delta_h, \Lambda^s]u\right\|_{L^2_\omega} \leqslant c(s)\|u\|_{H^s_\omega}. \tag{4.40}$$

Now, consider $\tilde{J}_s$. We know from the proof of theorem 4.7, that $\tilde{J}_s \leqslant I_s + J_s$, where we recall that

$$I_s = \sum_{i=0}^{r}\sum_{j=1}^{N_i} h^{2i}\left\|[(X^{ij})^*, \Lambda^s]X^{i,j}u\right\|_{L^2_\omega},$$

and that

$$J_s = \sum_{i=0}^{r}\sum_{j=1}^{N_i} h^{2i}\left\|(X^{ij})^*[X^{i,j}, \Lambda^s]u\right\|_{L^2_\omega}.$$

We deal with I_s first. In (4.19), we proved that

$$I_s \leqslant c(s)\sum_{i=0}^{r}\sum_{j=1}^{N_i} h^{2i}\left(\left\|X^{i,j}\Lambda^s u\right\|_{L^2_\omega} + C(s)\|u\|_{H^s_\omega}\right).$$

Using proposition 4.11, $\exists c_1(s)\forall\zeta, \exists C_1(s,\zeta), \forall|h| \leqslant h_1, \forall u \in \mathcal{C}^\infty(M)$,

$$I_s \leqslant c_1(s)\zeta\left\|\tilde{\Delta}_h\Lambda^s u\right\|_{L^2_\omega} + C_1(s,\zeta)\|u\|_{L^2_\omega}. \tag{4.41}$$

We deal now with J_s. In (4.21), we proved that $J_s \leqslant K_s + c(s)\|u\|_{H^s_\omega}$, where

$$K_s = \sum_{i=0}^{r}\sum_{j=1}^{N_i} h^{2i}\left\|[X^{i,j},\Lambda^s](X^{ij})^* u\right\|_{L^2_\omega}.$$

We then proved in (4.23) that

$$K_s \leqslant c(s)\sum_{i=0}^{r}\sum_{j=1}^{N_i} h^{2i}\left(\left\|X^{i,j}\Lambda^s u\right\|_{L^2_\omega} + \sup_{m\in M}|c^{ij}(M)|C(s)\|u\|_{H^s_\omega}\right).$$

Therefore, again by proposition 4.11, $\exists c_2(s), \forall\zeta, \exists C_2(s,\zeta), \forall|h| \leqslant h_1, \forall u \in \mathcal{C}^\infty(M)$,

$$J_s \leqslant c_2(s)\zeta\left\|\tilde{\Delta}_h\Lambda^s u\right\|_{L^2_\omega} + C_2(s,\zeta)\|u\|_{L^2_\omega}. \tag{4.42}$$

Then, by (4.41) and (4.42), we conclude that $\exists c_1(s), c_2(s), \forall\zeta, \exists C_1(s,\zeta), C_2(s,\zeta), \forall|h| \leqslant h_1, \forall u \in \mathcal{C}^\infty(M)$,

$$\tilde{J}_s \leqslant (c_1(s)+c_2(s))\zeta\left\|\tilde{\Delta}_h\Lambda^s u\right\|_{L^2_\omega} + (C_1(s,\zeta)+C_2(s,\zeta))\|u\|_{L^2_\omega}. \tag{4.43}$$

Now, plug (4.40) and (4.43) in (4.39), we get that $\exists c_3(s), \forall\zeta, \exists C_3(s,\zeta), \forall|h| \leqslant h_1, \forall u \in \mathcal{C}^\infty(M)$,

$$\left\|\tilde{\Delta}_h\Lambda^s u\right\|_{L^2_\omega} \leqslant c_3(s)\zeta\left\|\tilde{\Delta}_h\Lambda^s u\right\|_{L^2_\omega} + C_3(s,\zeta)\|u\|_{L^2_\omega} + \left\|\tilde{\Delta}_h u\right\|_{H^s_\omega}. \tag{4.44}$$

Choose ζ small enough so that $c_3(s)\zeta < 1$, and denote it ζ_1, we get that

$$\left\|\tilde{\Delta}_h\Lambda^s u\right\|_{L^2_\omega} \leqslant \frac{C_3(s,\zeta_1)}{1-c_3(s)\zeta_1} + \frac{1}{1-c_3(s)\zeta_1}\left\|\tilde{\Delta}_h u\right\|_{H^s_\omega}. \tag{4.45}$$

Finally, we conclude by plugging (4.45) in (4.38). □

As a consequence, we get

Corollary 4.12.1. *The following holds true:* $\exists\epsilon > 0, \forall n \in \mathbb{N}, \exists C(n) > 0, \forall u \in \mathcal{C}^\infty(M), \forall h \in [-h_1, h_1]$, *we have,*

$$\|u\|_{H_1^{n\epsilon}} \leqslant C(n)\left(\left\|\tilde{\Delta}_h^n u\right\|_{L^2_\omega} + \left\|\tilde{\Delta}_h^{n-1} u\right\|_{L^2_\omega} + ...\left\|\tilde{\Delta}_h u\right\|_{L^2_\omega} + \|u\|_{L^2_\omega}\right). \tag{4.46}$$

4.3 Convergence Of The Spectrum

In this section, we answer the question we have been seeking. We start with a fixed volume form case.

4.3.1 First Case: Fixed Volume Form

We almost have all the tools we need to proceed with the proof. We need one more observation.

Proposition 4.13. *The operator Δ_h converges to Δ_0 in $\mathcal{L}$.*

Proof. For any $u \in H^2_\omega(M)$, we compute

$$\|(\Delta_h - \Delta_0)u\|_{L^2_\omega(M)} = \left\| \sum_{i=1}^{r} \sum_{j=1}^{N_i} h^{2i} (X^{ij})^* X^{ij} u \right\|_{L^2_\omega(M)} \leqslant \sum_{i=1}^{r} \sum_{j=1}^{N_i} h^{2i} \left\| (X^{ij})^* X^{ij} u \right\|_{L^2_\omega(M)}.$$

Since $(X^{ij})^* X^{ij}$ is of order 2 and so using theorem A.12 we get that

$$\|(\Delta_h - \Delta_0)u\|_{L^2_\omega(M)} \leqslant c \sum_{i=1}^{r} \sum_{j=1}^{N_i} h^{2i} \|u\|_{H^2_\omega(M)} \leqslant c \left(\sum_{i=1}^{r} \sum_{j=1}^{N_i} h^{2i} \right) \|u\|_{H^2_\omega(M)}.$$

Therefore, we get that

$$\|(\Delta_h - \Delta_0)\|_{\mathcal{L}(H^2_\omega(M), L^2(M))} \leqslant c \sum_{i=1}^{r} \sum_{j=1}^{N_i} h^{2i} \to 0. \tag{4.47}$$

□

Now, we have the family of Riemannian operators that converge in the operator norm to Δ_0. We need one final lemma before we state and prove our main theorems.

Lemma 4.14. *Let $(u_n)_{n\geqslant 0}$ be a bounded sequence in $H^l_\omega(M)$ for all l that converges strongly in $L^2_\omega(M)$ to some u. Then, the sequence $(u_n)_{n\geqslant 0}$ converges strongly to u in $H^l_\omega(M)$ for any l.*

Proof. Fix some arbitrary l. Let $(u_{n_k})_{k\geqslant 0}$ be an arbitrary subsequence of $(u_n)_{n\geqslant 0}$. We prove that $(u_{n_k})_{k\geqslant 0}$ has a subsequence that converges to u in $H^l_\omega(M)$.
Fix $l' > l$. The sequence $(u_n)_{n\geqslant 0}$ is a bounded sequence in $H^{l'}_\omega(M)$ then $(u_{n_k})_{k\geqslant 0}$ is. So, $(u_{n_k})_{k\geqslant 0}$ has a subsequence that converges weakly in $H^{l'}_\omega(M)$ to some v. By compact embedding of $H^{l'}_\omega(M)$ in $H^{l'}_\omega(M)$ as $l < l'$, this subsequence converges strongly in $H^l_\omega(M)$ and $L^2_\omega(M)$ to v. By the convergence of the original sequence to u in $L^2_\omega(M)$, and by the uniqueness of limit in this space, we get that $u = v$.
Thus, for any l, every subsequence of $(u_n)_{n\geqslant 0}$ has a subsequence that converges to u strongly in $H^l_\omega(M)$. This implies that $(u_n)_{n\geqslant 0}$ converges to u strongly in $H^l_\omega(M)$ (one can show this by contradiction or by the fact that this implies that every convergent subsequence has the same limit).
Finally, as l is arbitrary, we conclude. $\square$

Recall theorem 2.8.

Theorem 4.15. *Let $(h_n)_{n\geqslant 0}$ be a sequence that goes to 0 and $(u_n)_{n\geqslant 0}$ be a sequence of normalized eigenfunctions of Δ_{h_n}. Let $(\mu_n)_{n\geqslant 0}$ be the associated sequence of eigenvalues. Assume that the sequence $(\mu_n)_{n\geqslant 0}$ is bounded. Then, the following assertions hold true.*

1. *There exist a subsequence $(\mu_{n_k})_{k\geqslant 0}$ that converges to an eigenvalue of Δ_0, say λ.*
2. *Up to extracting a subsequence, $(u_{n_k})_{k\geqslant 0}$ (that corresponds to $(\mu_{n_k})_{k\geqslant 0}$) converges to v_0 in $H^l_\omega(M)$ for any l, and v_0 is an eigenfunction of Δ_0 associated to λ.*

Proof. 1. Since the sequence $(\mu_n)_{n\geqslant 0}$ is bounded, then by Bolzano–Weierstrass theorem, it has a subsequence $(\mu_{n_k})_{k\geqslant 0}$ that converges to some λ.
Now, $(u_{n_k})_{k\geqslant 0}$ is a sequence of smooth functions as they are eigenfunctions of elliptic Riemannian Laplace operators. If we apply the uniform estimate (4.27) to u_{n_k} we get that for any l,

$$\left\|u_{n_k}\right\|_{H^l_\omega} \leqslant C\left\|(\Delta_h+1)^{(r+1)l}u_{n_k}\right\|_{L^2_\omega} = (1+\mu_{n_k}^{(r+1)l})\left\|u_{n_k}\right\|_{L^2_\omega} = (1+\mu_{n_k}^{(r+1)l}). \tag{4.48}$$

Again, as $(\mu_n)_{n\geqslant 0}$ is bounded, it implies that the sequence $(u_{n_k})_{k\geqslant 0}$ is bounded in $H^l_\omega(M)$ for any l. In particular in $H^2_\omega(M)$

Now we compute

$$\begin{aligned}\left\|(\Delta_0-\lambda)u_{n_k}\right\|_{L^2_\omega(M)} &= \left\|(\Delta_0-\Delta_{h_n})u_{n_k}\right\|_{L^2_\omega(M)} + \left\|(\Delta_{h_n}-\lambda)u_{n_k}\right\|_{L^2_\omega(M)} \\ &\leqslant \|\Delta_0-\Delta_{h_n}\|_{\mathcal{L}}\left\|u_{n_k}\right\|_{H^2_\omega(M)} + |\mu_{n_k}-\lambda|.\end{aligned} \tag{4.49}$$

Thus, we get that

$$\lim_{k\to\infty}\left\|(\Delta_0-\lambda)u_{n_k}\right\|_{L^2_\omega(M)} = 0.$$

Therefore, by theorem 2.8, we get that λ is an eigenvalue of Δ_0.

2. We prove that $(u_{n_k})_{n\geqslant 0}$ has a subsequence that converge to some v_0 in any Sobolev space $H^l_\omega(M)$, and then prove that this v_0 is an eigenfunction corresponding to λ.
   The sequence $(u_{n_k})_{k\geqslant 1}$ is bounded by (4.48) in $H^1_\omega(M)$, then, it has a subsequence, say $(u_{n_{k_j}})_{j\geqslant 1}$, that converges weakly in $H^1_\omega(M)$ to some v_0. As $H^1_\omega(M)$ is compactly embedded in $L^2_\omega(M)$, it implies that $(u_{n_{k_j}})_{j\geqslant 1}$ converges strongly to v_0 in $L^2_\omega(M)$. Moreover, by (4.48), $(u_{n_{k_j}})_{j\geqslant 1}$ is bounded in any Sobolev space $H^l_\omega(M)$. Thus, by lemma 4.14, $(u_{n_{k_j}})_{j\geqslant 1}$ is convergent to v_0 in any $H^l_\omega(M)$.
   Now, for all $\varphi\in\mathcal{C}^\infty(M)$, we compute

$$\begin{aligned}|\langle(\Delta_0-\lambda)v_0,\varphi\rangle_{L^2_\omega}| \leqslant{}& \|(\Delta_0-\Delta_{h_n})v_0\|_{L^2_\omega}\|\varphi\|_{L^2_\omega} + \left\|v_0-u_{n_{k_j}}\right\|_{L^2_\omega}\|\Delta_{h_n}\varphi\|_{L^2_\omega} \\ &+ \left[\mu_{n_{k_j}}\langle u_{n_{k_j}},\varphi\rangle_{L^2_\omega} - \lambda\langle v_0,\varphi\rangle_{L^2_\omega}\right].\end{aligned} \tag{4.50}$$

   The right-hand side converges to 0 as $k\to\infty$. We conclude that for any $\varphi\in\mathcal{C}^\infty(M)$,

$$\langle\Delta_0 v_0,\varphi\rangle_{L^2_\omega} = \lambda\langle v_0,\varphi\rangle_{L^2_\omega}.$$

   This implies, as Δ_0 is self-adjoint, that v_0 is an eigenfunction of Δ_0 with corresponding eigenvalue λ.

□

We prove now, the convergence of the ordered spectrum.

Theorem 4.16. *Denote by $(\lambda_k)_{k\geqslant 0}$ and $(\lambda_k(h))_{k\geqslant 0}$ the ordered spectrum of Δ_0 and Δ_h respectively, counted with multiplicities. Then, for any $k \geqslant 0$ fixed, we have*

$$\lim_{h\to 0} \lambda_k(h) = \lambda_k. \tag{4.51}$$

Proof. Let $(h_n)_{n\geqslant 0}$ be an arbitrary sequence that converges to 0 as $n \to \infty$. Fix $k \geqslant 0$. For n large enough, the sequence $(\lambda_k(h_n))_{n\geqslant 0}$ is bounded above by $\lambda_k(1)$. Indeed, denote by V_1 the vector space spanned by k eigenfunctions of Δ_1 corresponding to the first k eigenvalues; $V_1 = \operatorname{span}\{u_1(1), ..., u_k(1)\}$. For any $u \in V_1$ and any $h \in [0,1]$, we have

$$\langle \Delta_h u, u\rangle_{L^2_\omega} \leqslant \langle \Delta_1 u, u\rangle_{L^2_\omega} \leqslant \lambda_k(1) \sum_{i=1}^{k} |\langle u, u_i(1)\rangle_{L^2_\omega}|^2 \leqslant \lambda_k(1)\|u\|_{L^2_\omega}.$$

Then, we deduce the bound using min-max theorem.
Now, by theorem 4.15, the sequence $(\lambda_k(h_n))_{n\geqslant 0}$ has a subsequence, say $(\lambda_k(h_{n_j}))_{j\geqslant 0}$, that converges to an eigenvalue λ of Δ_0. Remains to prove that $\lambda = \lambda_k$.
We first observe that

$$\lambda_k(h_{n_j}) \leqslant \lambda_k + o(1) \qquad \text{as } j \to \infty. \tag{4.52}$$

Indeed, let $\{u_1, ..., u_k\}$ be an orthonormal set of eigenfunctions that corresponds to the first k eigenvalues of Δ_0. Let $V_0 = \operatorname{span}\{u_1, ..., u_k\}$. For any $v \in V_0$, we compute

$$\begin{aligned}
\langle \Delta_{h_{n_j}} v, v\rangle_{L^2_\omega} &= \langle \Delta_0 v, v\rangle_{L^2_\omega} + \sum_{i=1}^{r}\sum_{j=1}^{N_i} h_{n_j}^{2i} \langle (X^{ij})^* X^{ij} v, v\rangle_{L^2_\omega} \\
&\leqslant \sum_{i=1}^{k} \lambda_i |\langle v, u_i\rangle_{L^2_\omega}|^2 + c\sum_{i=1}^{r}\sum_{j=1}^{N_i} h_{n_j}^{2i} \|v\|_{L^2_\omega} \|v\|_{H^2_\omega} \\
&\leqslant \lambda_k \|v\|^2_{L^2_\omega} + o(1)\|v\|_{L^2_\omega}\|v\|_{H^2_\omega} \qquad \text{as } j \to 0.
\end{aligned}$$

Taking maximum over all normal vectors $v \in V_0$ (satisfying $\|v\|_{L^2_\omega} = 1$), then minimum over all subspaces of dimension k, we get (4.52) by min-max theorem.
Inequality (4.52) implies that

$$\lim_{j\to\infty} \lambda_k(h_{n_j}) \in \{\lambda_1, ..., \lambda_k\}.$$

Now, let $\{u_k(h_{n_j})\}_{k\geqslant 1}$ be an orthonormal sequence of eigenfunctions corresponding to $(\lambda_k(h_{n_j}))_{j\geqslant 0}$ for any j.
For $k=1$, we have $\lim_{j\to\infty}\lambda_1(h_{n_j})=\lambda_1$. For $k=2$, we have by orthogonality that

$$\begin{aligned}\langle\lim_{j\to\infty}u_2(h_{n_j}),u_1\rangle_{L^2_\omega} &= \langle\lim_{j\to\infty}u_2(h_{n_j}),\lim_{n\to\infty}u_1(h_{n_j})\rangle_{L^2_\omega}\\ &= \lim_{j\to\infty}\langle u_2(h_{n_j}),u_1(h)\rangle_{L^2_\omega}=0.\end{aligned}$$

Then the eigenfunction corresponding to $\lim_{j\to\infty}\lambda_2(h_{n_j})$ is orthogonal to u_1, and so

$$\lim_{j\to\infty}\lambda_2(h_{n_j})=\lambda_2.$$

Proceeding the same way, we conclude that the sequence $(\lambda_k(h_{n_j}))_{j\geqslant 0}$ converges to λ_k as $j\to 0$.
Finally the sequence h_n was arbitrary, so, any subsequence of $\lambda_k(h)$ has a subsequence (the $(\lambda_k(h_{n_j}))_{j\geqslant 0}$) that converges to λ_k. We deduce (4.51). $\square$

Of course, this all holds for any fixed smooth volume form on M. In particular, in an equiregular case, we have the convergence if we consider the volume form $(1/\sqrt{f(x)}dx)$.

Theorem 4.16 is about the convergence of ordered eigenvalues. This doesn't imply the convergence of eigenbranches in general. As a counterexample, one can consider

Example 4.1. *The family of Laplace operators Δ_t defined on a finite cylinder in $\mathbb{R}_x\times\mathbb{R}_\theta$ (x,t and θ variables represent the length of the cylinder, the radius of the cylinder and the angle parameter of the cylinder respectively). Direct calculations show that the eigenfuncions of Δ_t are given by*

$$sin(kx)e^{\dfrac{2i\pi l\theta}{t}} \qquad\qquad k\leqslant 1, l\in\mathbb{Z},$$

and that the corresponding eigenbranch is given by

$$\lambda(t)=k^2+\frac{4\pi^2l^2}{t^2}.$$

As expected, for small t, the n^{th} eigenvalue $\lambda_n(t)=n^2$, obtained for $l=0$ converges to the n^{th} eigenvalue of the Laplacian Δ_0 defined on a segment, *which is well-known to be n^2. However, for $l \neq 0$, the eigenbranch $\lambda(t)$ diverges as $t \to 0$.*

In this case, however, we have

Proposition 4.17. *Let $\lambda(h)$ be an eigenbranch of Δ_h. Then, $\lambda(h)$ converges to an eigenvalue of Δ_0.*

Proof. Denote by ϱ_h the positive quadratic form given by

$$\varrho_h(u) = \sum_{i=1}^{r}\sum_{j=1}^{N_i} h^{2i}\left\|X^{ij}u\right\|^2_{L^2_\omega},$$

and by $\mathcal{N}$ the usual norm operator ($\mathcal{N}(u) = \|u\|_{L^2_\omega}$). So, Δ_h is the Riemannian Laplace operator associated to $(\varrho_h, \mathcal{N})$. Thus, as $\mathcal{N}$ is independent of h, we have $\dot{\mathcal{N}} = 0$ and we get

$$\dot{\lambda}(h) = \dot{\varrho}_h(u(h)) + \lambda(h)\dot{\mathcal{N}}(u(h)) = \dot{\varrho}_h(u(h)) = \sum_{i=1}^{r}\sum_{j=1}^{N_i}(2i)h^{2i-1}\left\|X^{ij}u\right\|_{L^2_\omega} > 0.$$

Since $\lambda(h)$ is positive (bounded below) and increasing, and we are studying the behavior of $\lambda(h)$ near $h = 0$, it is convergent as $h \to 0$ (and thus bounded). We then follow theorem 4.15. □

Remark 4.18. *All the results above hold true for an arbitrary fixed (reference) volume form $d\omega$. In particular, everything works for Popp's volume.*

4.3.2 Second Case: Equiregular Case

We suppose now that the reference volume form is the volume form induced from the approximation scheme of chapter 3; that is $d\omega = d\mathcal{P}_o$. So now, the sublaplacian is defined with respect to $d\mathcal{P}_o$, which can be written locally in coordinates x as $d\mathcal{P}_o = (1/\sqrt{f(x)})dx$ (one of the main results of chapter 3). We prove in this section that the spectrum of $\tilde{\Delta}_h$ converges to the spectrum of $(\Delta_0, d\mathcal{P}_o)$.
First observe that $\tilde{\Delta}_h$ converge to Δ_0.

Proposition 4.19. *The operator $\tilde{\Delta}_h$ converges to Δ_0 in $\mathcal{L}$.*

Proof. We compute

$$\left\|\tilde{\Delta}_h - \Delta_0\right\|_{\mathcal{L}} \leqslant \left\|\tilde{\Delta}_h - \Delta_h\right\|_{\mathcal{L}} + \left\|\Delta_h - \Delta_0\right\|_{\mathcal{L}}.$$

The first term converges to zero by corollary 4.9.1, and the second one converges to zero by proposition 4.13. We conclude. □

Theorem 4.20. *Let $(h_n)_{n\geqslant 0}$ be a sequence that goes to 0 and $(u_n)_{n\geqslant 0}$ be a sequence of normalized eigenfunctions of $\tilde{\Delta}_{h_n}$. Let $(\mu_n)_{n\geqslant 0}$ be the associated sequence of eigenvalues. Assume that the sequence $(\mu_n)_{n\geqslant 0}$ is bounded. Then, the following assertions hold true.*

1. *There exist a subsequence $(\mu_{n_k})_{k\geqslant 0}$ that converges to an eigenvalue of Δ_0, say λ.*
2. *Up to extracting a subsequence, $(u_{n_k})_{k\geqslant 0}$ (that corresponds to $(\mu_{n_k})_{k\geqslant 0}$) converges to v_0 in $H^l_\omega(M)$ for any l, and v_0 is an eigenfunction of Δ_0 associated to λ.*

Proof. The proof exactly follows the proof of theorem 4.15.

1. The convergence of $\tilde{\Delta}_{h_n}$ to Δ_0 in $\mathcal{L}$, the convergence of $(\mu_{n_k})_{k\geqslant 0}$ to λ, and the boundedness of $(u_{n_k})_{k\geqslant 0}$ in $H^2_\omega(M)$ (which is implied by the subelliptic estimate (4.46)) imply that
$$\lim_{k\to\infty} \left\|(\Delta_0 - \lambda)u_{n_k}\right\|_{L^2_\omega(M)} = 0.$$
Therefore, by theorem 2.8, we get that λ is an eigenvalue of Δ_0.
2. The same subsequence argument as in part 2 of theorem 4.15 implies that there is an eigenfunction v_0 of λ such that $(u_{n_k})_{k\geqslant 0}$ has a subsequence that converges to v_0 in $H^l_\omega(M)$ for any l.

□

As we just saw, the proofs of theorem 4.20 and theorem 4.15 are the same, because the proof depends on the convergence of the operators and the boundedness of eigenfunctions which are valid in both cases. However, the proof of theorem 4.16 depends directly on the fact that we are dealing with fixed volume form. Due to the uniformity between the spaces $L^2_\omega(M)$ and $L^2_h(M)$, the convergence of the ordered spectrum works in this case and follows the same idea of the proof.

Theorem 4.21. *Denote by $(\lambda_k)_{k\geqslant 0}$ and $(\tilde{\lambda}_k(h))_{k\geqslant 0}$ the ordered spectrum of Δ_0 and $\tilde{\Delta}_h$ respectively, counted with multiplicities. Then, for any $k \geqslant 0$ fixed, we have*

$$\lim_{h\to 0} \tilde{\lambda}_k(h) = \lambda_k. \tag{4.53}$$

Proof. Let $(h_n)_{n\geqslant 0}$ be an arbitrary sequence that converges to 0 as $n \to \infty$. Fix $k \geqslant 0$. There exists a constant c such that for n large enough, the sequence $(\tilde{\lambda}_k(h_n))_{n\geqslant 0}$ is bounded above by $c\lambda_k(1)$. It is enough to observe that for any u,

$$\langle \tilde{\Delta}_{h_n} u, u\rangle_{L^2_h} = \sum_{i=0}^{r}\sum_{j=1}^{N_i} h^{2i}\left\|X^{ij}u\right\|^2_{L^2_h} \leqslant c\sum_{i=0}^{r}\sum_{j=1}^{N_i} h^{2i}\left\|X^{ij}u\right\|^2_{L^2_\omega} = c\langle \Delta_{h_n} u, u\rangle_{L^2_\omega}.$$

Min-max theorem implies that $\tilde{\lambda}_k(h_n) \leqslant \lambda_k(h_n) \leqslant c\lambda_k(1)$.
Now, by theorem 4.20, the sequence $(\tilde{\lambda}_k(h_n))_{n\geqslant 0}$ has a subsequence, say $(\tilde{\lambda}_k(h_{n_j}))_{j\geqslant 0}$, that converges to an eigenvalue λ of Δ_0. Remains to prove that $\lambda = \lambda_k$.
We first observe that

$$\tilde{\lambda}_k(h_{n_j}) \leqslant \lambda_k + o(1) \qquad \text{as } j \to \infty. \tag{4.54}$$

Indeed, let $\{u_1, ..., u_k\}$ be an orthonormal set of eigenfunctions that corresponds to the first k eigenvalues of Δ_0. Let $V_0 = \text{span}\{u_1, ..., u_k\}$. For any $v \in V_0$, we compute

$$\begin{aligned}
\langle \tilde{\Delta}_{h_{n_j}} u, u\rangle_{L^2_h} &= \langle \Delta_0 u, u\rangle_{L^2_h} + \sum_{i=1}^{r}\sum_{j=1}^{N_i} h_{n_j}^{2i}\langle (X^{ij})^{*_h} X^{ij}u, u\rangle_{L^2_h} \\
&= \langle \Delta_0 u, u\rangle_{L^2_h} + \sum_{i=1}^{r}\sum_{j=1}^{N_i} h_{n_j}^{2i}\left\|X^{ij}u\right\|^2_{L^2_h} \\
&\leqslant \langle \Delta_0 u, u\rangle_{L^2_h} + c\sum_{i=1}^{r}\sum_{j=1}^{N_i} h_{n_j}^{2i}\left\|X^{ij}u\right\|^2_{L^2_\omega} \\
&\leqslant \langle \Delta_0 u, u\rangle_{L^2_\omega} + \int_U u\Delta_0 u\left(\frac{1}{\sqrt{f_{h_{n_j}}(m)}} - \frac{1}{\sqrt{f(m)}}\right)dm \\
&+ c_1\|u\|^2_{H^1_\omega}\sum_{i=1}^{r}\sum_{j=1}^{N_i} h_{n_j}^{2i}.
\end{aligned} \tag{4.55}$$

By the (point-wise) convergence of f_h to f, we have the following point-wise convergence

$$\lim_{j\to\infty}\left[u(m)\Delta_0 u(m)\left(\frac{1}{\sqrt{f_{h_{n_j}}(m)}}-\frac{1}{\sqrt{f(m)}}\right)\right]=0.$$

Moreover, we have

$$\left|u\Delta_0 u\left(\frac{1}{\sqrt{f_{h_{n_j}}(m)}}-\frac{1}{\sqrt{f(m)}}\right)\right|\leqslant u\Delta_0 u\left(\frac{c}{\sqrt{f(m)}}\right)\in L^1_\omega(M).$$

This implies by the Lebesgue dominated convergence theorem that

$$\int_U u\Delta_0 u\left(\frac{1}{\sqrt{f_{h_{n_j}}(m)}}-\frac{1}{\sqrt{f(m)}}\right)dm=o(1).$$

Applying min-max to (4.55), we conclude (4.54).
Finally, the same argument on the orthogonality of the limiting eigenfunction implies that $\lambda=\lambda_k$. we conclude (4.53). □

Remark 4.22. *Theorem 4.21 also holds for Popp's volume as it is equal to our volume form up to a constant.*

Remark 4.23. *Proposition 4.17 is not necessarily true anymore as now, the norm operator depends on h and its derivative is not necessarily 0. However, whenever the eigenbranch is increasing, the result holds true.*

Appendix A

Spectral Theory

Here, we state some well-known theorems in spectral theory that played an important role in this manuscript. The reader can refer to [34][47][73][74] for more details.

A.1 General Spectral Theorems

In this section, $\mathcal{H}$ denotes a general Hilbert space. We start with the min-max theorem. This theorem is very standard in spectral theory and can be found in many places. For instance, see [63, Chapter 12] [77, Chapter 1][80, Chapter 4].

Theorem A.1 (Min-Max Theorem For Matrices)**.** *Let A be $d \times d$ hermitian matrix with eigenvalues $\lambda_1 \leqslant ... \leqslant \lambda_d$. Then for any $1 \leqslant k \leqslant d$,*

$$\lambda_k = \min_{dim(U)=k} \left(\max_{\substack{u \in U \\ \|u\|=1}} \langle Au, u \rangle \right).$$

Corollary A.1.1 (Continuity Of Spectrum)**.** *Let A, B be $d \times d$ hermitian matrices such that $\|A - B\| \leqslant \epsilon$ for some ϵ. Then, for any $1 \leqslant k \leqslant d$,*

$$|\lambda_k(A) - \lambda_k(B)| \leqslant \epsilon.$$

Proof. Let U be a subspace of $\mathbb{R}^d$ of dimension k. Then for $u \in U$ with $\|u\| = 1$, we have by min-max that

$$\langle Au, u \rangle = \langle Bu, u \rangle + \langle (A - B)u, u \rangle \leqslant \langle Bu, u \rangle + \epsilon \|u\|^2 .$$

Taking infimum over all normal $u \in U$ and then over all U of dimension k we get one direction. By symmetry, we get the inequality. $\square$

Theorem A.2 (Min-Max Theorem For Operators)**.** *Let T be a semibounded self adjoint operator with compact resolvent with domain $D(T) \subset \mathcal{H}$, and let $\{\lambda_m\}_{m\geqslant 1}$ be its increasing sequence of eigenvalues. Then,*

$$\lambda_m = \min_{\substack{dim F = m \\ F \subset D(T)}} \{\Lambda(F)\},$$

with

$$\Lambda(F) = \max_{\substack{u \in F \\ u \neq 0}} \left\{ \frac{\langle Tu, u\rangle}{\|u\|_{\mathcal{H}}^2} \right\}. \tag{A.1}$$

Definition A.3. *Let T with domain $D(T)$ be a closable operator in $\mathcal{H}$. A core of T is a subset D of $D(T)$ such that the closure of the restriction of T to D is T.*

Theorem A.4. *Let T be a semibounded self adjoint operator with compact resolvent with domain $D(T) \subset \mathcal{H}$, and let $\{\lambda_m\}_{m\geqslant 1}$ be its increasing sequence of eigenvalues. Let D be a core for T. Then*

$$\lambda_m = \min_{\substack{dim F = m \\ F \subset D}} \{\Lambda(F)\},$$

with $\Lambda(F)$ defined by (A.1).

Proof. Let

$$\tilde{\lambda}_m = \inf_{\substack{dim F = m \\ F \subset D(T)}} \max_{\substack{u \in F \\ u \neq 0}} \left\{ \frac{\langle Tu, u\rangle_{\mathcal{H}}}{\|u\|_{\mathcal{H}}^2} \right\}.$$

Required to prove $\lambda_m = \tilde{\lambda}_m$. Clearly, since $D \subset D(T)$, we have $\lambda_m \leqslant \tilde{\lambda}_m$.

Now, Let $U = \text{vect}\{u^1, ..., u^m\}$ be a set of orthonormal eigenvectors of T that corresponds to the first m eigenvalues of T. By definition A.3, there exist m sequences $\{(u_n^1)_{n\geqslant 1}, ..., (u_n^m)_{n\geqslant 1}\}$ such that $u_n^i \to u^i$ and $Tu_n^i \to Tu^i$, in $\mathcal{H}$ for all $i \in \{1, ..., m\}$. Let $U_n = \text{vect}\{u_n^1, ..., u_n^m\}$. For $u_n \in U_n$, we write,

$$u_n = \sum_{i=1}^m \alpha_i u_n^i = \sum_{i=1}^m \alpha_i u_n^i + u_\infty - u_\infty,$$

with $u_\infty = \sum_{i=1}^m \alpha_i u^i$. We have

$$\|u_\infty\|_{\mathcal{H}}^2 = \sum_{i=1}^m a_i. \tag{A.2}$$

Then, using the Cauchy Schwartz inequality after the triangle inequality, we get

$$\|u_n - u_\infty\|_{\mathcal{H}} = \left\| \sum_{i=1}^m \alpha_i (u_n^i - u^i) \right\|_{\mathcal{H}} \leqslant \|u_\infty\|_{\mathcal{H}} \, \epsilon_n,$$

where

$$\epsilon_n = \left(\sum_{i=1}^m \left\| u_n^i - u^i \right\|_{\mathcal{H}}^2 \right)^{1/2}.$$

The last inequality gives us

$$\|u_\infty\|_{\mathcal{H}} (1 - \epsilon_n) \leqslant \|u_n\|_{\mathcal{H}} \leqslant \|u_\infty\|_{\mathcal{H}} (1 + \epsilon_n). \tag{A.3}$$

Now, we compute

$$\langle T u_n, u_n \rangle_{\mathcal{H}} = \sum_{i,j=1}^m |\alpha_i \alpha_j| \langle T u_n^i, u_n^i \rangle_{\mathcal{H}} = \sum_{i,j=1}^m |\alpha_i \alpha_j| \delta_{i,j}^n,$$

with $\delta_{i,j}^n \to \lambda_i \delta_{i,j}$ as $n \to \infty$. So, we get,

$$|\langle T u_n, u_n \rangle_{\mathcal{H}} - \sum_{i=1}^m \lambda_i \alpha_i^2| \leqslant \sum_{i=1}^m |\alpha_i|^2 |\delta_{i,i}^n - \lambda_i| + \sum_{i \neq j}^m |\alpha_i \alpha_j \delta_{i,j}^n|.$$

The right-hand side converges to 0 at ∞, so, for every $\epsilon > 0$, there exist n_0 such that for all $n > n_0$, we have

$$|\langle T u_n, u_n \rangle_{\mathcal{H}} - \sum_{i=1}^m \lambda_i \alpha_i^2| \leqslant \epsilon.$$

Hence, for all $n > n_0$, we have

$$\langle T u_n, u_n \rangle_{\mathcal{H}} \leqslant \sum_{i=1}^m \lambda_i \alpha^2 + \epsilon \leqslant \lambda_m \|u_\infty\|_{\mathcal{H}} + \epsilon \leqslant \frac{\lambda_m}{(1 - \epsilon_n)} \|u_n\|_{\mathcal{H}} + \epsilon.$$

Therefore, for all $n > n_0$, for every $u_n \in U_n$,

$$\frac{\langle Tu_n, u_n\rangle_{\mathcal{H}}}{\|u_n\|_{\mathcal{H}}^2} \leqslant \frac{\lambda_m}{(1-\epsilon_n)^2} + \frac{\epsilon}{\|u_n\|_{\mathcal{H}}^2}.$$

The right-hand side converges to $\lambda_m + \dfrac{\epsilon}{\|u_\infty\|_{\mathcal{H}}}$. Thus, for every $\tilde{\epsilon} > 0$, there exist $n_1 > 0$, such that for all $n \geqslant \max\{n_0, n_1\}$,

$$\frac{\langle Tu_n, u_n\rangle_{\mathcal{H}}}{\|u_n\|_{\mathcal{H}}^2} \leqslant \lambda_m + \frac{\epsilon}{\|u_\infty\|_{\mathcal{H}}} + \tilde{\epsilon},$$

for all $u_n \in U_n$. Therefore, there exist $U_n \subset D$ of dimension m such that

$$\Lambda(U_n) \leqslant \lambda_m + \frac{\epsilon}{\|u_\infty\|_{\mathcal{H}}} + \tilde{\epsilon},$$

which implies that

$$\tilde{\lambda}_m \leqslant \lambda_m + \frac{\epsilon}{\|u_\infty\|_{\mathcal{H}}} + \tilde{\epsilon}$$

for any arbitrary $\epsilon, \tilde{\epsilon}$, and thus the result follows. □

We give, in corollary A.5.1, a sufficient condition for an operator T to be essentially self-adjoint. It is a direct corollary of the following theorem, which is completely taken from [73, Theorem X.1].

Theorem A.5. *Let T be a closed symmetric negative operator on a Hilbert space $\mathcal{H}$. Then, T is self-adjoint if and only if*

$$\dim(\ker(\lambda I - T^*)) = 0,$$

on the upper and the lower half-plane.

Corollary A.5.1. *Let T be a closed symmetric operator on a Hilbert space $\mathcal{H}$. Then, T is self-adjoint if and only if there are no eigenvectors with positive eigenvalue in the domain of T^*.*

We state now, without giving the proof, the Rellich-Kondrachov theorem followed by the Kondrachov embedding theorem. Refer to [1, Chapter 6] (see also [79]).

Theorem A.6. *Let Ω be an open bounded Lipschits domain of R^n and set $p^* = \frac{np}{n-p}$. Then the Sobolev space $W^{1,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$ for every $1 \leqslant q < p^*$. In particular, $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$.*

Theorem A.7. *Let M be a compact manifold with C^1 boundary. Then if $k > l$ and $k - \frac{n}{p} > l - \frac{n}{q}$, then the Sobolev space $W^{k,p}(M)$ is compactly embedded in $W^{l,q}(M)$. In particular, $H^k(M)$ is compactly embedded in $H^l(M)$ for every $k > l$.*

A.2 The Schrödinger Operator

We handle now the operator $-\Delta + V$. Let $X \subset \mathbb{R}^n$ and let $\mathcal{H} = L^2(X)$. Denote by $\mathcal{C}_c^\infty(X)$ the space of smooth functions on X of compact support. Let V be a non-negative smooth[1] function, that converges to infinity at infinity. It is well-known that $-\Delta + V$ is a self-adjoint operator with compact resolvent on $\mathcal{H}$ (see [34][47] for instance).
We give a proof for the sake of completion.

Theorem A.8. *The operator $T = -\Delta + V$ with domain $\mathcal{C}_c^\infty(X)$ is essentially self-adjoint. Its unique self-adjoint extension is the operator T with domain*

$$D \subset H_V^1(X) := \{u \in H^1(X), V^{1/2}u \in \mathcal{H}\}.$$

Proof. We will first prove that $D \subset H_V^1(X)$. The space $H_V^1$is equipped with the following norm

$$\|u\|_{H_V^1}^2 = \|\nabla u\|_{\mathcal{H}}^2 + \left\|V^{1/2}u\right\|_{\mathcal{H}}^2.$$

For any $u \in \mathcal{C}_c^\infty(X)$, we have

$$\langle Tu, u \rangle_{\mathcal{H}} \geqslant \|\nabla u\|_{\mathcal{H}}^2 + \left\|V^{1/2}u\right\|_{\mathcal{H}}^2. \tag{A.4}$$

Now, let $u \in D$ and let $u_n \in C_c^\infty(X)$ such that $u_n \to u$ and $Tu_n \to Tu$. Then, (A.4) implies that u_n is Cauchy in $H^1(X)$ and $V^{1/2}u_n$ is Cauchy in $\mathcal{H}$. This implies that $u \in H_V^1(X)$.

[1] Continuity is enough for the results in this section to hold.

Now, let $f \in \big(\mathrm{Ran}(T+i)\big)^{\perp}$, where the $\perp$ is taken with respect to $\|.\|_{\mathcal{H}}$. Then, for all $u \in \mathcal{C}_c^{\infty}(X)$, $\langle (T+i)u, f\rangle_{\mathcal{H}} = 0$. We have

$$0 = \langle (T+i)u, f\rangle_{\mathcal{H}} = \langle (\Delta + V + i)\bar{f}, u\rangle_{\mathscr{D}'(X)\mathscr{D}(X)}.$$

So $(-\Delta + V + i)\bar{f} = 0$. Since $f \in L^2(X)$ and $\Delta f \in L^2_{loc}(X)$, it implies that $f \in H^2_{loc}(X)$ (let χ be a cutoff function. Compute $(\Delta+1)\chi f$ and see that it is an element of $H^{-1}(X)$. Then, $\chi f \in H^1(X)$ which gives that $f \in H^1_{loc}(X)$. This will imply that $(\Delta+1)\chi f \in L^2(M)$).

Now, pick $\xi \in \mathcal{C}_c^{\infty}(X)$ such that $0 \leqslant \xi \leqslant 1$, $\xi = 1$ on $B(0,1)$ and $supp(\xi) \subset B(0,2)$. Denote by $\xi_k(x) = \xi\left(\dfrac{x}{k}\right)$. For any $u \in \mathcal{C}_c^{\infty}(X)$ we have

$$\langle \nabla(\xi_k f), \nabla(\xi_k u)\rangle_{\mathcal{H}} + \int_X \xi_k^2 V u f = \langle f, T(\xi_k^2 u)\rangle_{\mathcal{H}} + \int_X |\nabla \xi_k|^2 u f + \sum_j \int_X (f\partial_j u - u\partial_j f)\xi_k \partial_j \xi_k.$$

This formula can be extended by density to $u \in H^2_{loc}(X)$. Taking $u = f$, we get

$$\|\nabla(\xi_k f)\|^2_{\mathcal{H}} + \int_X \xi_k^2 V f^2 = \langle f, T(\xi_k^2 f)\rangle_{\mathcal{H}} + \int_X |\nabla \xi_k|^2 f^2.$$

Since $\langle f, T(\xi_k^2 f)\rangle_{\mathcal{H}} = 0$, we get

$$\int_X |f\xi_k|^2 \leqslant \int_X |\nabla \xi_k|^2 |f|^2.$$

By the Lebesgue domination theorem, we get $f = 0$. Therefore $\big(\mathrm{Ran}(T + i)\big)^{\perp} = \{0\}$. This implies that $\mathrm{Ran}(T+i)$ is dense in $\mathcal{H}$. We conclude. $\square$

Lemma A.9. *If $H^1_V(X)$ is compactly embedded in $\mathcal{H}$, then (T, D) has a compact resolvent.*

Proof. Let $\zeta < 0$. Note that

$$(T-\zeta)^{-1} : \mathcal{H} \to D \subset H^1_V(X) \xhookrightarrow{compact} \mathcal{H}.$$

Let $u \in \mathcal{H}$ and let $f = (T-\zeta)^{-1}u$. We compute,

$$\|f\|_{H^1_V(X)} = \int_X \left(|\nabla f|^2 + (V-\zeta)|f|^2\right) dx \leqslant \langle ((T-\zeta)f, f)\rangle_{\mathcal{H}} = \langle u, f\rangle_{\mathcal{H}} \leqslant \|u\|_{\mathcal{H}} \|f\|_{\mathcal{H}}.$$

Then the map $(T-\zeta)^{-1} : \mathcal{H} \to D$ is continuous. We conclude. $\square$

Theorem A.10. *The operator (T, D) is of compact resolvent.*

Proof. Let $\chi \in \mathcal{C}_c^\infty(X)$ such that $0 \leqslant \chi \leqslant 1$, $\chi = 1$ near 0 and $supp(\chi) \subset B(0, 1)$. Define the following mappings

$$\begin{aligned} \mathcal{R}_n : H_V^1(X) &\to H^1(B(0,n)) \\ u &\mapsto \chi(./n)u(.) \end{aligned}$$

$$\begin{aligned} \mathcal{F}_n : H^1(B(0,n)) &\to \mathcal{H} \\ u &\mapsto u \end{aligned}$$

By theorem A.6, $\mathcal{F}_n$ is compact. Since $\mathcal{R}_n$ is continuous, then the map $\mathcal{J}_n := \mathcal{F}_n \circ \mathcal{R}_n$ is compact. Let

$$\begin{aligned} \mathcal{J} : H_V^1(X) &\to \mathcal{H} \\ u &\mapsto u \end{aligned}$$

We compute, for $u \in H_V^1(X)$,

$$\begin{aligned} \|(\mathcal{J} - \mathcal{J}_n)u\|_{\mathcal{H}} &= \|(1-\chi)u\|_{\mathcal{H}} \\ &= \int_X |(1-\chi)(x/n)|^2 |u(x)|^2 dx \\ &= \int_X |(1-\chi)(x/n)|^2 (V(x))^{-1} V(x) |u(x)|^2 dx \\ &\leqslant \left(\sup_{y \geqslant (n/2)} |(V(y))^{-1}| \right) \int_X V(x)|u(x)|^2 dx \\ &\leqslant \left(\sup_{|y| \geqslant (n/2)} |(V(y))^{-1}| \right) \|u\|^2_{H_V^1(X)} . \end{aligned}$$

Hence,

$$\|\mathcal{J} - \mathcal{J}_n\|_{\mathcal{L}(H_V^1(X),\mathcal{H})} \leqslant \sup_{|y| \geqslant (n/2)} |(V(y))^{-1}| \to 0.$$

Therefore, $\mathcal{J}$ is compact. and by theorem A.9, we conclude. $\square$

A.3 Pseudo-differential operators

Pseudo-differential operators are a type of operator that generalizes the concept of differential operators in calculus. The general theory of pseudo-differential operators is not necessary here though. We give some essential theorems without proofs, for these theorems are basic in the context of pseudo-differential operators and can be found in several places like [5][25] and the references within. We briefly recall that

Definition A.11. *A pseudo-differential operator is an operator which is defined by $u \mapsto Op(a)u$ as:*

$$Op(a)u(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\xi} a(x,\xi) \hat{u}(\xi) d\xi.$$

Here, a is a $\mathcal{C}^\infty$ symbol on $\mathbb{R}^{2d}$, which admits, as $\xi \to \infty$, an expansion

$$a(x,\xi) = \sum_{j \geqslant 0} a_{p-j}(x,\xi),$$

with $a_l(x,\lambda\xi) = \lambda^l a_l(x,\xi)$ for all $\lambda > 0$ and every $\xi \in \mathbb{R}^d \backslash \{0\}$. We call p the degree of the symbol (or of the pseudo-differential operator).

Theorem A.12. *Let a be a symbol of order $p \in \mathbb{R}$. The operator $Op(a)$ is bounded from $H^s(\mathbb{R}^d)$ to $H^{s-p}(\mathbb{R}^d)$ for all $d \geqslant 1$ and all $s \in \mathbb{R}$. In particular, pseudo-differential operators of order 0 are bounded in any Sobolev space.*

Theorem A.12 is true also for compact manifolds (see [69]).

Corollary A.12.1. *For M being a compact manifold, a pseudo-differential operator of non-positive order is bounded in $L^2(M)$.*

Theorem A.13. *Let P_1 and P_2 be two pseudo-differential operators with the same principle symbol (thus same order p). Then, $P_1 - P_2$ is of order $p-1$. In particular, if P_1, P_2 are two pseudo-differential operators of orders p_1 and p_2 respectively, then $[P_1, P_2]$ is a pseudo-differential operators of order $p_1 + p_2 - 1$.*

Appendix **B**

Analytic Perturbation Theory In Finite Dimension

We give Kato's proof for the analyticity of eigen-quantities in the finite-dimensional cases.

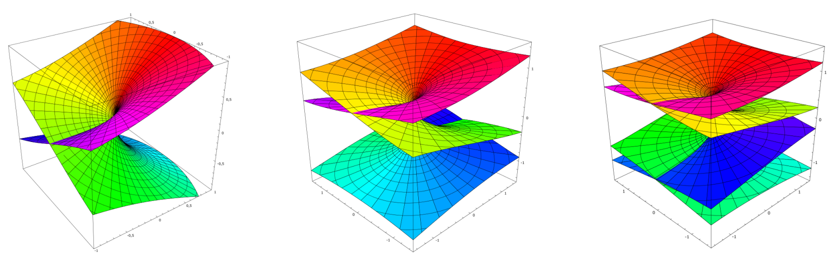
B.1 Introduction

Perturbation Theory operates under the premise that if we cannot ascertain precise solutions to a problem, we can derive an approximate solution by utilizing solutions to an approximate version of the problem that can be treated exactly. In chapter two of [54], Kato studied perturbations of linear operators in finite-dimensional spaces. He uses the theory of Knopp [56] to prove the analyticity of eigenvalues, but he didn't mention the details. We'll explain a little as this theory was the base for a lot of our work.

We just note that another proof can be found in the first section of chapter 3 in [77], however, we choose to explain the proof using algebraic equations (Knopp's theory).

Let us first give a brief review of Riemannian surfaces. A Riemann surface is a connected one-dimensional complex manifold that locally near every point looks like patches (sheets) of the complex plane and globally looks like several sheets glued together. Riemann surfaces are nowadays considered the natural setting for studying the global behavior of multi-valued functions such as the square root and other algebraic functions, or the logarithm. Important examples of Riemann surfaces are provided by analytic continuation are given in figure B.1.

Consider for instance the function $f(z) = z^{\frac{1}{2}}$ near the branch point 0 in figure B.1. The two horizontal axes represent the real and imaginary parts of z, while the vertical axis represents the real part of $\sqrt{z}$. The imaginary part of $\sqrt{z}$ is represented by the coloration of the points. Starting from the right on the red region, and moving anti-clockwise, we pass by the yellow region and reach after a while the place where the sheet is suddenly purple, and this line (between the change of colors) is the place where the two sheets were glued. If we pass down to the green region and continue moving anti-clockwise, we will reach again this line. Passing to the purple region, we get back to our initial position (see figure B.2 and figure B.3).



(a) $f(z) = z^{\frac{1}{2}}$ (b) $f(z) = z^{\frac{1}{3}}$ (c) $f(z) = z^{\frac{1}{4}}$

Figure B.1: Riemann Surface Of The Function $f(z)$

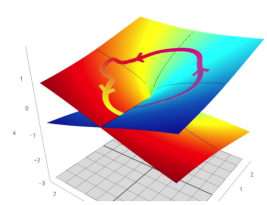


Figure B.2: The Journey On The Riemann Surface Of $z^{\frac{1}{2}}$

In general, for $w = \sqrt[p]{z}$, the corresponding $p-$sheet Riemann surface is mapped injectively to the 1-sheet representing w. So, let a be a branch point, and suppose that f is a regular function that goes into itself after $p - fold$ continuation (of the way described before). Then f can be written as a power series in $(z-a)^{\frac{1}{p}}$. Indeed, if we define the function $\tilde{f}$ such that

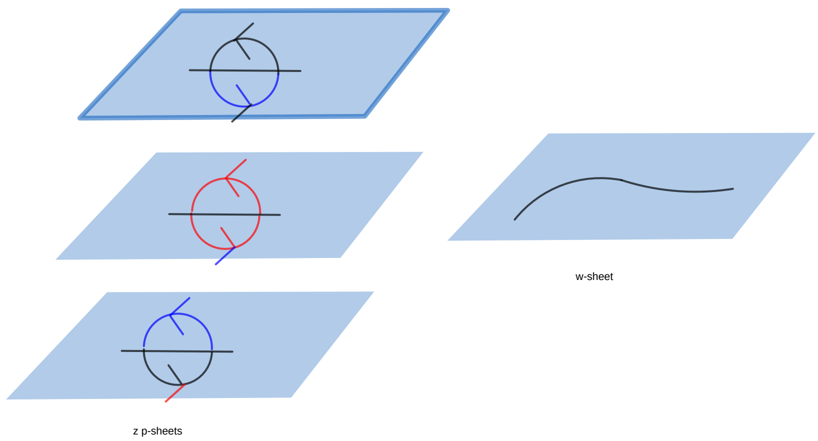


Figure B.3: The Behaviour Of $f(z)$ When It Hit The Intersection Line

$\tilde{f}(\tilde{z}) = f(z)$ where $\tilde{z}^p := z-a$, then $\tilde{f}$ is single-valued regular (hence analytic) function in a neighborhood of $\tilde{z} = 0$ and can be written in Laurent series as $\tilde{f}(\tilde{z}) = \sum_{-\infty}^{\infty} c_n \tilde{z}^n$.

Now, we introduce the setup. Let X be a finite-dimensional vector space[1] of dimension N. Suppose that $T(x)$ is an operator-valued function that is holomorphic in the complex plane. The operator $T = T(0)$ is called the unperturbed operator and $xT^{(1)} + x^2T^{(2)} + ...$ is the perturbation (as $T(x)$ can be written as $T(x) = T + xT^{(1)} + x^2T^{(2)} + ...$). The fundamental result in perturbation theory is the following:

Theorem B.1 (Kato 1995)**.** *The eigenvalues and the eigenfunctions of $T(x)$ are branches of analytic functions of x with only algebraic singularities (singularities exhibited by a radical function).*

[1]Whenever necessary, X will be considered as a normed space with a convenient norm.

B.2 Analyticity Of Eigenvalues

Let us first study the first part of the theorem. It is well-known that the eigenvalues of $T(x)$ are the roots of the characteristic equation:

$$G(x, \zeta) := \det(T(x) - \zeta) = 0, \tag{B.1}$$

which is an algebraic equation with coefficients that are analytic in x. We write G in the form

$$g_0(x) + g_1(x)\zeta + g_2(x)\zeta^2 + ... + g_N(x)\zeta^N,$$

with $g_i(x)$ analytic, depending only on x. Knopp in his book [56], in chapter 5 specifically, studied this algebraic equation (B.1) and proved the analyticity of the roots. We state a very important theorem of his and give the proof as in [56]. Denote by $D(x)$ the discriminant of $G(x, \zeta)$.

Theorem B.2. *Let (x_0, ζ_0) be a solution of (B.1) such that x_0 is a root of multiplicity β. For all K_ϵ disc of sufficiently small radius $\epsilon > 0$ described at ζ_0, there exist a disc K_δ with radius $\delta > 0$ described at x_0 such that for any $x \neq x_0$ in K_δ, the equation $G(x, \zeta) = 0$ has β distinct roots in K_ϵ.*

Proof. If we set $\zeta = (\zeta - \zeta_0) + \zeta_0$, we may write

$$G(x, \zeta) = \tilde{g}_0(z) + \tilde{g}_1(x)(\zeta - \zeta_0) + ... + \tilde{g}_N(x)(\zeta - \zeta_0)^N,$$

with $\tilde{g}_0(x_0) = \tilde{g}_1(x_0) = ... = \tilde{g}_\beta(x_0) = 0, \tilde{g}_\beta(x_0) \neq 0$. By inverse function theorem, we can describe a small disc $K_{\delta'}$ around the point x_0, such that $D(x)$ and $\tilde{g}_\beta(x)$ are not zero within $K_{\delta'}$ and on its boundary. We have

$$G(x, \zeta) = \tilde{g}_\beta(x)(\zeta - \zeta_0)^\beta[1 + A + B],$$

where

$$A = A(x, \zeta) = \frac{\tilde{g}_{\beta+1}}{\tilde{g}_\beta}(\zeta - \zeta_0) + ... + \frac{\tilde{g}_N}{\tilde{g}_\beta}(\zeta - \zeta_0)^{N-\beta},$$

and

$$B = B(x, \zeta) = \frac{\tilde{g}_{\beta-1}}{\tilde{g}_\beta}\frac{1}{\zeta - \zeta_0} + ... + \frac{\tilde{g}_0}{\tilde{g}_\beta}\frac{1}{(\zeta - \zeta_0)^\beta}.$$

Let M be an upper bound for all $|\tilde{g}_i(x)|$ in $K_{\delta'}$, and set $c = \inf_{x \in K_{\delta'}}\{|\tilde{g}_\beta(x)|\}$. Now, take

$$\epsilon < \frac{c}{4M}$$

Then, for all $x \in K_{\delta'}$ and all $\zeta \in K_\epsilon$, $|A| < \dfrac{1}{2}$. Now, choose δ small enough that in the interior of K_δ, we have $\tilde{g}_i(x) < \mu$ for all $0 \leqslant i \leqslant \beta - 1$, with

$$\mu = \frac{c}{2(\frac{1}{\epsilon} + \frac{1}{\epsilon^2} + ... + \frac{1}{\epsilon^\beta})}.$$

Then we get that $|B| < \frac{1}{2}$ for all x and ζ satisfying $|x - x_0| < \delta$ and $|\zeta - \zeta_0| < \epsilon$. Finally, we let x_1 be arbitrary in K_δ. For all ζ on the boundnary of K_ϵ, we have

$$|\tilde{g}_\beta(x_1)(\zeta - \zeta_0)^\beta| > |\tilde{g}_\beta(x_1)(\zeta - \zeta_0)^\beta(A(x_1, \zeta) + B(x_1, \zeta))|.$$

Apply Rouches theorem, to conclude that $G(x_1, \zeta)$ has precisely the same number of zeros in the interior of K_ϵ as $|\tilde{g}_\beta(x_1)(\zeta - \zeta_0)^\beta|$, β zeros. All are distinct because at x_1, which is in $K_{\delta'}$, $D(x_1) \neq 0$. □

Now, we suppose further that $D(x_0) \neq 0$, that is $\beta = 1$ at x_0. Then, for every x in K_δ, there is one and only one root of $G(x, \zeta)$ in $K\epsilon$. Consequently, this root is a single-valued continuous function $f_1(x)$ of x. Moreover, we have

Theorem B.3. *The function $\zeta = f_1(x)$ is regular in K_δ.*

Proof. Let x_1 be an arbitrary point in the interior of K_δ and let $\zeta_1 = f_1(x_1)$ so that $G(x_1, \zeta_1) = 0$. Let $x_1 + \xi$ be a point in the interior of K_δ such that $f_1(x_1 + \xi) = \zeta_1 + \omega, G(x_1 + \xi, \zeta_1 + \omega) = 0$. By continuity of f_1, $\omega \to 0$ as $\xi \to 0$. We have

$$f_1'(x_1) = \lim_{\xi \to 0} \left(\frac{f_1(x_1 + \xi) - f_1(x_1)}{\xi} \right) = \lim_{\xi \to 0} \left(\frac{\omega}{\xi} \right).$$

We write

$$G(x_1 + \xi, \zeta_1 + \omega) = G(x_1, \zeta_1) + \xi G_x(x_1, \zeta_1) + \omega G_\zeta(x_1, \zeta_1) + \textit{ second order terms } .$$

This implies that

$$\frac{\omega}{\xi} = -\frac{G_x(x_1, \zeta_1)}{G_\zeta(x_1, \zeta_1)} + \textit{ term that converges to 0 as } \xi \to 0 \ .$$

□

Therefore, there exist N analytic functions $f_{1,x_0}, ..., f_{N,x_0}$ defined on K_δ, such that $f_{1,x_0}(x), ..., f_{N,x_0}(x)$ are the N solutions of $G(x, \zeta) = 0$. These functions can be continued -by uniqueness of combination in K_δ- analytically over every path not containing a critical point. Imagine that there is a semiline issued from each critical point and denote by Y the union of these semilines. Since $\mathbb{C}\backslash Y$ is simply connected, monodromy theorem (principle of analytic continuation) implies the existence of analytic continuation of $f_{1,x_0}, ..., f_{N,x_0}$ to $\mathbb{C}\backslash L$; the existence of N analytic single-valued regular functions, say $F_1, ..., F_N$ such that for all $x \in \mathbb{C}\backslash Y$, $\{F_1(x), ..., F_N(x)\}$ are the N solutions of $G(x, \zeta) = 0$.

Consider now a critical point, without loss of generality let's say 0, with K being a circle around 0. If we move K_δ continuously then the functions $f_{1,x_0}, ..., f_{N,x_0}$, for some regular point x_0, can be continued along K, and they undergo a permutation between each other. We group them as cycles:

$$\{F_1(x), ..., F_p(x)\}, \{F_{p+1}(x), ..., F_{p+q}(x)\}, ...$$

Each group (cycle at $x = 0$), undergoes a cyclic permutation (of period equals to the number of elements in the cycle). The elements of a cycle of period p constitute a branch of an analytic function with an algebraic singularity (branch point) at $x = 0$, and we have the Püisseux series

$$F_j(x) = F(0) + c_1\omega^j x^{1/p} + c_2\omega^{2j} x^{2/p} + ..., \tag{B.2}$$

$j = 1, ..., p$, with $\omega = e^{\frac{2\pi i}{p}}$ and c_k are real constants for all k.

Proposition B.4. *If the operator $T(x)$ is self-adjoint, then c_k in (B.2) is zero for k not multiple of p.*

Proof. If $T(x)$ is self-adjoint, then $F_j(x)$ is real for any x near 0. If $k = np$ for some $n \in \mathbb{N}$, then $c_k\omega^{kj} x^{k/p} = c_k x^n$ is real. If k is not a multiple of p, then for $F_j(x)$ to be real, c_k must be 0 as $\omega^{kj} x^{k/p}$ is not real. □

B.3 Analyticity Of Eigenfunctions

The eigenprojection is defined using the resolvent operator so let's first consider the perturbation of the resolvent.

B.3.1 Analyticity Of The Resolvent

For any $\zeta \notin \mathrm{spec}(T(x))$, we define the resolvent of $T(x)$ as the function

$$R(\zeta, x) = (T(x) - \zeta)^{-1}.$$

Let ζ_0 be such that $\zeta_0 \notin \mathrm{spec}(T)$. If we denote by

$$A(x) := T(x) - T = \sum_{i=1}^{\infty} x^i T^{(i)},$$

and by $R(\zeta) := R(\zeta, 0) = (T - \zeta)^{-1}$, then we have

$$T(x) - \zeta = \big(1 - (\zeta - \zeta_0 - A(x))R(\zeta_0)\big)\,(T - \zeta_0).$$

For $|x|$ and $|\zeta - \zeta_0|$ small enough, we have $|\zeta - \zeta_0| - \|A(x)\| \leqslant \|R(\zeta_0)\|^{-1}$ and so, $\big(1 - (\zeta - \zeta_0 - A(x))R(\zeta_0)\big)^{-1}$ exists and can be written as a convergent Neumann series. Then, in a neighborhood of $\zeta = \zeta_0$ and $x = 0$, $R(\zeta, x)$ is holomorphic and can be written as

$$R(\zeta, x) = R(\zeta) + \sum_{n=1}^{\infty} x^n R^{(n)}(\zeta). \tag{B.3}$$

B.3.2 Analyticity Of Eigenprojection

Let λ be an eigenvalue of T of multiplicity α, and let Γ be a closed curve in the resolvent set of T enclosing no eigenvalues but λ. Continuity of eigenvalues implies that for x small enough, Γ contained no eigenvalues of $T(x)$. We define the operator

$$P(x) = -\frac{1}{2\pi i}\int_\Gamma R(\zeta, x)d\zeta.$$

This is a projection, that equals the sum of eigenprojections for all eigenvalues of $T(x)$ inside Γ. Integrating (B.3), we get

$$P(x) = P - \frac{1}{2\pi i}\sum_{n=1}^{\infty} x^n \int_\Gamma R^{(n)}(\zeta)d\zeta = P + \sum_{n=1}^{\infty} x^n P^{(n)}. \tag{B.4}$$

The series (B.4) is convergent for x small enough, and so $P(x)$ is holomorphic near $x = 0$. Moreover, this gives that the range of $P(x)$ and P are isomorphic and that $\dim(P(x)) = \dim(P)$. So if $x = 0$ is a non-critical point, then $\zeta(x)$ is the only eigenvalue of $T(x)$ and is of multiplicity α. So, on $\mathcal{C}\backslash L$, the eigenprojections $P_i(x)$ (that correspond to $F_i(x)$) are holomorphic for $i = 1, ..., N$.

Now, near a critical point, again, the family $\{P_i(x)\}$ undergo, after one revolution around the critical point, a permutation that Kato proved to be identical for the two families (eigenbranches and eigenfunction branches).

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Mohammad Hussein Harakeh

Méthodes spectrales en géométrie sous-riemannienne

Résumé : Ce manuscrit traite de deux sujets qui convergent vers une idée : l'utilisation des méthodes spectrales en géométrie sous-riemannienne. Tout d'abord, nous étudions la validité des inégalités de concentration pour les fonctions propres de l'opérateur généralisé de Baouendi-Grushin sur un cylindre infini. Nous démontrons que généralement, les valeurs propres de l'opérateur de Baouendi-Grushin ont une multiplicité de 2, que nous prouvons être une condition suffisante pour la validité de l'inégalité de concentration. Ensuite, nous étudions les structures sous-riemanniennes en les approximant par des structures riemanniennes. Nous introduisons un schéma d'approximation et prouvons qu'il induit une forme de volume qui coïncide - à une constante près - avec le volume de Popp. Nous démontrons ensuite que le spectre de la famille des Laplaciens riemanniens associés au schéma d'approximation converge vers le spectre du sous-Laplacien. **Mots-clés :** Théorie spectrale - Inégalité de concentration - Opérateurs sous-elliptiques - Théorie de la perturbation - Géométrie sous-riemannienne.

Spectral Methods In SubRiemannian Geometry

Abstract : This manuscript handles two subjects that meet at one idea: using spectral methods in subRiemannian geometry. First, we study the validity of concentration inequalities for eigenfunctions, for the generalised Baouendi Grushin operator on an infinite cylinder. We prove that generically, the eigenvalues of the Baouendi Grushin operator has multiplicity 2 which we prove to be a sufficient condition to the validity of the concentration inequality. Second, we study subRiemannian structures by approximating these structures with Riemannian ones. We introduce an approximation scheme and prove that it induces a volume form that coincides -up to a constant- with the Popp's volume. We then prove that the spectrum of the family of Riemannian Laplacians associated to the approximation scheme converges to the spectrum of the subLaplacian. **Keywords** : Spectral theory- concentration inequality- Subelliptic operators- Perturbation theory- SubRiemannian geometry.

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