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Conditioning and penalization of marked Galton-Watson trees

Conditionnement et pénalisation d'arbres de Galton-Watson marqués

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Conditioning and penalization of marked Galton Watson trees

under the direction of Romain Abraham and Pierre Debs

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Sonia Boulal

Conditionnement et pénalisation d'arbres de Galton-Watson marqués

Résumé :

Le processus de Galton-Watson est un modèle très simple d'évolution de population où chaque individu se reproduit aléatoirement et indépendamment de tous les autres selon une même loi fixée. Il est connu que dans le cas où le nombre moyen d'enfants par individu est inférieur ou égale à 1 (sous-critique ou critique), la population s'éteint en temps fini presque sûrement. Certains auteurs ont décrit cette population (ou plutôt l'arbre généalogique associé) lorsqu'elle est anormalement grande et ont mis en évidence l'évènement exceptionnel permettant d'obtenir ce comportement particulier. D'un point de vue mathématique, cela consiste à conditionner la loi de l'arbre T par un évènement de la forme $\{A(T) = n\}$ où A est une fonctionnelle de l'arbre T (par exemple, $A(T)$ est la dernière génération survivante, ou bien le nombre total d'individus dans la population,...) et à regarder la limite (en loi) lorsque n tend vers l'infini.

Dans un récent travail, Abraham, Bouaziz et Delmas ont considéré d'autres types d'évènements faisant intervenir une source d'aléa supplémentaire. Dans leur article, chaque individu est marqué aléatoirement avec une probabilité dépendant de son nombre d'enfants. Ils ont conditionné par l'évènement "Le nombre d'individus marqués est égal à n". En utilisant des techniques similaires aux cas classiques, ils ont obtenu la limite en loi de l'arbre conditionné dans le cas critique. Une partie de ma thèse poursuit ce travail pour obtenir des résultats dans le cas sous-critique. Ainsi, j'ai pu décrire la limite en loi de l'arbre marqué dans les cas critique et sous-critique, sous certaines conditions de régularité concernant la loi de reproduction et la loi de marquage.

Une deuxième partie de ma thèse consiste à considérer une autre façon d'obtenir des arbres avec un nombre anormal de sommets marqués. Une technique possible est de faire un changement de mesure via une transformation de Girsanov avec une martingale. Afin de trouver une telle martingale, nous avons utilisé une méthode générale appelée *pénalisation* qui a été introduite par Roynette, Vallois et Yor en 2006 pour le mouvement Brownien. J'ai adapté cette technique dans le cadre des arbres discrets. En effet, j'ai biaisé la loi de l'arbre par une fonction du nombre de marques, qui soit les favorise, soit les défavorise, et j'ai décrit l'arbre obtenu sous cette nouvelle loi. J'ai traité les trois cas possibles, sous-critique, critique et sur-critique (le nombre moyen d'enfants par individu est strictement supérieur à 1). Dans les deux derniers cas, j'obtiens des arbres limites déjà connus, tandis que dans le premier cas, je décris un nouvel arbre, que l'on nommera "arbre à poids". Ce dernier est un arbre multi-type marqué, caractérisé par une notion de masse se transmettant de génération en génération. Il faut noter que, cet arbre n'est pas un arbre de Galton-Watson marqué.

Mots clés : Arbre de Galton-Watson marqué · Pénalisation · Transformation de Girsanov · Comportement asymptotique · Arbres aléatoires · Conditionnement

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Conditioning and penalization of marked Galton-Watson trees

Abstract :

The Galton–Watson process is a very simple model of population evolution where each individual reproduces randomly and independently of all others according to the same given distribution. It is well known that in the case where the average number of offspring per individual is less than 1 (sub-critical or critical case), the population becomes extinct in finite time. Many authors have focused on the description of this population (more precisely, the genealogical tree associated with it) under the condition that it reaches an abnormally large size. From a mathematical point of view, this consists in conditioning the distribution of the tree T by some event of the form $\{A(T) = n\}$ where A is a functional of the tree T (for instance, $A(T)$ is the height of the last surviving generation, or the total number of individuals in the population,...) and to study the limit (in distribution) of the tree when n tends to infinity.

In a recent paper of Abraham, Bouaziz and Delmas, the authors explored alternative types of conditioning involving an additional source of randomness. In their paper, each individual is randomly marked with a probability depending on their number of children. They then conditioned the tree on the event "The number of marked individuals is equal to n". By adapting techniques from the classical framework, they obtained the limiting distribution of the conditioned tree in the critical case. Part of my PhD research extends this work to the sub-critical regime. Rather than studying the limiting distribution of the tree T, my focus is on the marked tree with its marks, allowing me to describe its asymptotic behavior in both critical and sub-critical cases. These results were obtained under certain regularity assumptions on both the offspring distribution and the marking mechanism.

The second part of my thesis investigates a different approach to generating trees with an unusual number of marked vertices. One natural method is to perform a change of measure using a martingale, through a Girsanov-type transformation. To find such a martingale, I used a general method called *penalization*, introduced by Roynette, Vallois, and Yor in 2006 for Brownian motion. I adapt this method to the discrete tree setting by biasing the distribution of the tree according to a function of the number of marks, either favoring or disfavoring them, and then analyzing the obtained trees under this new probability measure. I treated all three regimes : sub-critical, critical, and supercritical (where the average number of offspring per individual exceeds 1). In the critical and supercritical cases, I recovered known limiting objects. In contrast, the sub-critical case led me to identify a new limiting structure, which I refer to as the "weighted tree". This is a multi-type marked tree, characterized by a notion of mass that is transmitted from one generation to the next. Note that, it does not correspond to a marked Galton-Watson tree.

Keywords : Marked Galton–Watson tree · Penalization · Girsanov transformation · Asymptotic behavior · Random trees · Conditioning

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Première partie

État de l’art et résultats principaux de la thèse

Chapitre 1

Arbre de Galton-Watson et conditionnement

Les processus de Galton-Watson ont été introduits en 1845 par Bienaymé et indépendamment par Galton en 1873 afin d'étudier la pérennité des noms de famille. Pour cela, on attribuait à une famille une loi de reproduction, c'est-à-dire qu'à chaque génération, on déterminait la probabilité qu'un individu ait un certain nombre de garçons avec cette même loi.

Mathématiquement parlant, ce processus est très simple : chaque individu donne naissance indépendamment des uns des autres à un nombre aléatoire d'enfants selon une loi de probabilité $\mathbf{p}$ que l'on appelle loi de reproduction. Autrement dit, à chaque génération, chaque individu peut avoir k enfants avec probabilité $\mathbf{p}(k)$. Pour éviter des cas dégénérés, on suppose que la loi $\mathbf{p}$ satisfait l'hypothèse suivante :

$$\mathbf{p}(0) > 0,\ \mathbf{p}(0) + \mathbf{p}(1) < 1 \text{ et } \mu(\mathbf{p}) < +\infty, \tag{1.1}$$

avec $\mu(\mathbf{p}) = \sum_{n=0}^{+\infty} n\mathbf{p}(n)$ sa moyenne. Si $\mu(\mathbf{p}) < 1$ (resp. $\mu(\mathbf{p}) = 1$, $\mu(\mathbf{p}) > 1$), on dit que la loi de reproduction est sous-critique (resp. critique, sur-critique). Dans les cas critique et sous-critique, on a presque sûrement extinction de la population.

En 1986, Neveu a introduit la notion d'arbre de Galton Watson (GW) [29]. C'est un arbre généalogique, noté τ, qui décrit l'évolution de la population ayant pour loi de reproduction $\mathbf{p}$.

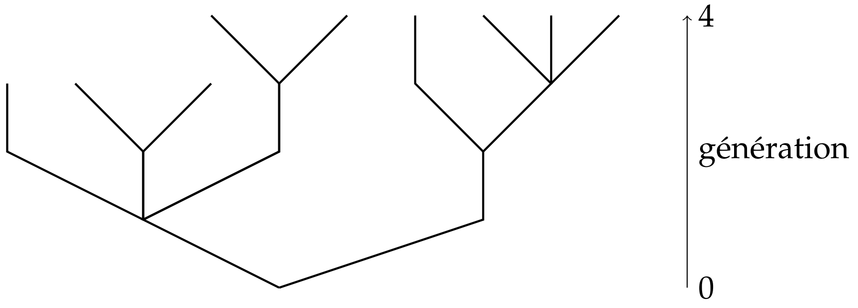


FIGURE 1.1 – Un arbre de généalogie à 5 générations.

Le conditionnement d'arbres de Galton-Watson a commencé en 1986 avec les travaux de Kesten [24], qui s'est intéressé à des limites locales sur ces derniers. Cette idée repose sur le fait qu'au lieu d'analyser la structure globale d'un grand arbre aléatoire, ce qui peut être très complexe, on se concentre sur sa structure locale, c'est-à-dire le voisinage autour d'un sommet aléatoire. On observe ainsi ce qui évolue quand la taille de l'arbre tend vers l'infini. Dans les cas critique et sous-critique, l'arbre est presque sûrement fini. Cependant, dans [24], Kesten considérait la limite locale d'un arbre critique ou sous-critique conditionné à avoir une hauteur plus grande que n. Quand n tend vers l'infini, cet arbre conditionné converge en loi pour la topologie locale, décrite précisément à la Section 8.1, vers un arbre de Kesten : cet arbre est constitué d'une branche infinie à laquelle est greffé un nombre aléatoire d'arbres de Galton Watson, ayant pour loi de reproduction $\mathbf{p}$. Cet arbre limite peut être vu comme un arbre de Galton Watson conditionné à ne jamais s'éteindre.

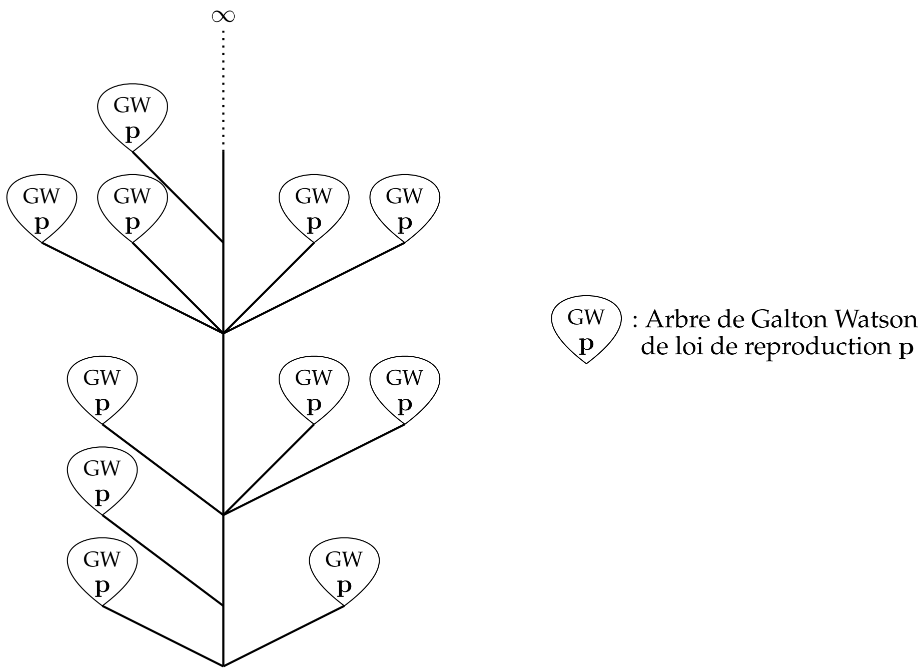


FIGURE 1.2 – Arbre de Kesten.

Différents conditionnements d'arbres de Galton Watson ont été considérés : grand nombre d'individus (voir Kennedy [23] et Geiger et Kaufman [18]), grand nombre de feuilles (voir Curien et Kortchemski [11]). Dans les cas critiques et sous-critiques, Rizzolo [30], mais aussi Kortchemski [26], conditionnent afin d'avoir un grand nombre d'individus tel que leur nombre d'enfants appartienne à un ensemble $\mathcal{A}$. Dans le cas critique, l'arbre limite obtenu est un arbre de Kesten, voir Abraham et Delmas [7]. Dans [6], Abraham et Delmas obtiennent la convergence vers un arbre de condensation dans le cas sous-critique et non générique ; tandis que dans le cas sous-critique générique, ils obtiennent une convergence vers un arbre de Kesten mais avec une loi de reproduction différente.

En effet, plusieurs lois de reproduction peuvent donner la même loi pour des arbres conditionnés, ainsi être générique signifie que l'on peut se rapprocher d'une certaine façon à une loi critique. À l'inverse, une loi non générique est une loi pour laquelle cela est impossible. En d'autres termes, $\mathbf{p}$ est générique lorsqu'il existe une (unique) loi biaisée, dépendante de $\mathbf{p}$, critique.

Contrairement à l'arbre de Kesten qui est anormalement grand, l'arbre de condensation est anormalement large. En effet, sur cet arbre, à chaque génération, un individu se reproduit suivant une loi biaisée lui permettant d'avoir une infinité d'enfants. De ce fait, cet individu est choisi uniformément parmi les enfants de l'individu de loi biaisée à la génération précédente. Les autres individus se reproduisent suivant la loi initiale $\mathbf{p}$. Dès qu'un individu a une infinité d'enfants, tous les individus des générations suivantes se reproduisent normalement. Cet arbre a été introduit par Jonsson et Stefanson [20] ainsi que par Janson [19].

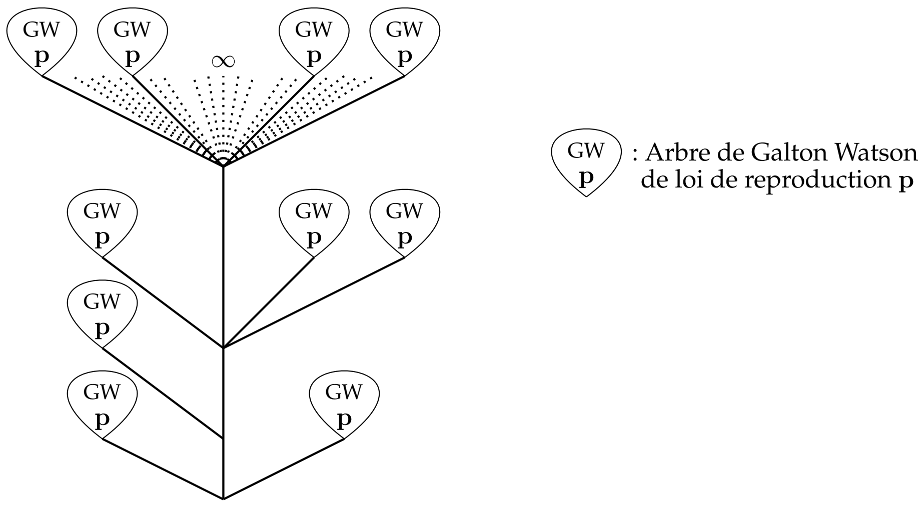


FIGURE 1.3 – Arbre de condensation.

Comme cet arbre est infiniment large, il est, alors, aussi infiniment grand. En effet, si l'on se fixe un entier n grand, parmi les enfants, de l'individu en ayant une infinité, il en existe un, tel que l'arbre de Galton-Watson greffé fini soit de hauteur supérieure à n. Ce raisonnement fonctionne pour tout entier n positif, l'arbre de condensation est ainsi infiniment grand.

Chapitre 2

Arbres marqués

2.1 L'ensemble des arbres discrets

Soit $\mathbb{N}$ l'ensemble des entiers positifs et $\mathbb{N}^*$ l'ensemble des entiers strictement positifs. On rappelle le formalisme de Neveu [29] qui ordonne les nœuds d'un arbre. Soit $\mathcal{U} = \bigcup_{n\geq 0}(\mathbb{N}^*)^n$ l'ensemble des suites finies d'entiers strictement positifs, avec, par convention, $(\mathbb{N}^*)^0 = \{\emptyset\}$. Pour $n \geq 0$ et $u = (u_1, ..., u_n) \in \mathcal{U}$, on pose $|u| = n$, la génération de u et $|u|_\infty = \max\{|u|, u_1, u_2, ..., u_{|u|}\}$ avec par convention $|\emptyset| = |\emptyset|_\infty = 0$. On dit que $|u|_\infty$ est la norme de u bien que cela ne soit pas une norme puisque $\mathcal{U}$ n'est pas un espace vectoriel. Si u et v sont deux suites de $\mathcal{U}$, on note uv leur concaténation, avec, par convention, $uv = u$ si $v = \emptyset$ et $uv = v$ si $u = \emptyset$. L'ensemble des ancêtres de la racine $\emptyset$ est $A_\emptyset = \{\emptyset\}$ et celui de $u \neq \emptyset$ est :

$$A_u = \{v \in \mathcal{U};\ \text{il existe } w \in \mathcal{U}, w \neq \emptyset,\ \text{tel que } u = vw\}.$$

L'ancêtre commun le plus récent d'une partie s de $\mathcal{U}$, que l'on note $MRCA(s)$, est l'unique élément v de $\bigcap_{u\in s} A_u$ telle que la génération $|v|$ soit maximale.

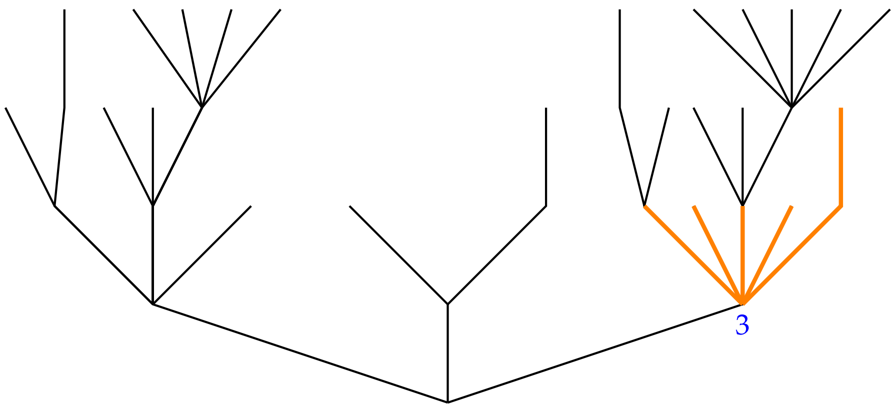


FIGURE 2.1 – Le sous-arbre s en gras et orange, et $MRCA(s) = 3$ en bleu .

Pour $u, v \in \mathcal{U}$, on note $u < v$, l'ordre lexicographique sur $\mathcal{U}$, i.e. $u < v$ si $u \in A_v$, ou si on pose $w = MRCA(\{u, v\})$, alors $u = wiu'$ et $v = wjv'$ avec $i, j \in \mathbb{N}^*$, $i < j$.

Dans la suite du document, afin d'alléger la lecture et s'il n'y a pas de confusion, on notera $u = u_1 ... u_n$ pour $u = (u_1, ..., u_n) \in \mathcal{U}$. Par exemple (1), sera noté 1 ; si (121), $(1, 21)$ et $(12, 1)$ n'existent pas on notera $(1, 2, 1)$ par 121.

Un arbre $\mathbf{t}$ est un sous-ensemble de $\mathcal{U}$ qui satisfait :

— $\emptyset \in \mathbf{t}$;

— si $u \in \mathbf{t}$, alors $A_u \subset \mathbf{t}$;

— pour tout $u \in \mathbf{t}$, il existe $k_u(\mathbf{t}) \in \mathbb{N} \cup \{+\infty\}$ tel que, pour tout entier positif i, $ui \in \mathbf{t}$ si et seulement si $1 \leq i \leq k_u(\mathbf{t})$.

L'entier $k_u(\mathbf{t})$ représente le nombre d'enfants du nœud $u \in \mathbf{t}$. Si $k_u(\mathbf{t}) = 0$, on dit que le nœud u est une feuille et si $k_u(\mathbf{t}) = +\infty$, u est appelé nœud infini. Le nœud $\emptyset$ est appelé racine de $\mathbf{t}$.

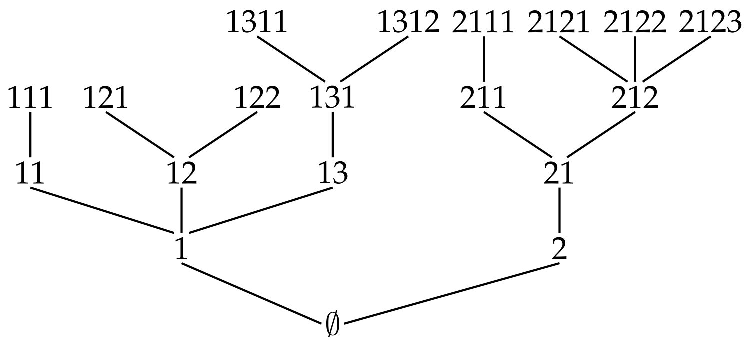


FIGURE 2.2 – Un arbre $\mathbf{t}$ à 5 générations.

On note

— $\mathbb{T}_\infty$ l'ensemble des arbres discrets, planaires, enracinés ;

— $\mathbb{T}_0 := \{\mathbf{t} \in \mathbb{T}_\infty, |\mathbf{t}| < +\infty\}$, le sous-ensemble des arbres finis, avec $|\mathbf{t}|$ le cardinal de $\mathbf{t}$;

— $\mathbb{T} := \{\mathbf{t} \in \mathbb{T}_\infty ; k_u(\mathbf{t}) < +\infty \ \forall u \in \mathbf{t}\}$, le sous-ensemble des arbres ne possédant pas de nœuds infinis.

Soit $h \in \mathbb{N}$, la fonction de restriction $r_{h,\infty}$ de $\mathbb{T}_\infty$ dans $\mathbb{T}_\infty$ est définie par :

$$r_{h,\infty}(\mathbf{t}) := \{u \in \mathbf{t}, |u|_\infty \leq h\},$$

et la fonction de restriction r_h de $\mathbb{T}$ dans $\mathbb{T}$ est définie par :

$$r_h(\mathbf{t}) := \{u \in \mathbf{t}, |u| \leq h\}.$$

Dans le premier cas, la fonction coupe l'arbre dès que l'on a atteint la génération h ou dès qu'un individu a plus de h enfants, alors on garde seulement les h premières branches. Dans le second cas, la fonction coupe l'arbre à la génération h. Autrement dit, elle décrit l'arbre $\mathbf{t}$ jusqu'à la génération h. Si l'on reprend l'exemple de la figure 2.2, on obtient les arbres suivants lorsque l'on regarde la restriction de niveau 2 :

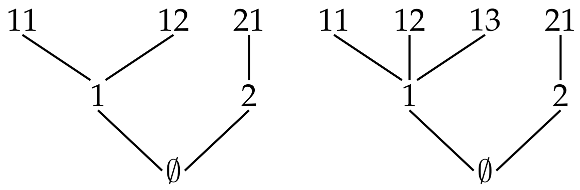


FIGURE 2.3 – $r_{2,\infty}(\mathbf{t})$ à gauche et $r_2(\mathbf{t})$ à droite.

On munit l'ensemble $\mathbb{T}_\infty$ (resp. $\mathbb{T}$), de la distance ultra-métrique

$$d_\infty(\mathbf{t}, \mathbf{t}') := 2^{-\max\{h\in\mathbb{N}, r_{h,\infty}(\mathbf{t})=r_{h,\infty}(\mathbf{t}')\}} \quad \left(\text{resp. } d(\mathbf{t}, \mathbf{t}') := 2^{-\max\{h\in\mathbb{N}, r_h(\mathbf{t})=r_h(\mathbf{t}')\}} \right).$$

Une suite d'arbres $(\mathbf{t}_n)_{n\in\mathbb{N}}$ converge vers un arbre $\mathbf{t}$ sous la distance d_∞ (resp. d) si, et seulement si, pour tout $h \in \mathbb{N}$,

$$r_{h,\infty}(\mathbf{t}_n) = r_{h,\infty}(\mathbf{t}) \ \ (\text{ resp. } r_h(\mathbf{t}_n) = r_h(\mathbf{t})) \text{ pour } n \text{ assez grand.}$$

La tribu de Borel associée à la distance d_∞ (resp. d) est la plus petite tribu contenant les singletons pour lesquels les fonctions de restrictions $(r_{h,\infty}, h \in \mathbb{N})$ (resp. $(r_h, h \in \mathbb{N})$) sont mesurables.
Avec cette distance, la fonction de restriction est contractante. D'après [20] $\mathbb{T}_0$ est dense dans $\mathbb{T}_\infty$ (resp. $\mathbb{T}_0 \cap \mathbb{T} \in \mathbb{T}$) et puique $(\mathbb{T}_\infty, d_\infty)$ (resp. $(\mathbb{T}, d)$) est complet, on obtient que $(\mathbb{T}_\infty, d_\infty)$ (resp. $(\mathbb{T}, d)$) est un espace métrique Polonais. De plus, d'après [20], $(\mathbb{T}_\infty, d_\infty)$ est compact.

2.2 Marquage sur les arbres de Galton-Watson

En 2017, dans [2], Abraham, Bouaziz et Delmas généralisent le conditionnement à avoir un grand nombre d'individus tels que leur nombre d'enfants appartienne à un ensemble $\mathcal{A}$, en marquant les nœuds aléatoirement. Dire que l'on conditionne par rapport au nombre d'individus ayant leur nombre de descendants appartenant à $\mathcal{A}$, revient à étudier pour tout $u \in \mathbf{t}$ si $\mathbb{1}_{k_u(\mathbf{t})\in\mathcal{A}}$ vaut 1 avec $k_u(\mathbf{t})$ le nombre d'enfants de l'individu u.
Par exemple, si $\mathcal{A} = \{0\}$, on retrouve l'étude du conditionnement par rapport au nombre de feuilles de l'arbre, et si $\mathcal{A} = \mathbb{N}$ on étudie le conditionnement par rapport au nombre total d'individus. Ainsi, l'étude par rapport à $\mathcal{A}$ était déjà une généralisation de précédents conditionnements.

Toutefois, ce conditionnement est déterministe. Assurément, chaque individu u sera marqué si $\mathbb{1}_{k_u(\mathbf{t})\in\mathcal{A}}$ vaut 1 sinon non. Conditionnellement à un arbre $\mathbf{t}$ donné, $\mathcal{A}$ étant fixé, il n'y a pas d'aléatoire dans le choix des nœuds marqués. L'approche de Abraham, Bouaziz et Delmas, est d'introduire davantage d'aléatoire dans ce marquage, de manière à ce que la fonction de marquage, qui précédemment était une indicatrice, puisse prendre différentes valeurs comprises entre 0 et 1.

En effet, on fixe un arbre $\mathbf{t}$, et on marque aléatoirement ses nœuds, conditionnellement à $\mathbf{t}$, chaque nœud est marqué indépendamment les uns des autres avec une probabilité dépendant de son nombre de descendants.

Plus précisément, indépendamment les uns des autres, chaque individu donne naissance à k enfants et est

- marqué avec probabilité $\mathbf{p}_0(k,1) := \mathbf{p}(k)\mathbf{q}(k)$;
- non marqué avec probabilité $\mathbf{p}_0(k,0) := \mathbf{p}(k)(1-\mathbf{q}(k))$

où $\mathbf{q} := (\mathbf{q}(k))_{k\geq 0}$ est une suite de nombres dans $[0,1]$ que l'on appelle fonction de marquage. La loi $\mathbf{p}_0$ sur $\mathbb{N}\times\{0,1\}$ est appelée loi de reproduction-marquage du Galton-Watson marqué (MGW) associé. On suppose que l'on a la condition suivante :

$$\exists k \in \mathbb{N}^*,\ \mathbf{p}(k)\mathbf{q}(k) > 0. \tag{2.1}$$

Cela signifie que chaque individu d'un MGW est marqué avec une probabilité positive.

On note $\mathbf{t}^*$ un arbre marqué, il est défini par un arbre $\mathbf{t} \in \mathbb{T}_\infty$ et une marque $\eta_u \in \{0,1\}$ pour tous $u \in \mathbf{t}$, c'est à dire :

$$\mathbf{t}^* = \big(\mathbf{t}, (\eta_u)_{u\in\mathbf{t}}\big).$$

Un nœud $u \in \mathbf{t}$ est dit marqué si $\eta_u = 1$ et non marqué si $\eta_u = 0$.

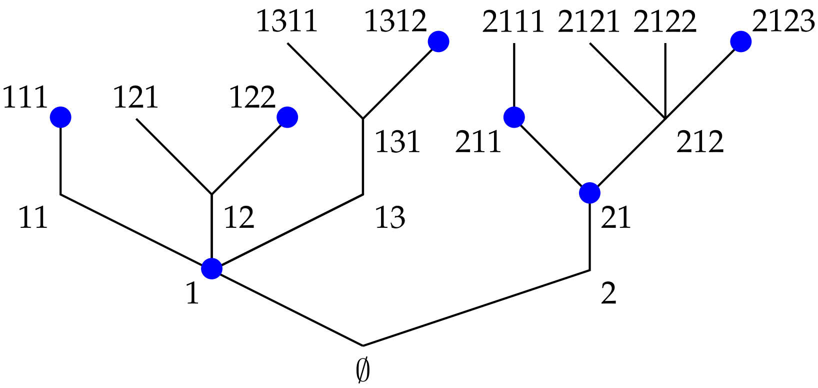


FIGURE 2.4 – Un arbre marqué $\mathbf{t}^*$ à 5 générations.

On note $\mathbb{T}^*_\infty$ l'ensemble des arbres planaires, enracinés, marqués. L'entièreté des notations définies pour $\mathbf{t}$ sont similaires pour $\mathbf{t}^*$. Par exemple, $\mathbb{T}^*_0 := \{(\mathbf{t}, (\eta_u(\mathbf{t}))_{u\in\mathbf{t}}), \mathbf{t} \in \mathbb{T}_0\}$.
On pose $M(\mathbf{t}^*) := \sum_{u\in\mathbf{t}} \eta_u$ le nombre de nœuds marqués sur un arbre $\mathbf{t}^* \in \mathbb{T}^*_\infty$. Comme dans la Section 2.1, pour tout $h \in \mathbb{N}$, on définit les fonctions de restrictions

$$r^*_h : \mathbb{T}^* \longrightarrow \mathbb{T}^*, \qquad \text{et} \qquad r^*_{h,\infty} : \mathbb{T}^*_\infty \longrightarrow \mathbb{T}^*_\infty$$

pour tous $\mathbf{t}^* = \big(\mathbf{t}, (\eta_u)_{u\in\mathbf{t}}\big) \in \mathbb{T}^*$, par

$$r^*_h(\mathbf{t}^*) = \big(r_h(\mathbf{t}), (\eta^h_u)_{u\in r_h(\mathbf{t})}\big), \qquad r^*_{h,\infty}(\mathbf{t}^*) = \big(r_{h,\infty}(\mathbf{t}), (\eta^h_u)_{u\in r_{h,\infty}(\mathbf{t})}\big)$$

avec

$$\eta^h_u = \begin{cases} \eta_u & \text{si } |u| \leq h-1, \\ 0 & \text{si } |u| = h. \end{cases}$$

Ces fonctions de restrictions sont bien définies. On fixe $h \in \mathbb{N}$.

Pour $r_h^*(.)$, la restriction de l'arbre marqué contient tous les individus jusqu'à la génération h, et contient les marques seulement jusqu'à la génération précédente.
En effet, puisque la marque dépend du nombre d'enfants de l'individu, et que l'on coupe l'arbre à la génération h, on ne connaît pas le nombre d'enfants des individus de la génération h. On ne peut donc pas déterminer si les individus de cette dernière génération sont marqués ou non. C'est pourquoi, la restriction est caractérisée à l'aide de η_u^h et non η_u, pour tout individu u défini jusqu'à la génération h.

Avec $r_{h,\infty}^*(.)$, la restriction de l'arbre marqué, ne coupe pas seulement l'arbre à la génération h mais le coupe aussi lorsqu'un individu a plus de h enfants tout en gardant sa marque s'il en avait une. Autrement dit, si l'individu a $h+x$ enfants, on garde les h premiers et les x autres disparaissent. En effet, contrairement au cas précédent, cette restriction permet d'étudier la convergence locale pour des arbres pouvant avoir un noeud infini, il est donc nécessaire de pouvoir savoir si un tel noeud est marqué ou non.

En reprenant la figure 2.4, on obtient les restrictions suivantes à l'ordre 2 :

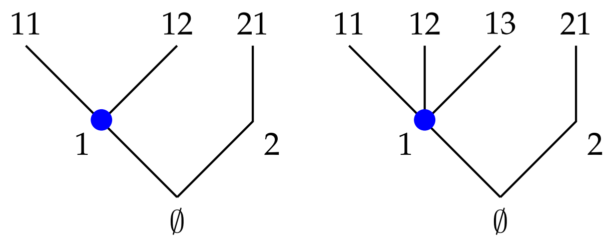


FIGURE 2.5 – $r_{2,\infty}^*(\mathbf{t}^*)$ à gauche et $r_2^*(\mathbf{t}^*)$ à droite.

On peut attribuer à l'ensemble $\mathbb{T}_\infty^*$ (resp. $\mathbb{T}^*$) la tribu $\mathscr{F}$ générée par la famille d'ensembles $(\{\mathbf{t}^* \in \mathbb{T}_\infty^*,\ u \in \mathbf{t}\})_{u\in\mathcal{U}}$ $\big($resp. $(\{\mathbf{t}^* \in \mathbb{T}^*,\ u \in \mathbf{t}\})_{u\in\mathcal{U}}\big)$ et ainsi définir un espace mesurable $(\mathbb{T}_\infty^*, \mathscr{F})$ (resp. $(\mathbb{T}^*, \mathscr{F})$).
On attribue aussi à $\mathbb{T}_\infty^*$ (resp. $\mathbb{T}^*$) la filtration $(\mathscr{F}_n)_{n\geq 0}$ où $\mathscr{F}_n$ est générée par les fonctions de restrictions $r_{n,\infty}^*$ (resp. r_n^*). Notons que $\mathscr{F} = \bigvee_{n\geq 0} \mathscr{F}_n$ et pour tout $u \in \mathcal{U}$,

$$\{\mathbf{t}^* \in \mathbb{T}_\infty^*,\ u \in \mathbf{t}\} \in \mathscr{F}_{|u|} \text{ (resp. } \{\mathbf{t}^* \in \mathbb{T}^*,\ u \in \mathbf{t}\} \in \mathscr{F}_{|u|}).$$

Soit $\mathbf{t}^* = (\mathbf{t}, (\eta_u)_{u\in\mathbf{t}}) \in \mathbb{T}_\infty^*$ $(resp.\ \mathbb{T}^*)$ un arbre marqué, on a $\displaystyle\sum_{u\in\mathbf{t}} k_u(\mathbf{t}) = |\mathbf{t}| - 1$. Pour tout $n \in \mathbb{N}$, on définit :

— $z_n(\mathbf{t}^*)$ l'ensemble des nœuds de $\mathbf{t}$ à la génération n :

$$z_n(\mathbf{t}^*) = \{u \in \mathbf{t},\ |u| = n\},$$

— $Z_n(\mathbf{t}^*)$ le nombre d'individus à la génération n :

$$Z_n(\mathbf{t}^*) = \mathrm{Card}\big(z_n(\mathbf{t}^*)\big),$$

— $M_n(\mathbf{t}^*)$ le nombre de marques sur l'arbre jusqu'à la génération $n-1$: $M_0(\mathbf{t}^*) = 0$ et

$$M_n(\mathbf{t}^*) = \sum_{u\in\mathbf{t}, |u|\leq n-1} \eta_u \text{ for } n \geq 1.$$

Notons que les fonctions $z_n,\ Z_n,\ M_n$ sont toutes $\mathscr{F}_n$-mesurables. Maintenant que les outils sont bien définis, intéressons nous plus précisément à $\mathbf{p}_0 = (\mathbf{p}_0(k,\eta))_{k\in\mathbb{N},\eta\in\{0,1\}}$ la mesure de probabilité sur $\mathbb{N}\times\{0,1\}$. Il existe une unique mesure de probabilité $\mathbb{P}_{\mathbf{p}_0}$ sur $(\mathbb{T}^*,\mathscr{F})$ telle que, pour tout $h\in\mathbb{N}^*$ et tout $\mathbf{t}^* = (\mathbf{t},(\eta_u)_{u\in\mathbf{t}})\in\mathbb{T}^*_\infty$ (resp. $\mathbb{T}^*$), si τ^* est une variable aléatoire sur $\mathbb{T}^*_\infty$ (resp. $\mathbb{T}^*$),

$$\mathbb{P}_{\mathbf{p}_0}(r_h^*(\tau^*) = r_h^*(\mathbf{t}^*)) = \prod_{u\in r_{h-1}(\mathbf{t})} \mathbf{p}_0(k_u(\mathbf{t}),\eta_u),$$

$$\mathbb{P}_{\mathbf{p}_0}(r_{h,\infty}^*(\tau^*) = r_{h,\infty}^*(\mathbf{t}^*)) = \prod_{u\in\mathbf{t},|u|_\infty<h} \mathbf{p}_0(k_u(\mathbf{t}),\eta_u) \prod_{\substack{u\in r_{h,\infty}(\mathbf{t}),\\ k_u(r_{h,\infty}(\mathbf{t}))=h}} \sum_{k\geq h}\mathbf{p}_0(k,\eta_u).$$

On dit que la variable aléatoire τ^* sous $\mathbb{P}_{\mathbf{p}_0}$ est un arbre de Galton-Watson marqué (MGW) de loi de reproduction-marquage $\mathbf{p}_0$. Pour τ^* défini sur $\mathbb{T}_0^*$, on a pour tout $\mathbf{t}^*\in\mathbb{T}_0^*$,

$$\mathbb{P}(\tau^*=\mathbf{t}^*) = \prod_{u\in\mathbf{t}}\mathbf{p}_0(k_u(\mathbf{t}),\eta_u) = \mathbb{P}(\tau=t)\prod_{u\in\mathbf{t},\eta_u=1}\mathbf{q}(k_u(\mathbf{t}))\prod_{u\in\mathbf{t},\eta_u=0}(1-\mathbf{q}(k_u(\mathbf{t}))).$$

De façon équivalente, la loi d'un arbre de Galton-Watson marqué ayant pour loi de reproduction-marquage $\mathbf{p}_0$ est aussi caractérisée par une loi de reproduction $\mathbf{p}$ et une fonction de marquage $\mathbf{q}:\mathbb{N}\longrightarrow[0,1]$ avec

$$\mathbf{p}(k) = \mathbf{p}_0(k,0)+\mathbf{p}_0(k,1), \quad \mathbf{q}(k) = \frac{\mathbf{p}_0(k,1)}{\mathbf{p}(k)} \text{ if } \mathbf{p}(k)\neq 0$$

(la valeur de $\mathbf{q}(k)$ n'a pas d'importance si $\mathbf{p}(k)=0$), ou de manière équivalente

$$\mathbf{p}_0(k,1) = \mathbf{p}(k)\mathbf{q}(k), \qquad \mathbf{p}_0(k,0) = \mathbf{p}(k)(1-\mathbf{q}(k)). \tag{2.2}$$

Cette approche consiste à tirer dans un premier temps un arbre de Galton-Watson de loi de reproduction $\mathbf{p}$ puis conditionnellement à l'arbre, ajouter des marques sur chaque nœud, indépendamment les uns des autres avec probabilité $\mathbf{q}(k)$ où k est le nombre d'enfants du nœud. On écrit alors $\mathbb{P}_{\mathbf{p}_0}$ ou $\mathbb{P}_{\mathbf{p},\mathbf{q}}$ en fonction du point de vue adopté, ou simplement $\mathbb{P}$ quand le contexte est clair.

Sous $\mathbb{P}_{\mathbf{p},\mathbf{q}}$, l'arbre aléatoire τ^* satisfait les propriétés de branchement suivantes :

— La variable aléatoire $k_\emptyset(\tau^*)$ est distribuée suivant la loi $\mathbf{p}$;

— Conditionnellement à $\{k_\emptyset(\tau^*)=j\}$, la racine est marquée avec probabilité $\mathbf{q}(j)$;

— Conditionnellement à $\{k_\emptyset(\tau^*)=j\}$, les j sous-arbres attachés à la racine sont des i.i.d. arbres aléatoires marqués de loi $\mathbb{P}_{\mathbf{p},\mathbf{q}}$.

Par conséquent, si on note $Z_n := Z_n(\tau^*)$ et $M_n := M_n(\tau^*)$, on a pour tous $n,p\in\mathbb{N}^*$, les égalités suivantes sous $\mathbb{P}_{\mathbf{p},\mathbf{q}}$,

$$M_{n+p} \overset{(d)}{=} M_n + \sum_{i=1}^{Z_n}\tilde{M}_{i,p},\ Z_{n+p} \overset{(d)}{=} \sum_{i=1}^{Z_n}\tilde{Z}_{i,p},$$

où $(\tilde{M}_{i,p})_{i\geq1}$ est une suite de copies i.i.d de M_p, $(\tilde{Z}_{i,p})_{i\geq1}$ est une suite de copies i.i.d de Z_p et sont indépendants de $\mathscr{F}_n$.

2.3 Arbres particuliers

2.3.1 Arbres à poids

Soit $\mathbf{t}^* = (\mathbf{t}, (\eta_u)_{u\in\mathbf{t}}) \in \mathbb{T}^*$ un arbre marqué, on définit pour tout $u \in \mathbf{t}$ une nouvelle quantité que l'on appelle masse de u et que l'on note m_u. Cet arbre particulier, n'est pas défini sur $\mathbb{T}^*_\infty$ seulement sur $\mathbb{T}^*$. En effet, l'étude sous $\mathbb{T}^*_\infty$ fait intervenir la restriction $r^*_{h,\infty}(.)$ qui tronque les individus s'ils ont plus de h enfants. Cette troncature pose problème dans la distribution de la masse qui dépend du nombre d'enfants à chaque génération. Toutes les masses changeraient lorsque h augmente pas seulement celles des individus tronqués.

On fixe $m_\emptyset = 0$ et les masses sont définies récursivement de la manière suivante. Soit $n \in \mathbb{N}$ on suppose que les masses $(m_u)_{|u|\leq n}$ des individus des générations précédentes sont connues et construites. Soit $v \in z_{n+1}(\mathbf{t}^*)$ qui peut s'écrire $v = uj$ avec $u \in \mathbf{t}$ et $1 \leq j \leq k_u(\mathbf{t})$. Alors on obtient,

$$m_{uj} = \frac{m_u + \eta_u}{k_u(\mathbf{t})} + \frac{\displaystyle\sum_{w\in z_n(\mathbf{t}^*), k_w(\mathbf{t})=0} (m_w + \eta_w)}{Z_{n+1}(\mathbf{t})}.$$

Cette formule peut être interprétée de la manière suivante :

— Le premier terme correspond au fait que u transmet sa masse, additionnée de un s'il est marqué, uniformément à tous ses enfants ;

— Le second terme signifie que tous les individus sans enfant à la génération n transmettent leur masse, additionnée de un s'il est marqué, uniformément à tous les individus de la génération $n+1$.

Par exemple, si on considère l'arbre suivant à 4 générations (Figure 2.6), pour obtenir la masse de l'individu 21 on a :

$$m_{21} = \frac{m_2 + \eta_2}{2} + \frac{m_3 + \eta_3}{Z_2} = \frac{\frac{1}{3} + 1}{2} + \frac{\frac{1}{3}}{3} = \frac{4}{6} + \frac{1}{9} = \frac{7}{9}.$$

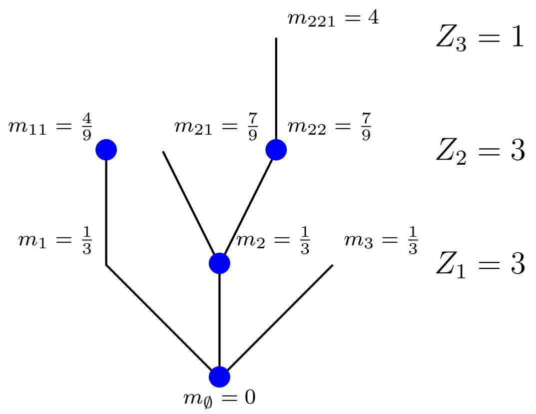


FIGURE 2.6 – Exemple d'un arbre marqué et ses masses associées.

Le parent 2 étant marqué, il a transmis, sa masse additionnée de 1, uniformément à ses deux enfants (terme $\frac{\frac{1}{3}+1}{2}$). A la génération 1, l'individu 3 n'a pas d'enfant et n'a pas de marque, il transmet sa masse, uniformément aux trois individus de la génération 2 (terme $\frac{\frac{1}{3}}{3}$).

Notons que le plan $\mathbf{t}^* \mapsto (m_u)_{u \in z_n(\mathbf{t}^*)}$ est $\mathscr{F}_n$-mesurable. De plus, par construction, on a le lien suivant entre la masse et le nombre de marques jusqu'à un certain seuil :

Proposition 2.1. *Soient $n \in \mathbb{N}$ et $\mathbf{t}^* \in \mathbb{T}_0^*$. Conditionnellement à $\mathscr{F}_n$, on a*

$$\sum_{u \in z_n(\mathbf{t}^*)} m_u = M_n(\mathbf{t}^*).$$

2.3.2 L'arbre de Kesten et l'arbre de condensation

On considère $\tau_l(\mathbf{p})$, représentant un arbre aléatoire limite dans la suite du document, défini par :

1. deux types de nœuds : normal ou spécial ;
2. la racine est spéciale ;
3. les nœuds normaux ont pour loi de reproduction $\mathbf{p}$;
4. les nœuds spéciaux ont pour loi de reproduction la loi biaisée $\check{\mathbf{p}}$ sur $\mathbb{N} \bigcup \{+\infty\}$ définie par :
$$\check{\mathbf{p}}(k) := \begin{cases} k\mathbf{p}(k) & \text{si } k \in \mathbb{N}, \\ 1 - \mu & \text{si } k = +\infty. \end{cases}$$
5. Le nombre d'enfants de tous les nœuds est indépendant de celui des uns des autres ;
6. tous les enfants d'un nœud normal sont normaux ;
7. quand un nœud spécial obtient un nombre fini d'enfants, l'un d'entre eux est sélectionné aléatoirement et uniformément pour être spécial tandis que les autres sont tous normaux ;
8. quand un nœud spécial obtient un nombre infini d'enfants, ils sont tous normaux.

On définit un arbre infini aléatoire $\tau_K(\mathbf{p})$ (l'arbre de Kesten) dont la loi est décrite dans [2] à la Sous-section 3.2. Cet arbre possède une branche infinie et tous ses nœuds sont de degrés finis. On note $\tau_K^*(\mathbf{p}, \mathbf{q})$ l'arbre de Kesten marqué, ayant pour fonction de marquage celle associée à la loi de reproduction $\mathbf{p}$ puisque le marquage est indépendant de la loi de reproduction.

On définit l'arbre aléatoire infini $\tau_C(\mathbf{p})$ (l'arbre de condensation) décrit dans [6] à la Sous-Section 3.2. Cet arbre n'a pas de branche infinie mais possède un nœud avec une infinité d'enfants. Cet arbre a été considéré dans [19] et [20]. On note $\tau_C^*(\mathbf{p}, \mathbf{q})$ l'arbre de condensation marqué, ayant la même fonction de marquage associée à $\mathbf{p}$ puisque celles-ci sont indépendantes.

On peut noter que :

— si $\mathbf{p}$ est critique, $\tau_l(\mathbf{p}) = \tau_K(\mathbf{p})$;

— si $\mathbf{p}$ est sous-critique, $\tau_l(\mathbf{p}) = \tau_C(\mathbf{p})$.

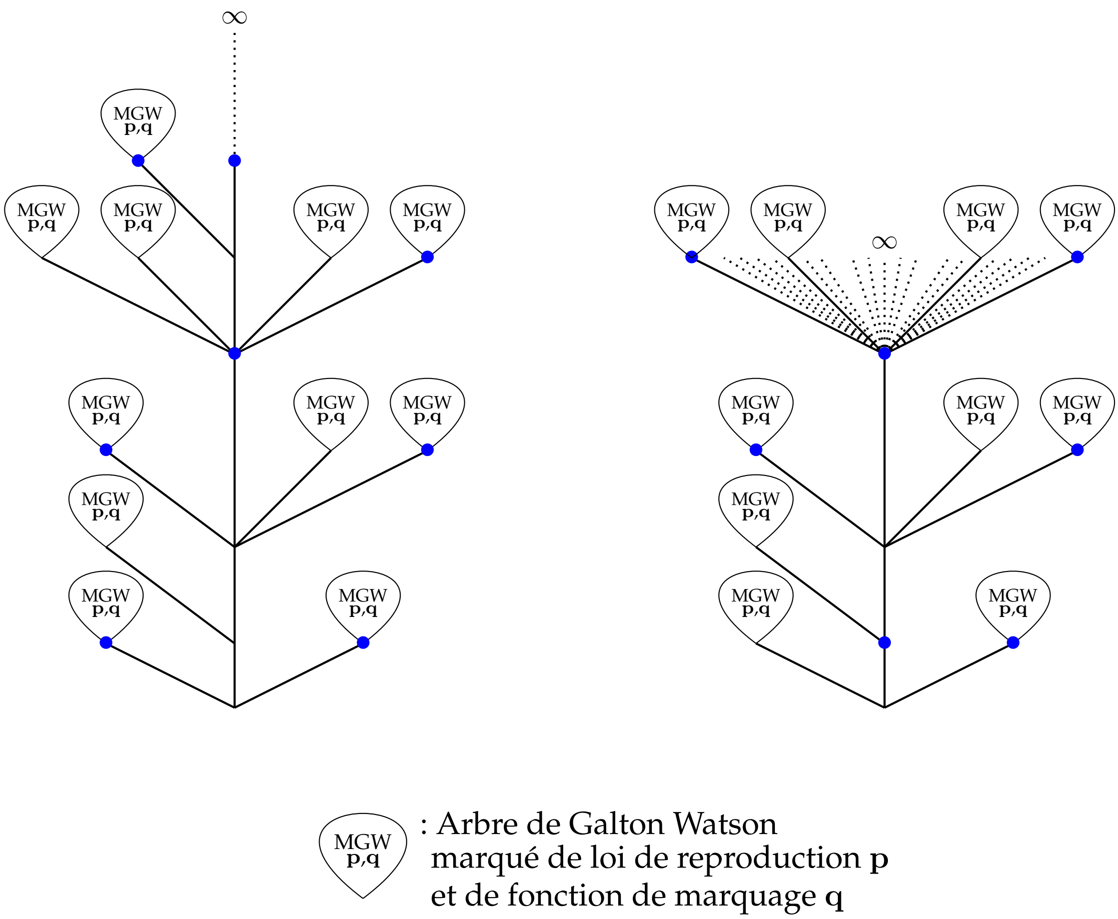


FIGURE 2.7 – Arbres de Kesten et de condensation marqués.

Chapitre 3

Conditionnement

En 2017, dans [2], Abraham, Bouaziz et Delmas ont conditionné le GW à avoir un très grand nombre de marques et prouvent une convergence locale vers un arbre de Kesten dans le cas critique. C'est à dire, qu'ils regardent la loi d'un GW τ, critique, conditionné à avoir un très grand nombre de marques et ils obtiennent à la limite la loi d'un Kesten.

Afin de poursuivre ces travaux, nous nous intéressons au cas sous-critique. On va alors distinguer deux cas, celui où l'on peut se comparer à l'étude d'un GW avec une loi biaisée critique et celui où l'on ne peut pas. Contrairement à Abraham, Bouaziz et Delmas, qui ont effectué une étude directe du conditionnement d'un GW par rapport au nombre de marques, nous allons passer par l'étude du conditionnement d'un MGW ayant un très grand nombre de marques. Puisque la loi d'un GW conditionné à avoir un grand nombre de marques peut être exprimée en fonction de la loi d'un MGW conditionné lui aussi à avoir un grand nombre de marques. Nous obtenons les résultats pour la loi d'un GW τ, sous-critique, conditionné.

L'ensemble des résultats, liés à ce chapitre, ont été publiés dans l'article "Local limits of conditioned marked Galton–Watson trees " [4], soumis dans un journal.

3.1 Conditionnement et loi biaisée

Lorsque l'on s'intéresse au cas sous-critique, il est parfois possible de se reporter à un cas critique, grâce à une loi biaisée. Soit $\theta > 0$, on considère $\mathbf{p}_\theta$ la fonction définie sur $\mathbb{N}$ par

$$\forall k \geq 0,\ \mathbf{p}_\theta(k) := \theta^{k-1}\mathbf{p}(k)\left(c_\theta \mathbf{q}(k) + 1 - \mathbf{q}(k)\right), \tag{3.1}$$

avec c_θ une constante de normalisation.
On note I l'ensemble des θ tel que $\mathbf{p}_\theta$ définisse une loi de probabilité sur $\mathbb{N}$ et $c_\theta > 0$. Notons que $1 \in I$ et $\mathbf{p}_1 = \mathbf{p}$.

On définit la fonction de marquage $\mathbf{q}_\theta$ sur $\mathbb{N}$ par

$$\mathbf{q}_\theta(k) = \frac{c_\theta \mathbf{q}(k)}{c_\theta \mathbf{q}(k) + 1 - \mathbf{q}(k)}, \text{ pour tout } k \geq 0.$$

On peut remarquer que l'on a pour tout $k \in \mathbb{N}$:

$$\mathbf{q}(k) > 0 \Leftrightarrow \mathbf{q}_\theta(k) > 0.$$

On a le résultat suivant,

Proposition 3.1. *Soit τ^* un MGW de loi de reproduction $\mathbf{p}$ sous-critique ($\mu(\mathbf{p}) < 1$) satisfaisant* (1.1) *et $\mathbf{q}$ sa fonction de marquage satisfaisant* (2.1).
Alors pour tout $n \in \mathbb{N}$, tel que $\mathbb{P}(M(\tau^*) = n) > 0$, la loi de τ^* sachant $\{M(\tau^*) = n\}$ et la loi de τ_θ^* sachant $\{M(\tau_\theta^*) = n\}$ sont les mêmes.

Grâce à ce résultat et à la définition suivante, on pourra revenir à l'étude d'un MGW critique :

Definition 3.1. *On dit que $\mathbf{p}$ est générique, s'il existe $\theta_c \in I$ telle que $\mathbf{p}_{\theta_c}$ soit critique.*

De plus, si θ_c existe, ce dernier est unique (voir la preuve du Lemma 8.4). On distinguera donc les cas sous-critiques génériques et les cas sous-critiques non-génériques.

3.2 Conditionnement par le nombre de marques

Dans le cadre du conditionnement de MGW critique, nous obtenons une extension du théorème de Abraham, Bouaziz et Delmas (2017,[2]).
Dans le cas sous-critique non générique, on ajoute quelques propriétés supplémentaires sur $\mathbf{p}$ et $\mathbf{q}$. On suppose qu'il existe $\alpha > 2$ et une fonction à variation lente $\mathscr{L}$ tels que, pour tout $k \geq 1$,

$$\mathbf{p}(k) = \mathscr{L}(k)k^{-(1+\alpha)}. \tag{3.2}$$

De plus, on suppose alors que $\mathbf{q}$ admet une limite à l'infini,

$$\lim_{n\to+\infty} \mathbf{q}(n) = \ell_{\mathbf{q}},\ \ell_{\mathbf{q}} \in [0,1]. \tag{3.3}$$

Si $\ell_{\mathbf{q}} = 1$, on suppose que $\mathbf{q}$ satisfait pour $k \in \mathbb{N}^*$,

$$1 - \mathbf{q}(k) = k^{-\beta}\mathcal{L}(k) \tag{3.4}$$

avec $\beta \geq 2$ et $\mathcal{L}$ une fonction à variation lente.
On note, $\mathrm{dist}(T)$ la distribution (la loi) de la variable aléatoire T et $\rho(\mathbf{p})$ le rayon de convergence de la fonction génératrice associée à la variable aléatoire X de loi $\mathbf{p}$.

L'un des principaux théorèmes est le suivant :

Théorème 3.1. *Soit $\tau^*(\mathbf{p},\mathbf{q})$ un MGW de loi de reproduction $\mathbf{p}$ satisfaisant* (1.1) *et ayant un moment d'ordre 2, et $\mathbf{q}$ satisfaisant* (2.1).
Dans le cas critique, on a

$$\mathrm{dist}(\tau^*(\mathbf{p},\mathbf{q})|M(\tau^*(\mathbf{p},\mathbf{q})) = n) \underset{n\to+\infty}{\longrightarrow} \mathrm{dist}(\tau_K^*(\mathbf{p},\mathbf{q})).$$

Dans le cas sous-critique générique ($\exists\theta_c \in I$ telle que $\mathbf{p}_{\theta_c}$ soit critique), tel que $\mathbf{p}_{\theta_c}$ soit critique et admette un moment d'ordre 2 (vrai si $\theta_c < \rho$), on a

$$\mathrm{dist}(\tau^*(\mathbf{p},\mathbf{q})|M(\tau^*(\mathbf{p},\mathbf{q})) = n) \underset{n\to+\infty}{\longrightarrow} \mathrm{dist}(\tau_K^*(\mathbf{p}_{\theta_c},\mathbf{q}_{\theta_c})).$$

Dans le cas sous-critique non générique ($\forall\theta \in I$ telle que $\mathbf{p}_\theta$ soit sous-critique), si $\mathbf{p}$ satisfait (3.2), *$\mathbf{q}$ satisfait* (3.3) *et, si $\ell_{\mathbf{q}} = 1$,* (3.4) *on a*

$$\mathrm{dist}(\tau^*(\mathbf{p},\mathbf{q})|M(\tau^*(\mathbf{p},\mathbf{q})) = n) \underset{n\to+\infty}{\longrightarrow} \mathrm{dist}(\tau_C^*(\mathbf{p},\mathbf{q})).$$

Grâce au théorème précédent, nous pouvons déterminer la limite locale de la loi d'un GW $\tau(\mathbf{p})$, sous-critique, conditionné à avoir un très grand nombre de marques. On a alors le corollaire suivant :

Corollaire 3.1. *Soit $\tau(\mathbf{p})$ un GW de loi de reproduction $\mathbf{p}$ satisfaisant* (1.1) *avec un moment d'ordre 2 et $\mathbf{q}$ sa fonction de marquage satisfaisant* (2.1).
Dans le cas sous-critique générique, telle que $\mathbf{p}_{\theta_c}$ soit critique et admette un moment d'ordre 2 (vrai si $\theta_c < \rho$), on a :

$$\mathrm{dist}(\tau(\mathbf{p})|M(\tau(\mathbf{p})) = n) \underset{n\to+\infty}{\longrightarrow} \mathrm{dist}(\tau_K(\mathbf{p}_{\theta_c})).$$

Dans le cas sous-critique non générique, si $\mathbf{p}$ satisfait (3.2), *$\mathbf{q}$ satisfait* (3.3) *et, si $\ell_{\mathbf{q}} = 1$,* (3.4) *on a*

$$\mathrm{dist}(\tau(\mathbf{p})|M(\tau(\mathbf{p})) = n) \underset{n\to+\infty}{\longrightarrow} \mathrm{dist}(\tau_C(\mathbf{p})).$$

Contrairement à Abraham, Bouaziz et Delmas [2], qui n'ont besoin que d'un moment d'ordre 1 pour obtenir le corollaire précédent, nous avons la condition supplémentaire d'avoir un moment d'ordre 2. Cela est dû au MGW $\tau^*(\mathbf{p}, \mathbf{q})$, et au fait qu'on ait besoin d'un moment d'ordre 2 lors de l'étude des marches aléatoires.

Malheureusement, nous n'avons pas de résultats dans le cas général où $\mathbf{p}$ est sous-critique non-générique. Cependant, dans ce cas, nous conjecturons le résultat suivant :

Conjecture 3.1. *Soit τ^* un MGW sous-critique, non générique, de loi de reproduction $\mathbf{p}$ satisfaisant* (1.1), *de fonction de marquage $\mathbf{q}$ satisfaisant* (2.1), *et $\rho_l(\mathbf{p}, \mathbf{q}) = 1$. Alors :*

$$\mathrm{dist}(\tau^*|M(\tau^*) = n) \underset{n\to+\infty}{\longrightarrow} \mathrm{dist}(\tau_C^*(\mathbf{p}, \mathbf{q})). \tag{3.5}$$

Avec pour tout $\theta \in I$, $l(\theta) = \mathbb{E}[\theta^X(1 - \mathbf{q}(X))]$ et $\rho_l(\mathbf{p}, \mathbf{q})$ son rayon de convergence.

Pour obtenir ce résultat, nous avons besoin d'équivalents, pour les sommes aléatoires stoppées aléatoirement, de la forme de ceux obtenus dans [10], mais dans un cadre plus général.
Grâce à ce résultat, on obtient :

Corollaire 3.2. *Supposons que $\mathbf{p}$ satisfait* (1.1), *$\mu(\mathbf{p}) < 1$, est non-générique, $\mathbf{p}_{\rho(\mathbf{p})}$ admette un moment d'ordre 2, et $\mathbf{q}$ satisfait* (2.1) *tel que, $1 \leq \rho(\mathbf{p}) = \rho_l(\mathbf{p}, \mathbf{q}) = \theta_s$ et satisfait* (8.15). *Alors on a :*

$$\lim_{n\to+\infty} \mathrm{dist}(\tau^*|M(\tau^*) = n) = \mathrm{dist}(\tau_C^*(\mathbf{p}_{\rho(\mathbf{p})}, \mathbf{q}_{\rho(\mathbf{p})})),$$
$$\lim_{n\to+\infty} \mathrm{dist}(\tau|M(\tau) = n) = \mathrm{dist}(\tau_C(\mathbf{p}_{\rho(\mathbf{p})})).$$

Remarque 3.1. *Rappelons que :*

$$l(\theta) = \sum_{k\geq 0} \theta^k \mathbf{p}(k)(1 - \mathbf{q}(k)),$$

et si $(\mathbf{q}(k))_{k\geq 0}$ admet une limite finie $\ell_{\mathbf{q}}$, on peut distinguer deux cas :
— *$\ell_{\mathbf{q}} \neq 1$ et clairement $\rho(\mathbf{p}) = \rho_l(\mathbf{p}, \mathbf{q})$;*
— *sinon, comme on a* (3.4), *il existe $\beta \geq 2$ tel que :*

$$\mathbf{p}(k)(1 - \mathbf{q}(k)) = \frac{\mathbf{p}(k)\mathcal{L}(k)}{k^\beta},$$

implique encore que $\rho(\mathbf{p}) = \rho_l(\mathbf{p}, \mathbf{q})$.

Chapitre 4

Pénalisation

Une autre façon d'obtenir un arbre avec un nombre "anormal" de marques est de faire un changement de mesure grâce à une transformation de Girsanov à l'aide d'une martingale qui donnera plus de poids aux arbres avec un grand nombre de marques ou au contraire une très faible quantité de marques. Afin de trouver une telle martingale, une méthode générale appelée *pénalisation* a été introduite par Roynette, Vallois et Yor [31, 32, 33] dans le cas du mouvement Brownien à une dimension.

Dans cette partie, on généralise le conditionnement à avoir un grand nombre de marques en utilisant cette nouvelle méthode, en pénalisant notre arbre par rapport au nombre de marques sur le MGW. En 2020, Abraham et Debs [5], ont obtenu des résultats en pénalisant des GW sans marque, on a donc choisi de compléter ces recherches dans le cadre de MGW (voir Section 9.2).

On note $\mu = \mu(\mathbf{p})$, et on étudie, pour plusieurs fonctions mesurables positives $\phi(x, y)$, la limite suivante

$$\lim_{p \to +\infty} \frac{\mathbb{E}\left[\mathbb{1}_{\Lambda_n} \phi\left(Z_{n+p}, M_{n+p}\right)\right]}{\mathbb{E}\left[\phi\left(Z_{n+p}, M_{n+p}\right)\right]} \tag{4.1}$$

avec $\Lambda_n \in \mathscr{F}_n$, où Z_n, M_n et $\mathscr{F}_n$ ont été définies à la Section 2.2. Selon la fonction choisie, nous allons favoriser des arbres possédant certaines propriétés, i.e. mettre un poids plus important sur ces derniers, d'où le terme de pénalisation. Si cette limite existe, elle est de la forme $\mathbb{E}[\mathbb{1}_{\Lambda_n} B_n]$ avec $(B_n)_{n\in\mathbb{N}}$ une $\mathscr{F}_n$-martingale positive telle que $B_0 = 1$ (voir Section 1.2 of [34]). Cela permet de définir une nouvelle mesure de probabilité $\mathbb{Q}$:

$$\forall \Lambda_n \in \mathscr{F}_n,\ \mathbb{Q}\left(\Lambda_n\right) = \mathbb{E}[\mathbb{1}_{\Lambda_n} B_n],$$

et on étudie la loi de notre MGW τ^* sous $\mathbb{Q}$.
Une partie des résultats, liés à ce chapitre, ont été publiés dans l'article "Penalization of Galton–Watson Trees with Marked Vertices" [3], en août 2024 dans "Journal of Theoretical Probability".

4.1 Pénalisation polynomiale

On commence avec la fonction $\phi\left(Z_p, M_p\right) = M_p^\ell$, avec $\ell \in \mathbb{N}^*$, qui dans ce cas va donner du poids aux arbres possédant un grand nombre de marques. On rappelle que M_p représente le nombre de marques jusqu'à la génération $p-1$.

Théorème 4.1. *Soient* $\mathbf{p}$ *satisfaisant* (1.1) *et admettant un moment d'ordre $\ell \in \mathbb{N}^*$, et* $\mathbf{q}$ *une fonction de marquage satisfaisant* (2.1). *Alors, pour tout $n \in \mathbb{N}$ et tout $\Lambda_n \in \mathscr{F}_n$, on a*

$$\lim_{p\to+\infty} \frac{\mathbb{E}[\mathbb{1}_{\Lambda_n} M_{n+p}^\ell]}{\mathbb{E}[M_{n+p}^\ell]} = \begin{cases} \mathbb{E}[\mathbb{1}_{\Lambda_n} f_\ell(M_n, Z_n)] & si\ \mu < 1, \\ \mathbb{E}[\mathbb{1}_{\Lambda_n} Z_n] & si\ \mu = 1, \\ \mathbb{E}\left[\mathbb{1}_{\Lambda_n} \frac{P_\ell(Z_n)}{\mu^{\ell n}}\right] & si\ \mu > 1, \end{cases}$$

avec P_ℓ une fonction polynomiale de degré ℓ et f_ℓ est définie pour tous $s, z \in \mathbb{N}$ par :

$$f_\ell(s,z) := \frac{1}{\xi_\ell} \sum_{i=0}^{\ell} \binom{\ell}{i} s^{\ell-i} \sum_{t_1+\ldots+t_z=i} \binom{i}{t_1 \ldots t_z} \prod_{j=1}^{z} \xi_{t_j}, \tag{4.2}$$

où $\binom{j}{t_1 \cdots t_z}$ est le coefficient multinomiale et $\xi_j := \lim_{p\to+\infty} \mathbb{E}[M_p^j]$ pour tout $j \geq 0$.

Remarque 4.1. *Si $\mu < 1$, puisque* $\mathbf{p}$ *admet un moment d'ordre ℓ et est sous-critique, le nombre total d'individus Z_∞ admet un moment d'ordre ℓ. Ainsi, le nombre de total de marques total aussi et l'on obtient l'existence de ξ_j pour tout $j \leq \ell$.*

Les deux martingales obtenues pour $\mu \geq 1$ sont déjà connues et l'étude sous $\mathbb{Q}$ n'est donc pas nécessaire.

Pour $\mu = 1$, on obtient un arbre de Kesten marqué et pour $\mu > 1$, on obtient des arbres multi-types marqués. L'arbre multi-type, sans marque, est décrit dans [5], Définition 4.4., dans notre cas on a :

— Les types de noeuds vont de 0 à ℓ;

— La racine est de type ℓ;

— Un noeud de type i à la génération n a, indépendamment des autres noeuds, k enfants ayant pour type respectifs $(t_1, \ldots, t_k)$ tel que $t_1 + \ldots + t_k = i$ avec une certaine probabilité décrite dans [5];

— Conditionnellement à cet arbre, tous les noeuds sont marqués indépendamment les uns des autres et un noeud u avec k_u enfants a une marque avec probabilité $\mathbf{q}(k_u)$.

Remarque 4.2. *On peut noter que l'arbre limite obtenu aura ℓ lignées spéciales suivant la loi biaisée* $\mathbf{p}^*$ *définie dans la Sous-section 2.3.2 et le reste des individus suivront la loi initiale* $\mathbf{p}$, *avec pour loi de marquage* $\mathbf{q}$.

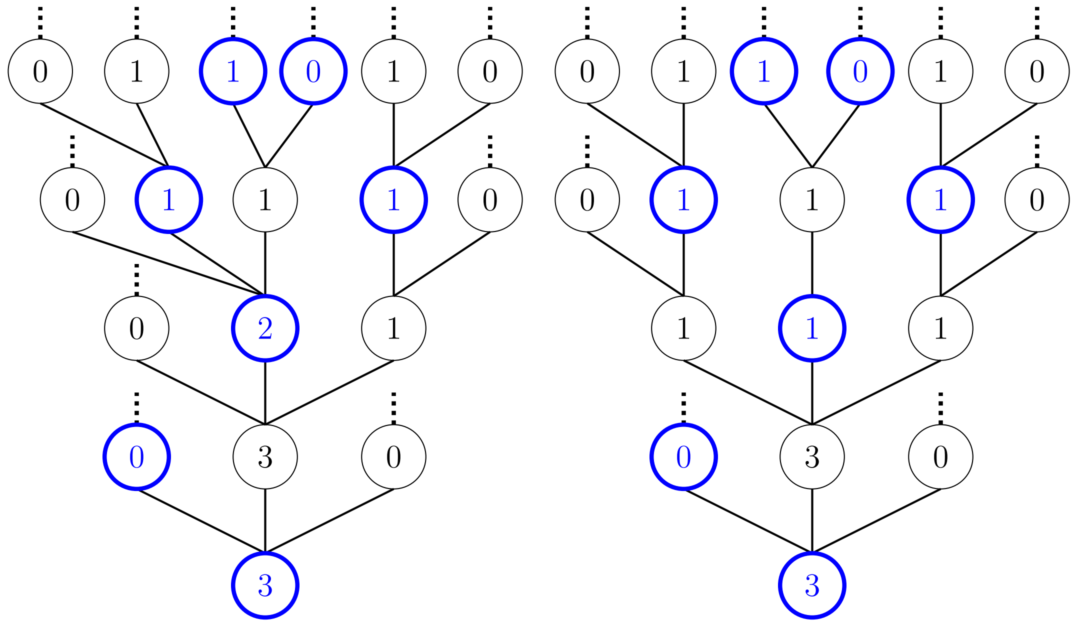


FIGURE 4.1 – Deux exemples d'arbres multitypes marqués (un noeud bleu est marqué, un noir ne l'est pas), dans le cas sur-critique pour $\ell = 3$.

Il reste à étudier le cas sous-critique ($\mu < 1$). Pour décrire la loi de reproduction de (Z, M) sous la nouvelle probabilité, on considère la *masse* d'un individu u notée m_u décrite dans la Section 2.2, cette quantité vaut zéro pour la racine et est transmise de génération en génération suivant certaines régles. Elle nous permet de décrire la nouvelle loi de reproduction et marquage sous $\mathbb{Q}$.

Nous pouvons ainsi décrire l'arbre à poids de type ℓ noté $\tau_\ell(\mathbf{p}, \mathbf{q})$. C'est un arbre multitype marqué :

— les types vont de 0 à ℓ ;

— à chaque génération, la somme des types doit être égale à ℓ ;

— le type de la racine est ℓ.

— S'il y a z individus à la génération n, ils ont respectivement les types $(t_1, \ldots, t_z)$ tels que $t_1 + \cdots + t_z = \ell$ avec une probabilité qui dépend de z, M_n et de la masse de chaque individu de la génération n (voir (9.17) pour la formule explicite).

De plus, si u est un individu de type $i \in [\![1, \ell]\!]$, sa loi de reproduction dépendra de la fonction f_i de la loi initiale $\mathbf{p}$, de la fonction de marquage $\mathbf{q}$ et de sa masse m_u, en effet u a k enfants et une marque $\eta \in \{0, 1\}$ avec probabilité :

$$\mathbf{p}_i(k, \eta) := \frac{\mathbf{p}_0(k, \eta) f_i(m_u + \eta, k)}{f_i(m_u, 1)}$$

avec $\mathbf{p}_0$ la loi de reproduction-marquage de l'arbre initial qui est aussi la loi de reproduction-marquage d'un nœud de type 0 (2.2).

Remarque 4.3. *Du fait que la masse se transmette de génération en génération, on perd la propriété d'indépendance et $\tau_\ell(\mathbf{p}, \mathbf{q})$ ne satisfait pas les propriétés de branchement.*

Remarque 4.4. *Si tous les noeuds en dessous de la génération n ne sont pas marqués, alors les masses m_u pour $u \in z_n$ sont nulles, et, de ce fait, pour tout $u \in z_n$ de type $i \geq 1$, on a*

$$\mathbf{p}_i(0,0) = 0.$$

Finalement, $\tau_\ell(\mathbf{p},\mathbf{q})$ ne peut pas s'éteindre sans noeuds marqués.

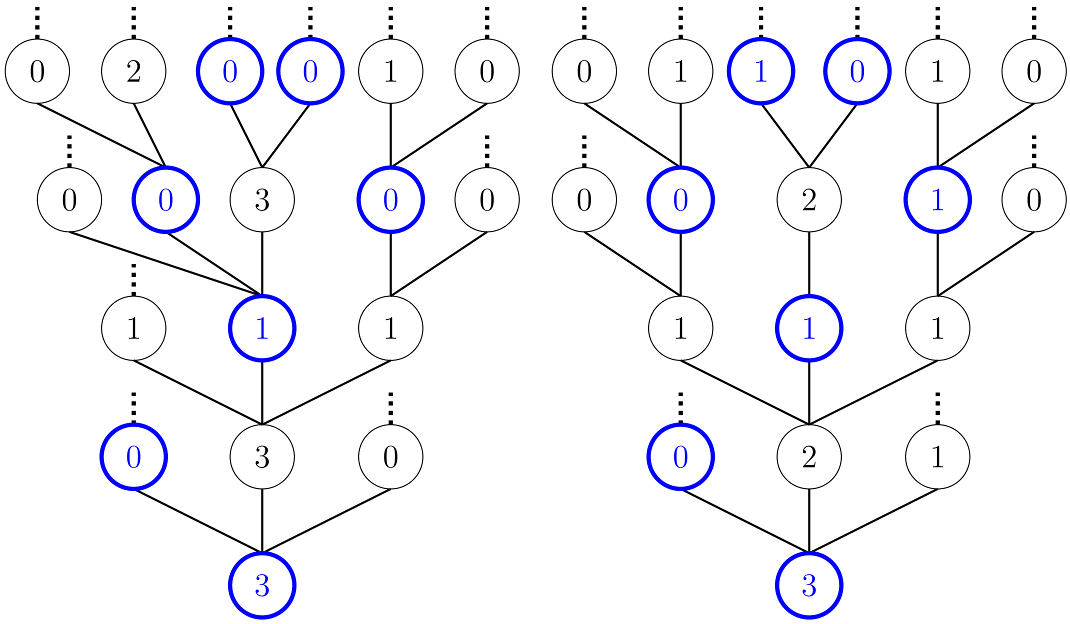


FIGURE 4.2 – Deux exemples d'arbres multitypes marqués, dans le cas sous-critique pour $\ell = 3$.

Remarque 4.5. *A la différence du cas sur-critique, le fait d'être d'un certain type n'influence pas le type des enfants. En effet, dans le cas sur-critique, si un individu est de type i à la génération n alors ses k enfants ont pour type respectifs $(t_1, \ldots, t_k)$ tel que $t_1 + \ldots + t_k = i$. Alors que dans le cas sous-critique, si on note $(t_1, \ldots, t_{z_{n+1}})$ les types des individus de la génération $n+1$, alors $t_1 + \ldots + t_{z_{n+1}} = \ell$.*
Autrement dit, dans le cas sur-critique un individu de type 0 peut donner naissance à seulement des individus de type 0 alors que dans le cas sous-critique, il peut avoir des enfants de type allant de 0 à ℓ.

Finalement, cet arbre nous permet de décrire le comportement des MGW sous $\mathbb{Q}$:

Théorème 4.2. *Soient $\mathbf{p}$ une loi de reproduction sous-critique satisfaisant (1.1) et admettant un moment d'ordre $\ell \in \mathbb{N}^*$, et $\mathbf{q}$ une fonction de marquage satisfaisant (2.1). Soit $\mathbb{Q}_\ell$ la mesure de probabilité définie par*

$$\forall n \in \mathbb{N},\ \frac{d\mathbb{Q}_\ell}{d\mathbb{P}}_{|\mathscr{F}_n} = f_\ell(M_n, Z_n).$$

Alors $\mathbb{Q}_\ell$ est la loi de l'arbre à poids de type ℓ.
Autrement dit, la loi de $\tau^*(\mathbf{p},\mathbf{q})$ sous $\mathbb{Q}$ est la même que celle de $\tau_\ell(\mathbf{p},\mathbf{q})$ sous $\mathbb{P}$.

Nous avons les propriétés suivantes sous $\mathbb{Q}_\ell$:

— La probabilité d'extinction de la population est égale à 1. En effet, en utilisant la définition de $\mathbb{Q}_\ell$, le théorème de convergence dominée et le fait que $\mathbb{P}$-presque sûrement $Z_\infty = 0$:

$$\begin{aligned}\lim_{n\to+\infty} \mathbb{Q}_\ell(Z_n = 0) &= \lim_{n\to+\infty} \mathbb{E}[\mathbb{1}_{Z_n=0} f_\ell(M_n, 0)] \\ &= \frac{1}{\xi_\ell} \lim_{n\to+\infty} \mathbb{E}[\mathbb{1}_{Z_n=0} M_n^\ell] = \frac{\mathbb{E}[M_\infty^\ell]}{\xi_\ell} = 1.\end{aligned}$$

— On peut noter que pour tout $n \in \mathbb{N}^*$, Z_n n'a pas nécessairement de moment d'ordre 1. En effet, supposons que Z_1 (et donc Z_n) n'admet pas de moment d'ordre $\ell + 1$ sous $\mathbb{P}$, alors,

$$\begin{aligned}\xi_\ell \mathbb{E}^{\mathbb{Q}_\ell}[Z_n] &= \xi_\ell \mathbb{E}[Z_n f(M_n, Z_n)] \\ &\geq \mathbb{E}\left[Z_n \sum_{\substack{t_1+\ldots+t_{Z_n}=\ell \\ t_i\leq 1, \forall i}} \binom{\ell}{t_1 \cdots t_{Z_n}} \prod_{i=1}^{Z_n} \xi_{t_i} \right] = \ell! \xi_1^\ell \mathbb{E}\left[Z_n \binom{Z_n}{Z_n - \ell} \right].\end{aligned}$$

$\mathbb{E}^{\mathbb{Q}_\ell}$ représente l'espérance sous la probabilité $\mathbb{Q}_\ell$.
Ainsi, Z_n n'a pas de moment d'ordre 1 sous $\mathbb{Q}_\ell$.

Par la suite, on s'intéresse à la fonction $\phi(Z_p, M_p) = M_p^\alpha Z_p^\beta$, avec $\alpha, \beta \in \mathbb{N}^*$. En effet, le cas $\alpha = 0$ ayant été étudié dans [5] et le cas $\beta = 0$ précédemment ainsi que dans [3], il m'est apparu naturel d'étudier le comportement lorsque l'on donne du poids à la fois sur le nombre de marques et sur le nombre d'individus.
Ici, on favorisera un arbre avec un très grand nombre de marques et/ou d'individus.

On a obtenu le résultat suivant :

Théorème 4.3. *Soient* $\mathbf{p}$ *une loi de reproduction satisfaisant* (1.1) *et admettant un moment d'ordre* $\alpha + \beta$ *avec* $\alpha, \beta \in \mathbb{N}^*$, *et* $\mathbf{q}$ *une fonction de marquage satisfaisant* (2.1). *Alors, pour tout* $n \in \mathbb{N}$ *et tout* $\Lambda_n \in \mathscr{F}_n$, *on a*

$$\lim_{p\to+\infty} \frac{\mathbb{E}[\mathbb{1}_{\Lambda_n} M_{n+p}^\alpha Z_{n+p}^\beta]}{\mathbb{E}[M_{n+p}^\alpha Z_{n+p}^\beta]} = \begin{cases} \mathbb{E}\left[\mathbb{1}_{\Lambda_n} \frac{Z_n}{\mu^n}\right] & si\ \mu < 1, \\ \mathbb{E}[\mathbb{1}_{\Lambda_n} Z_n] & si\ \mu = 1, \\ \mathbb{E}\left[\mathbb{1}_{\Lambda_n} \frac{P_{\alpha+\beta}(Z_n)}{\mu^{n(\alpha+\beta)}}\right] & si\ \mu > 1, \end{cases}$$

avec $P_{\alpha+\beta}$ *une fonction polynomiale de degré* $\alpha + \beta$.

L'étude des arbres marqués, sous les mesures de probabilités engendrées par les martingales obtenues, est déjà connue. Dans les cas sous-critique et critique, on étudie un arbre de Kesten marqué, dans le cas sur-critique on étudie des arbres marqués multi-types décrits dans [5].

Il est intéressant de noter qu'ayant mis du poids sur le nombre d'individus à la dernière génération, il n'est pas étonnant d'obtenir une convergence vers un Kesten dans les cas sous-critique et critique. Par ailleurs, la convergence vers un arbre multi-types décrits dans [5] semblait, aussi, naturelle dans le cas sur-critique. Néanmoins, il est étonnant de constater que la puissance obtenue dépend à la fois du nombre d'individus à la dernière génération, mais aussi du nombre total de marques.

4.2 Pénalisation exponentielle

Pour finir, nous avons choisi d'étudier la fonction $\phi\left(Z_p, M_p\right) = H_\ell(Z_p)s^{M_p}t^{Z_p}$, avec $s, t \in (0,1)$ et H_ℓ est le ℓ-ème polynôme de Hilbert défini par :

$$H_0(x) = 1 \text{ et } H_\ell(x) = \frac{1}{\ell!}\prod_{j=0}^{\ell-1}(x-j)\,, \text{ for } \ell \geq 1. \tag{4.3}$$

Contrairement aux études précédentes, cette fonction donne du poids aux arbres qui ont très peu de marques. On obtient les résultats suivants :

Théorème 4.4. *Soient* $\mathbf{p}$ *une loi de reproduction satisfaisant les deux dernier points de* (1.1) *et admettant un moment d'ordre* $\ell \in \mathbb{N}^*$*, et* $\mathbf{q}$ *une fonction de marquage satisfaisant* (2.1)*. On définit*

$$r = \min\{k \in \mathbb{N}, \mathbf{p}(k) > 0\}.$$

Alors pour tout $s \in (0,1)$*, si* $\kappa(s)$ *est l'unique point fixe sur* $[0,1]$ *de la fonction*

$$f_s(t) := \mathbb{E}[s^{M_1}t^{Z_1}],$$

on a pour tous $n \in \mathbb{N}$ *et* $\Lambda_n \in \mathscr{F}_n$*,*

- *si* $r = 0$*, pour tout* $t \in [0,1]$*,*

$$\lim_{p\to+\infty}\frac{\mathbb{E}[\mathbb{1}_{\Lambda_n}H_\ell(Z_{n+p})s^{M_{n+p}}t^{Z_{n+p}}]}{\mathbb{E}[H_\ell(Z_{n+p})s^{M_{n+p}}t^{Z_{n+p}}]} = \begin{cases} \mathbb{E}\left[\mathbb{1}_{\Lambda_n}s^{M_n}\kappa(s)^{Z_n-1}\right] & si\ \ell = 0, \\ E\left[\mathbb{1}_{\Lambda_n}\dfrac{s^{M_n}Z_n\kappa(s)^{Z_n-1}}{f_s'(\kappa(s))^n}\right] & si\ \ell > 0, \end{cases}$$

- *si* $r > 0$*, pour tout* $t \in (0,1]$*,*

$$\lim_{p\to+\infty}\frac{\mathbb{E}[\mathbb{1}_{\Lambda_n}H_\ell(Z_{n+p})s^{M_{n+p}}t^{Z_{n+p}}]}{\mathbb{E}[H_\ell(Z_{n+p})s^{M_{n+p}}t^{Z_{n+p}}]} = \begin{cases} \mathbb{E}\left[\mathbb{1}_{\Lambda_n}s^{M_n}\alpha_1(s)^{-n}\mathbb{1}_{Z_n=1}\right] & si\ r = 1, \\ \mathbb{E}\left[\mathbb{1}_{\Lambda_n}s^{M_n}\alpha_r(s)^{\frac{-(r^n-1)}{r-1}}\mathbb{1}_{Z_n=r^n}\right] & si\ r > 1, \end{cases}$$

avec $\alpha_r(s) = \mathbf{p}(r)(s\mathbf{q}(r) + 1 - \mathbf{q}(r))$*.*

Remarque 4.6. *Notons que le cas* $\mathbf{p}(0) = 0$ *implique* $\kappa(s) = 0$*. En effet, dans ce cas :*

$$f_s(0) = \sum_{k\geq 0} 0^k\mathbf{p}(k)(s\mathbf{q}(k) + 1 - \mathbf{q}(k)) = 0$$

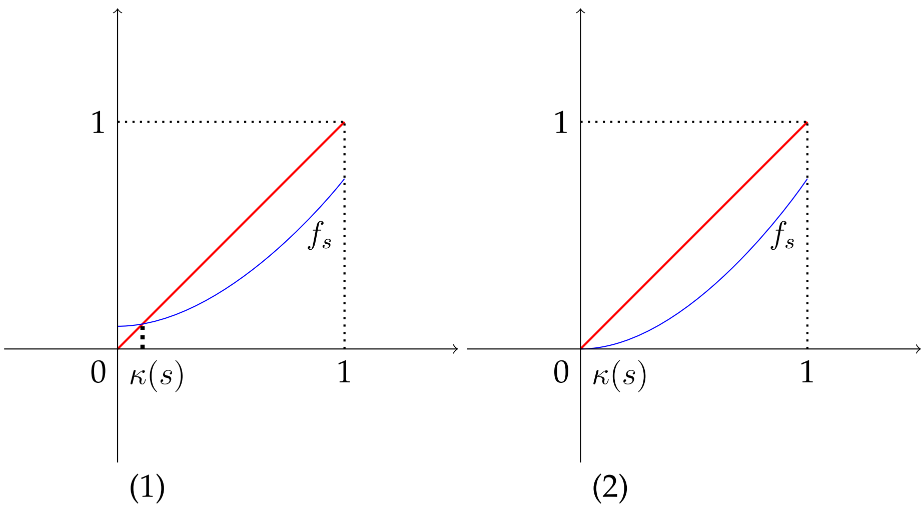


FIGURE 4.3 – Exemples de la fonction f_s et de son point fixe $\kappa(s)$. Cas $\mathbf{p}(0) > 0$ (1) et cas $\mathbf{p}(0) = 0$ (2).

Comme dans le premier cas de pénalisation que nous avons étudié, nous avons complètement décrit les arbres obtenus sous les nouvelles probabilités induites dans chacun des cas.

Théorème 4.5. *On suppose que $\mathbf{p}(0) > 0$. La mesure de probabilité $\mathbb{Q}_{s,1}$ définie sur $(\mathbb{T}^*, \mathscr{F})$ par*

$$\forall n \in \mathbb{N}, \Lambda_n \in \mathscr{F}_n, \quad \mathbb{Q}_{s,1}(\Lambda_n) = \mathbb{E}\left[\mathbb{1}_{\Lambda_n} s^{M_n} \kappa(s)^{Z_n - 1}\right],$$

est la loi d'un MGW ayant pour loi de reproduction-marquage $\mathbf{p}_{s,0}$ définie par :

$$\forall k \in \mathbb{N},\ \forall \eta \in \{0,1\}, \quad \mathbf{p}_{s,0}(k,\eta) = \mathbf{p}_0(k,\eta) s^{\eta} \kappa(s)^{k-1} \tag{4.4}$$

où $\mathbf{p}_0(k,\eta)$ est définie par (2.2)*.*

Remarque 4.7. *Nous pouvons remarquer que nous avons déjà obtenu cette loi de reproduction et de marquage. En effet, lorsque nous regardons le conditionnement à l'aide d'une loi biaisée 3.1. Il suffit de considérer $s = c_\theta$ et $\theta = \kappa(s)$ dans* (3.1)*.*

Remarque 4.8. *Par définition de la loi $\mathbf{p}_{s,0}$* (4.4)*, plus le nombre d'enfants est grand, plus la probabilité sera petite. Ainsi, contrairement au Théorème 4.1 qui favorise des arbres avec un grand nombre d'enfants et un grand nombre de marques. Ici, on favorisera un arbre avec très peu d'individus et très peu de marques.*

Afin de décrire la nouvelle mesure de probabilité $\mathbb{Q}_{s,2}$ défini sur $(\mathbb{T}^*, \mathscr{F})$ par :

$$\forall \Lambda_n \in \mathscr{F}_n,\ \mathbb{Q}_{s,2}(\Lambda_n) = \mathbb{E}\left[\mathbb{1}_{\Lambda_n} \frac{s^{M_n} Z_n \kappa(s)^{Z_n - 1}}{f_s'(\kappa(s))^n}\right], \tag{4.5}$$

on a besoin d'introduire le nouvel arbre aléatoire marqué suivant :

Definition 4.1. *L'arbre aléatoire marqué $\tau_{s,2}$ est un arbre aléatoire marqué multi-type décrit de la manière suivante :*

— *les nœuds sont normaux ou spéciaux ;*
— *la racine est spéciale ;*
— *la loi de reproduction-marquage d'un nœud normal est* $\mathbf{p}_{s,0}$ *(voir* (4.4)*) ;*
— *la loi de reproduction-marquage d'un nœud spécial est*

$$\mathbf{p}_{s,1}(k,\eta) = \mathbf{p}_{s,0}(k,\eta)\frac{k}{f_s'(\kappa(s))};$$

— *à chaque génération, il y a un unique nœud spécial choisi uniformément parmi tous les individus de cette génération.*

Remarque 4.9. *A la différence de l'arbre de Kesten, ici, le nœud spécial est choisi uniformément parmi tous les individus. Il n'y a donc pas de lignée spéciale.*

On a le résultat suivant :

Théorème 4.6. *On suppose* $\mathbf{p}(0) > 0$*, alors la probabilité* $\mathbb{Q}_{s,2}$ *définie par* (4.5) *est la loi de l'arbre aléatoire défini dans la Definition 4.1.*

Dans le cas où $\mathbf{p}(0) = 0$, on rappelle $r = \min\{k \in \mathbb{N},\ \mathbf{p}(k) > 0\}$, on a alors :

Théorème 4.7. *La mesure de probabilité* $\mathbb{Q}_{s,0,r}$ *définie sur* $(\mathbb{T}^*, \mathscr{F})$ *par*

$$\forall n \in \mathbb{N}, \Lambda_n \in \mathscr{F}_n, \quad \mathbb{Q}_{s,0,r}(\Lambda_n) = \mathbb{E}\left[\mathbb{1}_{\Lambda_n} B_{s,n,0,r}\right],$$

est la loi d'un arbre régulier r-aire où chaque nœud est marqué, indépendamment les uns des autres, avec la probabilité

$$\frac{s\mathbf{q}(r)}{s\mathbf{q}(r) + 1 - \mathbf{q}(r)}.$$

D'autre part, il est à noter que dans un souci de complétude, nous avons aussi traité le cas $s = 0$. Dans ce cas, on introduit $\psi(t) := \mathbb{E}\left[t^{Z_1}\mathbb{1}_{M_1=0}\right]$, $\tilde{r} := \min\{j \geq 0;\ \mathbf{p}(j)(1-\mathbf{q}(j)) > 0\}$ et $\tilde{\kappa}$ le plus petit point fixe positif de ψ.

Théorème 4.8. *Soient* $\mathbf{p}$ *une loi de reproduction satisfaisant les deux derniers points de* (1.1) *et admettant un moment d'ordre* $\ell \in \mathbb{N}^*$*, et* $\mathbf{q}$ *une fonction de marquage satisfaisant* (2.1). *On a pour tous* $n \in \mathbb{N}$ *et* $\Lambda_n \in \mathscr{F}_n$*,*

- *si* $\tilde{r} = 0$*, pour tout* $t \in (0,1]$*,*

$$\frac{\mathbb{E}\left[\mathbb{1}_{\Lambda_n} H_\ell(Z_{n+p}) t^{Z_{n+p}\mathbb{1}_{M_{n+p}=0}}\right]}{\mathbb{E}\left[H_\ell(Z_{n+p}) t^{Z_{n+p}}\mathbb{1}_{M_{n+p}=0}\right]} \underset{p\to+\infty}{\longrightarrow} \begin{cases} \mathbb{E}\left[\mathbb{1}_{\Lambda_n} B_{0,n,1}\right], & si\ \ell = 0 \\ \mathbb{E}\left[\mathbb{1}_{\Lambda_n} B_{0,n,2}\right], & sinon\ , \end{cases}$$

avec $B_{0,n,1} := \tilde{\kappa}^{Z_n - 1}\mathbb{1}_{M_n=0}$ *et* $B_{0,n,2} := \frac{Z_n \tilde{\kappa}^{Z_n-1}}{\psi'(\tilde{\kappa})^n}\mathbb{1}_{M_n=0}$.

- *si* $\tilde{r} \geq 1$*, pour tout* $t \in (0,1]$*,*

$$\frac{\mathbb{E}\left[\mathbb{1}_{\Lambda_n} H_\ell(Z_{n+p}) t^{Z_{n+p}\mathbb{1}_{M_{n+p}=0}}\right]}{\mathbb{E}\left[H_\ell(Z_{n+p}) t^{Z_{n+p}}\mathbb{1}_{M_{n+p}=0}\right]} \underset{p\to+\infty}{\longrightarrow} \begin{cases} \mathbb{E}\left[\mathbb{1}_{\Lambda_n} \mathbf{p}_0(1,0)^{-n}\mathbb{1}_{M_n=0, Z_n=1}\right], & si\ \tilde{r} = 1, \\ \mathbb{E}\left[\mathbb{1}_{\Lambda_n} \mathbf{p}_0(\tilde{r},0)^{\frac{-(\tilde{r}^n-1)}{\tilde{r}-1}}\mathbb{1}_{M_n=0, Z_n=\tilde{r}^n}\right], & si\ \tilde{r} > 1. \end{cases}$$

Sous la probabilité $\mathbb{Q}_{0,i}$ définie sur $(\mathbb{T}^*, \mathscr{F})$ par $\mathbb{Q}_{0,i}(\Lambda_n) = \mathbb{E}\left[\mathbb{1}_{\Lambda_n} B_{0,n,i}\right]$, pour :

— $i = 1$, τ^* est un MGW de loi de reproduction-marquage :

$$\mathbf{p}_{0,1}(k,\eta) = \mathbf{p}_0(k,\eta)\tilde{\kappa}^{k-1}\delta_{\eta 0};$$

— $i = 2$, τ^* est un arbre aléatoire marqué à deux types, il y a les nœuds normaux et les nœuds spéciaux. La loi de reproduction-marquage du premier est $p_{0,1}$ et celui du nœud spécial est :

$$\mathbf{p}_{0,2}(k,\eta) = \mathbf{p}_0(k,\eta)\frac{k\tilde{\kappa}^{k-1}}{\psi'(\tilde{\kappa})}\delta_{\eta 0},$$

et à chaque génération, un unique nœud spécial est choisi uniformément parmi tous les individus de cette génération.

Si $\tilde{r} \geq 1$, les probabilités obtenues, avec ces martingales, sont des mesures de Dirac sur des arbres réguliers $r-$aires sans marques.

Chapitre 5

Conclusion et perspectives

Le conditionnement d'arbres de Galton-Watson a commencé en 1986 et a perduré en se généralisant de plus en plus. Après avoir constaté que conditionner par rapport au nombre de feuilles ou au nombre total d'individus pouvait revenir à conditionner par rapport à l'évènement "le nombre d'individus ayant un nombre de descendants appartenant à un certain ensemble $\mathcal{A}$". Nous avons pu constater que ce conditionnement consistait à marquer des individus suivant si leur nombre de descendants appartenait à $\mathcal{A}$ ou non, et finalement, à conditionner par rapport au nombre de marques sur l'arbre. La suite logique était de mettre un peu d'aléatoire sur les individus qui allaient nous intéresser. Autrement dit, ce n'est pas parce qu'un individu à k enfants, avec $k \in \mathbb{N}$ que l'on sait déjà s'il est marqué ou non. En effet, nous avons introduit $\mathbf{q} : \mathbb{N} \to [0, 1]$, qui nous donne la probabilité qu'un individu ayant k enfants soit marqué ou non.

L'un des objectifs de ma thèse est de décrire la convergence locale lorsque l'on conditionne par rapport au nombre de marques (3).

J'ai commencé par m'intéresser à la loi d'un MGW conditionné à avoir un grand nombre de marques. En supposant que la loi de reproduction ait un moment d'ordre 2, on obtient, dans le cas critique, une convergence locale vers un arbre de Kesten marqué. Cet arbre étant un MGW, conditionné à ne pas s'éteindre, en possédant une lignée infinie. Ce dernier est donc infiniment grand.

Toujours sous l'hypothèse d'un moment d'ordre 2, j'ai considéré le cas sous-critique. On peut alors distinguer deux cas. Le cas générique, il existe une loi biaisée critique, dépendant de la loi de reproduction initiale, de sorte que l'on converge localement vers un Kesten marqué de loi de reproduction la loi biaisée. Le cas non-générique, il n'est alors pas possible de se ramener au cas critique. D'après l'étude faite, dans le cas déterministe, par Abraham et Delmas [6], le résultat attendu était une convergence locale vers un arbre de condensation marqué. Contrairement à l'arbre de Kesten, cet arbre n'est pas seulement de hauteur infinie mais est aussi infiniment large. En effet, l'un des individus de la lignée spéciale a la possibilité d'avoir une infinité d'enfants. Toutefois, pour obtenir cette convergence locale, dans le cadre des arbres marqués, il est nécessaire de comprendre comment marquer ce nœud "infini". C'est pourquoi, il a fallu ajouter des hypothèses supplémentaires à la fois sur la fonction de marquage et sur la loi de reproduction. En supposant que la loi de reproduction est α-stable et que la fonction de marquage admet une limite en l'infini, nous obtenons une convergence locale vers un arbre de condensation marqué.
A l'aide de ces résultats, j'ai pu compléter ceux obtenus par Abraham, Bouaziz et Delmas

[2]. En effet, dans le cas sous-critique, conditionné à avoir un grand nombre de marques, un GW générique converge vers un Kesten de loi de reproduction biaisée et un GW non générique converge vers un arbre de condensation. Contrairement au résultat obtenu dans [2], qui s'obtenait avec, seulement, un moment d'ordre 1, il est nécessaire d'avoir un moment d'ordre 2.

Une autre façon d'obtenir un arbre avec un nombre "anormal" de marques est de faire un changement de mesure grâce à une transformation de Girsanov à l'aide d'une martingale qui donnera plus de poids aux arbres avec un grand nombre de marques ou au contraire une très faible quantité de marques. Ainsi, on généralise le conditionnement à avoir un nombre anormal de marques, en pénalisant notre arbre par rapport au nombre de marques sur le MGW (4).

La première fonction de pénalisation utilisée est la suivante $\phi\left(Z_p, M_p\right) = M_p^\alpha Z_p^\beta$, avec $\alpha, \beta \in \mathbb{N}$, on suppose alors que la loi de reproduction admet un moment d'ordre $\alpha + \beta$.

Le cas $\alpha = 0$ étant traité dans [5], j'ai commencé par étudier le cas où $\beta = 0$. Je décris le comportement des MGW sous la nouvelle mesure de probabilité, définie à l'aide de la martingale obtenue. Dans le cas critique, le comportement des MGW sous la nouvelle mesure est le même que celui d'un Kesten marqué. Dans le cas sur-critique, le comportement est décrit par un arbre multi-type, dans [5], marqué avec la même fonction de marquage $\mathbf{q}$. Contrairement aux deux cas précédents, le cas sous-critique n'était pas connu. La difficulté a été d'interpréter la martingale obtenue afin de comprendre les arbres sous la nouvelle mesure de probabilité. On a donc eu besoin d'introduire la notion de masse (2.3.1). Ainsi l'arbre obtenu est un arbre multi-type marqué, où chaque type possède une loi de reproduction-marquage distincte. Autrement dit, l'individu est marqué en même temps qu'il a son nombre d'enfants. Du fait, que la masse intervienne, nous perdons l'indépendance entre chaque génération et l'arbre obtenu ne possède pas les propriétés de branchement.

Par la suite, je me suis intéressée au cas où $\alpha, \beta \in \mathbb{N}^*$. Dans les cas critique et sous-critique, nous obtenons le comportement d'un arbre de Kesten marqué et dans le cas sur-critique, nous obtenons de nouveau un arbre multi-type, décrit dans [5], marqué avec la même fonction de marquage $\mathbf{q}$.

Cette pénalisation favorise les arbres à avoir un grand nombre de marques. De ce fait, je me suis intéressée à une pénalisation qui favoriserait les arbres à avoir un faible nombre de marques. Ainsi, j'ai étudié la fonction $\phi\left(Z_p, M_p\right) = H_\ell(Z_p) s^{M_p} t^{Z_p}$, avec $s, t \in (0,1)$ et H_ℓ est le ℓ-ème polynôme de Hilbert, en supposant que la loi de reproduction admet un moment d'ordre ℓ. On distingue différents cas, suivant s'il est possible de ne pas avoir d'enfants ou non.

Si la probabilité de ne pas avoir d'enfant est strictement positive, alors nous obtenons deux types d'arbres possibles. Si $\ell = 0$, nous avons un MGW avec une loi de reproduction-marquage définie à l'aide des lois de reproduction et de marquage initiale.
Lorsque, $\ell > 0$ nous obtenons un arbre avec deux types. La loi de reproduction-marquage des nœuds normaux est la même que celle du cas précédent, et celle des nœuds spéciaux est biaisée. Contrairement au Kesten, il n'y a pas de lignée spéciale et à chaque génération le nœud spécial est choisi uniformément.

Dans le cas où la probabilité de ne pas avoir d'enfants est nulle, nous obtenons des arbres r-aires marqués, avec r le plus petit nombre de descendants possible. Nous nous sommes aussi intéressés au cas où $s = 0$, nous avons obtenu des arbres similaires avec la condition qu'il n'y ait pas de marques.

Travailler sur ces deux points de recherche m'a paru pertinent et intéressant, vis-à-vis de la littérature existante. D'autres questions se posent, que ce soit du côté de la pénalisation ou du conditionnement.

En effet, en ce qui concerne la partie conditionnement 3, nous n'avons malheureusement pas obtenu de résultat général dans le cas sous-critique non générique. Cependant, nous conjecturons l'obtention de résultats similaires. Pour ce faire, nous avons besoin de résultats plus généraux sur les probabilités locales de somme indexée aléatoirement. De plus, Abraham, Bi et Delmas, ont généralisé le cadre de [7, 6], qui présentaient un mono conditionnement pour déterminer des résultats de convergence locale dans le multi-conditionnement [1]. Ils considèrent $A = (A_i)_{i \in [\![1,J]\!]}$, avec J un entier strictement positif, une collection d'ensembles disjoints de $\mathbb{N}$ et $\alpha = (\alpha_i)_{i \in [\![1,J]\!]}$ une mesure de probabilité sur $[\![1, J]\!]$. Ils conditionnent des arbres de GW à avoir $\lfloor \alpha_i n \rfloor$ nœuds avec un nombre d'enfants appartenant à A_i pour tout $i \in [\![1, J]\!]$ quand n tend vers l'infini. Dans ce cadre, ils ont obtenu des convergences dans les cas critique et sous-critique générique vers un arbre de Kesten. J'aimerais regarder ce qu'il se passe dans le cas sous-critique non générique. Il semblerait que l'on obtiendrait une convergence vers un arbre de condensation. Ces deux arbres limites étant très différents, les méthodes le sont aussi, ainsi l'adaptation des travaux déjà effectués n'est pas directe. De plus, il serait intéressant de voir ce qu'il se passe lorsque l'on souhaite imposer plus de conditions sur les α_i. Par exemple, en souhaitant imposer la répartition limite en donnant plus de poids à certains α_i.

Pour finir, en ce qui concerne, la partie pénalisation 4, j'aimerais généraliser notre résultat dans le cadre réel. Une première étape serait d'obtenir les résultats de Abraham et Debs [5] avec Z_p^ℓ, ℓ réel, puis de l'adapter à M_p^ℓ. Une autre étude, qu'il serait intéressant de mener, est celle du comportement lorsque ℓ dépend de p et p tend vers l'infini.

Deuxième partie

Main Body of the Thesis

Chapter 6

Introduction

The Galton-Watson process was introduced by Bienaymé in 1845 and independently by Galton in 1873 in order to study the disappearance of surnames.

This process is a very simple model of population growth where all individuals give birth independently of each other to a random number of children according to the distribution $\mathbf{p}$. In other words, to each generation each individual could have k children with probability $\mathbf{p}(k)$. Thus $\mathbf{p}$ is called the offspring distribution.
Let $\mathbf{p} := (\mathbf{p}(n), n \in \mathbb{N})$ be an offspring distribution satisfying:

$$\mathbf{p}(0) > 0\ ,\ \mathbf{p}(0) + \mathbf{p}(1) < 1 \text{ and } \mu(\mathbf{p}) < +\infty, \tag{6.1}$$

$\mu(\mathbf{p}) := \sum_{n=0}^{+\infty} n\mathbf{p}(n)$ is its mean. If $\mu(\mathbf{p}) < 1$ (resp. $\mu(\mathbf{p}) = 1$, $\mu(\mathbf{p}) > 1$), we say that the offspring distribution is sub-critical (resp. critical, super-critical). In the sub-critical and critical case we have almost surely population extinction.

In 1986, Neveu introduced the notion of Galton-Watson tree (see [29]). This tree is a random genealogical tree, noted τ, that describes the population growth associated with offspring distribution $\mathbf{p}$. When $\mathbf{p}$ is sub-critical (resp. critical, super-critical), the Galton-Watson tree τ is also sub-critical (resp. critical, super-critical).

Conditioning critical or sub-critical Galton-Watson trees comes from the work of Kesten, in 1986, [24]. In the sub-critical or critical cases, the tree is almost surely finite, but Kesten considered in [24] the local limit of a sub-critical or critical tree conditioned to have height greater than n. When n goes to infinity, this conditioned tree converges in distribution to an infinite tree called here Kesten's tree. This tree has an infinite spine. A random number of independent Galton-Watson trees, with the same offspring distribution $\mathbf{p}$, are grafted to this spine. This limit tree can be seen as a Galton-Watson tree conditioned on non-extinction.

Since then, other ways of conditioning a critical Galton-Watson tree to be large has been considered: large total progeny (see Kennedy [23] and Geiger and Kaufman [18]), large number of leaves (see Curien and Kortchemski [11]). In the sub-critical and critical cases, Rizzolo [30] introduces the conditioning on having a large number of individuals with the number of offsprings belonging to a given set $\mathcal{A}$, it also appears in the paper of Kortchemski [26]. All these cases are contained in the general result of Abraham-Delmas [7] and the limiting tree in the critical case is always the Kesten's tree associated with $\mathbf{p}$. In [6], Abraham and Delmas obtained local convergence to a condensation tree in the sub-critical and non generic case; in the sub-critical and generic case, they obtained a convergence to a Kesten tree with a biased reproduction law. Indeed, some offspring distributions can give the same law for conditioned trees, thereby, to be generic signifies we can find a biased critical law, otherwise, if we cannot find a biased critical law, we are in the non generic case. In others words, $\mathbf{p}$ is generic, if it exists a (unique) biased distribution, depending on $\mathbf{p}$ and critical.

Condensation tree, contrary to Kesten tree, is abnormally large. In fact, at each generation an unique individual has a biased reproduction law that permits to him to have an infinity of children. The others individuals are the same original reproduction law $\mathbf{p}$. The moment, where an individual has an infinity of children, every individual are the same reproduction law $\mathbf{p}$. This tree was introduce by Jonnson and Stefanson [20] and also by Janson [19].

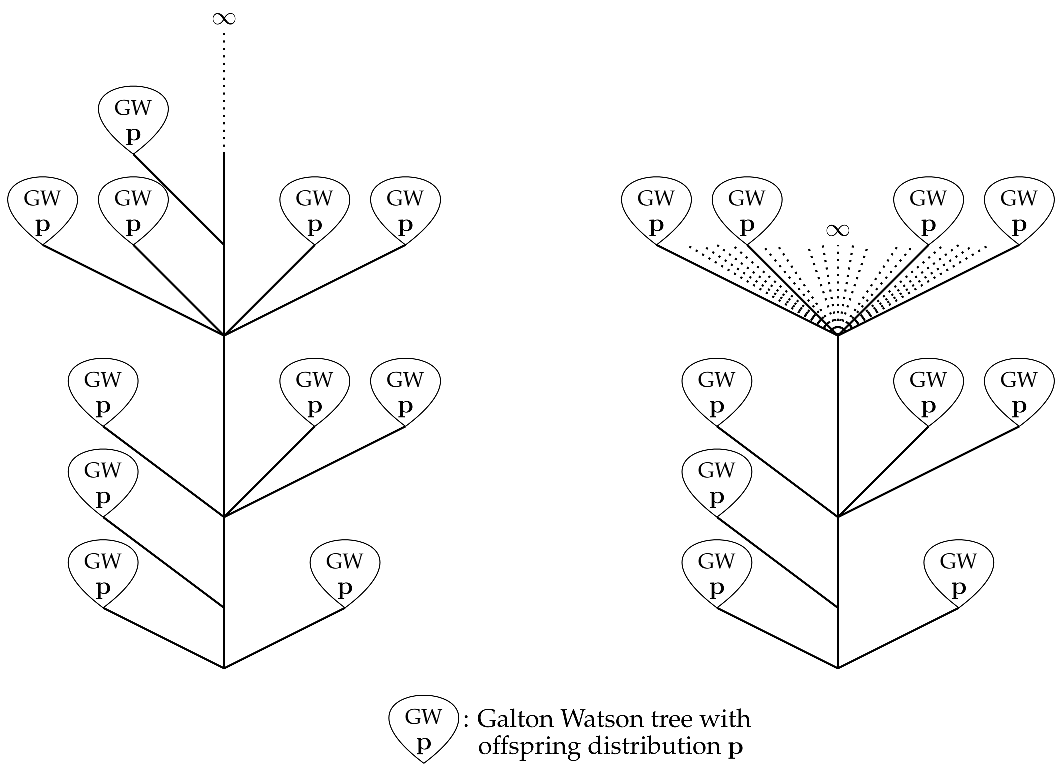


Figure 6.1 – Kesten and condensation trees

In 2017, in [2], Abraham, Bouaziz and Delmas generalized this approach by marking the nodes randomly where, conditionnally on the tree, the nodes are marked independently of each others with a probability that may depend on their out-degree. More precisely, independently of each other, each individual gives birth to k children and is

- marked with probability $\mathbf{p}_0(k_u, 1) := \mathbf{p}(k_u)\mathbf{q}(k_u)$;
- unmarked with probability $\mathbf{p}_0(k_u, 0) := \mathbf{p}(k_u)(1 - \mathbf{q}(k_u))$

where $\mathbf{q} := (\mathbf{q}(k))_{k\geq 0}$ is a sequence of numbers in $[0, 1]$ and is called the mark function. The probability distribution $\mathbf{p}_0$ on $\mathbb{N} \times \{0, 1\}$ is called here the marking-reproduction law of the associated marked Galton-Watson tree (MGW). We always suppose that the condition

$$\exists k \in \mathbb{N}^*,\ \mathbf{p}(k)\mathbf{q}(k) > 0 \tag{6.2}$$

holds so that every individual of a MGW is marked with positive probability.

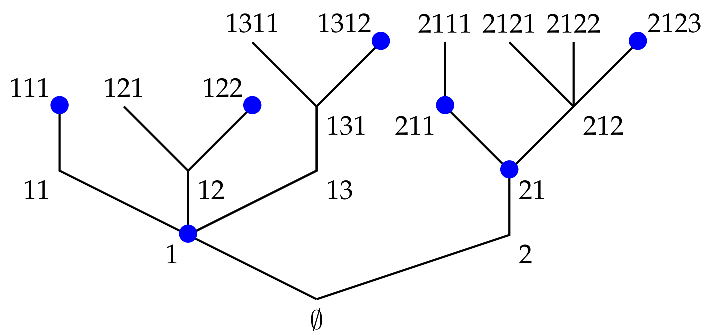


Figure 6.2 – A marked tree $\mathbf{t}^*$ with 5 generations.

Then they condition the tree on having a large number of marked nodes and prove a local convergence of this conditioned marked tree toward Kesten's tree (as for the other conditionnings) as the number of marked nodes tends to infinity.
The situation in the sub-critical case is more involved (see Janson [19] when conditioning on the total progeny, and Abraham-Delmas [6] when conditioning the number of nodes with offspring in a given set $\mathcal{A}$). Indeed two limits may appear: a Kesten's tree associated with a modified offspring distribution in the so-called generic case, or a condensation tree (a tree with a node with infinite out-degree) in the non-generic case. To this aim, we first need to introduce a one-parameter family of modified mark-reproduction law.

For every $\theta > 0$,

$$\forall k \geq 0,\ \mathbf{p}_\theta(k) := \theta^{k-1}\mathbf{p}(k)\left(c_\theta \mathbf{q}(k) + 1 - \mathbf{q}(k)\right), \tag{6.3}$$

where c_θ is a normalizing constant given by:

$$c_\theta := \frac{1 - \mathbb{E}\left[\theta^{X-1}(1 - \mathbf{q}(X))\right]}{\mathbb{E}\left[\theta^{X-1}\mathbf{q}(X)\right]}, \tag{6.4}$$

and X is a random variable distributed according to $\mathbf{p}$. We also define a new mark function denoted by $\mathbf{q}_\theta$:

$$\forall k \geq 0,\ \mathbf{q}_\theta(k) := \frac{c_\theta \mathbf{q}(k)}{c_\theta \mathbf{q}(k) + 1 - \mathbf{q}(k)}. \tag{6.5}$$

Let I the set of positive θ such that $\mathbf{p}_\theta$ defines a probability distribution on $\mathbb{N}$. If $\mathbf{p}$ is sub-critical, according to Lemma 8.4 either there exists a unique $\theta_c \in I$ such that $\mathbf{p}_{\theta_c}$ is critical, or $\theta_s := \max I \in I$ and $\mathbf{p}_{\theta_s}$ is sub-critical. We say that $\mathbf{p}$ is generic in the former case and that $\mathbf{p}$ is non-generic for the latter case. As in [6], we have the same properties of generic and non-generic case see Definition 8.1 . We have some properties in the non-generic case, explained in Lemma 8.5 .

For technical reasons, we need some additional assumptions on $\mathbf{p}$ and $\mathbf{q}$. We assume that there exists some $\alpha > 2$ and some slowly varying function (SV) $\mathscr{L}(.)$ such that, for all $k \geq 1$,

$$\mathbf{p}(k) = \mathscr{L}(k)k^{-(1+\alpha)}, \tag{6.6}$$

Moreover, we also assume that $\mathbf{q}$ admits a finite limit at infinity, so we have:

$$\lim_{n\to+\infty} \mathbf{q}(n) = \ell_{\mathbf{q}},\ \ell_{\mathbf{q}} \in [0,1]. \tag{6.7}$$

If $\ell_{\mathbf{q}} = 1$, we assume that the mark function $\mathbf{q}$ satisfies for $k \in \mathbb{N}^*$,

$$1 - \mathbf{q}(k) = k^{-\beta}\mathcal{L}(k) \tag{6.8}$$

with $\beta \geq 2$ and $\mathcal{L}(.)$ is a slowly varying function.

We present our two main results. We denote by $M(\mathbf{t}^*)$ the number of marked of this tree, and by $\rho(\mathbf{p})$ the convergence radius of the generative function associated to $\mathbf{p}$. Finally, we denote by $\mathrm{dist}(T)$ the distribution of the random variable T.
The first result, explained in Section 8.2 , is the following:

Proposition 6.1. *For every $\theta \in I$, if $\tau^*(\mathbf{p},\mathbf{q})$ is a sub-critical marked Galton-Watson tree with offspring distribution $\mathbf{p}$ and mark function $\mathbf{q}$, and $\tau^*(\mathbf{p}_\theta,\mathbf{q}_\theta)$ is a marked Galton-Watson tree with offspring distribution $\mathbf{p}_\theta$ and mark function $\mathbf{q}_\theta$, then the conditional distribution of $\tau^*(\mathbf{p},\mathbf{q})$ given $\{M(\tau^*(\mathbf{p},\mathbf{q})) = n\}$ and of $\tau^*(\mathbf{p}_\theta,\mathbf{q}_\theta)$ given $\{M(\tau^*_\theta(\mathbf{p}_\theta,\mathbf{q}_\theta)) = n\}$ are the same.*

The second result is explained in Sub-sections 8.5.1 and 8.5.2 . We denote by $\tau^*_K(\mathbf{p},\mathbf{q})$ a marked Kesten's tree and $\tau^*_C(\mathbf{p},\mathbf{q})$ a marked condensation tree (see Sub-section 7.4).

Theorem 6.1. *Let $\tau^*(\mathbf{p},\mathbf{q})$ be a marked Galton-Watson tree with offspring distribution $\mathbf{p}$ satisfying* (6.1), *and mark function $\mathbf{q}$ satisfying* (6.2).
In the critical case we have

$$\mathrm{dist}(\tau^*(\mathbf{p},\mathbf{q})|M(\tau^*(\mathbf{p},\mathbf{q})) = n) \underset{n\to+\infty}{\to} \mathrm{dist}(\tau^*_K(\mathbf{p},\mathbf{q})).$$

Generic sub-critical case ($\exists\theta_c \in I$ s.t. $\mathbf{p}_{\theta_c}$ critical), if $\mathbf{p}_{\theta_c}$ admits a moment of order 2 (always true for $\theta_c < \rho(\mathbf{p})$) we have

$$\mathrm{dist}(\tau^*(\mathbf{p},\mathbf{q})|M(\tau^*(\mathbf{p},\mathbf{q})) = n) \underset{n\to+\infty}{\to} \mathrm{dist}(\tau^*_K(\mathbf{p}_{\theta_c},\mathbf{q}_{\theta_c})).$$

Non-generic sub-critical case ($\forall\theta \in I$, $\mathbf{p}_\theta$ is sub-critical), if $\mathbf{q}$ satisfied (6.7) *and, if $\ell_{\mathbf{q}} = 1$* (6.8) *we have*

$$\mathrm{dist}(\tau^*(\mathbf{p},\mathbf{q})|M(\tau^*(\mathbf{p},\mathbf{q})) = n) \underset{n\to+\infty}{\to} \mathrm{dist}(\tau^*_C(\mathbf{p},\mathbf{q})).$$

Another way of getting a tree with an "abnormal" number of marked vertices is to make a change of measure via a Girsanov transformation with a martingale which gives more weight to trees with a large number of marks. In order to find such a martingale, a general method called penalization has been introduced by Roynette, Vallois and Yor [31, 32, 33] in the case of the one-dimensional Brownian motion to generate new martingales and to define, by change of measure, Brownian-like process conditioned on some specific zero-probability events.
We generalize the conditioning on having a large number of marked vertices, in the critical and sub-critical cases. We also decided to look the Galton-Watson process's behavior when we penalize by the total number of mark in the super-critical case. In 2020, Abraham and Debs [5], obtained some results about penalization without marks, we completed this research with marks in Section 9.2 .

Let us denote by M_n the number of marks until generation $n-1$, for $n \in \mathbb{N}^*$ and $M_0 = 0$, and let $(\mathscr{F}_n)_{n\geq 0}$ be the natural filtration generated by the tree up to generation n and marks up to generation $n-1$. Notice that we need to know the number of offspring of a node to decide whether it is marked or not. That is why, if we look at the tree up to generation n, we only have information for the marks up to generation $n-1$, which justifies the definition of the filtration $(\mathscr{F}_n)_{n\geq 0}$.

The aim of this part is the study, for some measurable positive functions $\phi(x,y)$, of the limit

$$\lim_{p\to+\infty} \frac{\mathbb{E}\left[\mathbb{1}_{\Lambda_n}\phi\left(Z_{n+p}, M_{n+p}\right)\right]}{\mathbb{E}\left[\phi\left(Z_{n+p}, M_{n+p}\right)\right]} \tag{6.9}$$

where $\Lambda_n \in \mathscr{F}_n$. If this limit exists, it has the form $\mathbb{E}[\mathbb{1}_{\Lambda_n} B_n]$ where $(B_n)_{n\in\mathbb{N}}$ is a positive $\mathscr{F}_n$-martingale such that $B_0 = 1$ (see Section 1.2 of [34]).
To show that B_n is a $\mathscr{F}_n$-martingale, we use the following property. Since $\Lambda_n \in \mathscr{F}_n \subset \mathscr{F}_{n+1}$ we have:

$$\frac{\mathbb{E}\left[\mathbb{1}_{\Lambda_n}\phi\left(Z_{n+p}, M_{n+p}\right)\right]}{\mathbb{E}\left[\phi\left(Z_{n+p}, M_{n+p}\right)\right]} \underset{p\to+\infty}{\longrightarrow} \mathbb{E}\left[\mathbb{1}_{\Lambda_n} B_{n+1}\right] = \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{\Lambda_n} B_{n+1}|\mathscr{F}_n\right]\right] = \mathbb{E}\left[\mathbb{1}_{\Lambda_n}\mathbb{E}\left[B_{n+1}|\mathscr{F}_n\right]\right]. \tag{6.10}$$

This enables to define a probability $\mathbb{Q}$ on $(\Omega, \mathscr{F})$, with $\mathscr{F} = \bigvee_{n\in\mathbb{N}} \mathscr{F}_n$, by:

$$\forall \Lambda_n \in \mathscr{F}_n,\ \mathbb{Q}\left(\Lambda_n\right) = \mathbb{E}_{\mathbf{p}_0}[\mathbb{1}_{\Lambda_n} B_n],$$

where $\mathbb{E}_{\mathbf{p}_0}$ denotes the expectation with respect to the probability distribution $\mathbb{P}_{\mathbf{p}_0}$ of a marked Galton-Watson tree with marking reproduction law $\mathbf{p}_0$ and we study the distribution of the tree under $\mathbb{Q}$. In what follows, we omit the reference to $\mathbf{p}_0$ and write simply $\mathbb{P}$ and $\mathbb{E}$.
We begin with the function $\phi\left(Z_p, M_p\right) = M_p^\ell$, for the natural number $\ell \geq 1$, which indeed gives weight to trees with a large number of marked nodes. Remark that Assumption (6.2) implies that $\mathbb{E}[M_{n+p}^\ell] > 0$ for $n+p > 0$.

Theorem 6.2. *Let $\mathbf{p}$ be an offspring distribution satisfying* (6.1) *and that admits a moment of order $\ell \in \mathbb{N}^*$, and let $\mathbf{q}$ be a mark function satisfying* (6.2)*. Then, for every $n \in \mathbb{N}$ and every $\Lambda_n \in \mathscr{F}_n$, we have*

$$\lim_{p\to+\infty} \frac{\mathbb{E}[\mathbb{1}_{\Lambda_n} M_{n+p}^\ell]}{\mathbb{E}[M_{n+p}^\ell]} = \begin{cases} \mathbb{E}[\mathbb{1}_{\Lambda_n} f_\ell(M_n, Z_n)] & \text{if } \mu < 1, \\ \mathbb{E}[\mathbb{1}_{\Lambda_n} Z_n] & \text{if } \mu = 1, \\ \mathbb{E}\left[\mathbb{1}_{\Lambda_n} \frac{P_\ell(Z_n)}{\mu^{\ell n}}\right] & \text{if } \mu > 1, \end{cases}$$

where P_ℓ is an explicit polynomial function of degree ℓ and f_ℓ is defined for all $s, z \in \mathbb{N}$ by:

$$f_\ell(s,z) := \frac{1}{\xi_\ell}\sum_{i=0}^{\ell}\binom{\ell}{i} s^{\ell-i} \sum_{t_1+\ldots+t_z=i} \binom{i}{t_1 \ldots t_z} \prod_{j=1}^{z} \xi_{t_j}, \tag{6.11}$$

where $\binom{j}{t_1\cdots t_z}$ is the multinomial coefficient defined in (9.10) *and $\xi_j := \lim_{p\to+\infty} \mathbb{E}[M_p^j]$ for all $j \geq 0$.*

Here, the more interesting case is the sub-critical one where the martingale depends of M_n. For instance, let us point out that, for $\ell = 1$, we get the martingale

$$f_1(M_n, Z_n) = Z_n + \frac{1}{\xi_1} M_n.$$

To describe the genealogical tree of (Z, M) under the new probability, we need to introduce the *mass* of a node u denoted by m_u: this quantity is zero for the root and is transmitted throughout the tree according to certain rules (see Sub-section 9.1.1) and permits us to exhibit the new reproduction law and marking procedure under $\mathbb{Q}$.

More precisely, let us define a random multi-type tree as follows.

- The types of the nodes run from 0 to ℓ.
- The sum of the types must be equal to ℓ.
- The root is type ℓ.
- If there is z individuals at generation n, they have the respective types $(t_1, \dots, t_z)$ such that $t_1 + \dots + t_z = \ell$ with a probability depending on z, M_n and the mass of each individual of generation n (see (9.17) for the explicit formula).
- If u is an individual of type $i \in [\![1, \ell]\!]$ with mass m_u, he has $k \in \mathbb{N}$ children and has mark $\eta \in \{0, 1\}$ with probability:

$$\mathbf{p}_i(k, \eta) := \frac{\mathbf{p}_0(k, \eta) f_i(m_u + \eta, k)}{f_i(m_u, 1)}$$

  where $\mathbf{p}_0$ is the reproduction-marking law of the initial tree but also the reproduction-mark law of a 0-type node.

We call a weighted tree of type ℓ a tree with these probabilities of reproduction and marking and denoted it by $\tau_\ell(\mathbf{p}, \mathbf{q})$.

Theorem 6.3. *Let $\mathbf{p}$ be a sub-critical offspring distribution satisfying* (6.1) *and that admits a moment of order $\ell \in \mathbb{N}^*$, and let $\mathbf{q}$ be a mark function satisfying* (6.2). *Let $\mathbb{Q}_\ell$ be the probability measure defined by*

$$\forall n \in \mathbb{N}, \; \frac{d\mathbb{Q}_\ell}{d\mathbb{P}}_{|\mathscr{F}_n} = f_\ell(M_n, Z_n). \tag{6.12}$$

Then $\mathbb{Q}_\ell$ is the distribution of a weighted tree of type ℓ.

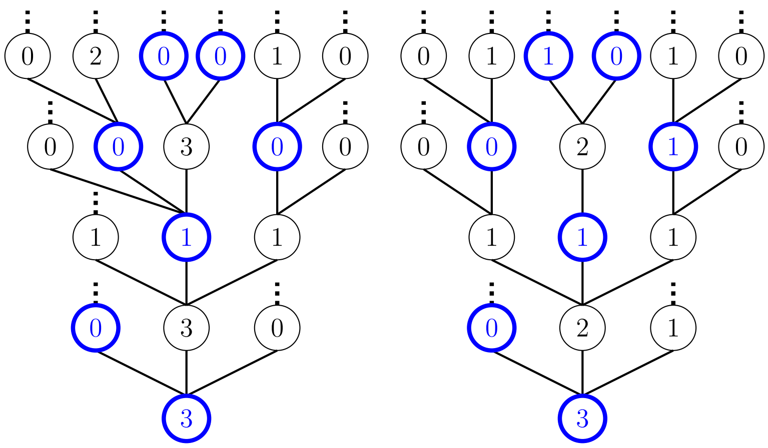


Figure 6.3 – Two examples of marked multi-type trees (marked in blue, unmarked in black), in the sub-critical case for $\ell = 3$.

The two martingales obtained for $\mu \geq 1$ are already known and the study under $\mathbb{Q}$ is, as a result, unnecessary (see Section 7.4 for $\mu = 1$ and [5] for $\mu > 1$).

Definition 4.4 of [5], give us that:

- The types of the nodes run from 0 to ℓ.
- The root is type ℓ.
- A node of type i at height n gives, independently of the other nodes, k offspring with respective types $(t_1, \ldots, t_k)$ such that $t_1 + \ldots + t_k = i$ with a probability describe in [5].
- Conditionally on this tree, all the nodes are marked independently of each other and a node u with k_u offspring has mark one with probability $\mathbf{q}(k_u)$.

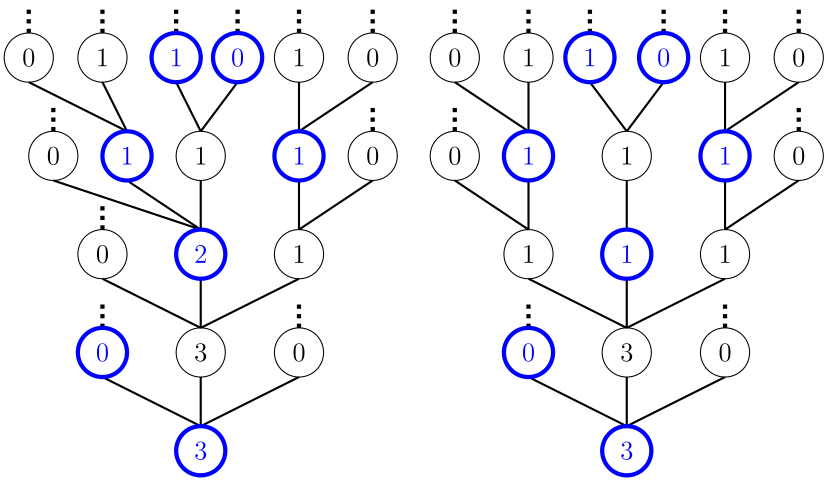


Figure 6.4 – Two examples of marked multi-type trees (marked in blue, unmarked in black), in the super-critical case for $\ell = 3$.

Now, we consider the following function $\phi(Z_p, M_p) = M_p^\alpha Z_p^\beta$, with $\alpha, \beta \in \mathbb{N}^*$. Indeed, the case $\alpha = 0$ was studied in [5] and the case $\beta = 0$ in the preceding case or in [3]. So, it is natural to continue with the general case, where the both variables give some weight. In this case we obtain,

Theorem 6.4. *Let* $\mathbf{p}$ *be an offspring distribution satisfying* (6.1) *which admits a moment of order* $\alpha + \beta$ *with* $\alpha, \beta \in \mathbb{N}^*$ *and let* $\mathbf{q}$ *be a mark function satisfying* (6.2). *For every* $n \in \mathbb{N}$ *and every* $\Lambda_n \in \mathscr{F}_n$, *we have:*

$$\lim_{p \to +\infty} \frac{\mathbb{E}\left[\mathbb{1}_{\Lambda_n} M_{n+p}^\alpha Z_{n+p}^\beta\right]}{\mathbb{E}\left[M_{n+p}^\alpha Z_{n+p}^\beta\right]} = \begin{cases} \mathbb{E}\left[\mathbb{1}_{\Lambda_n} \frac{Z_n}{\mu^n}\right] & \text{if } \mu < 1, \\ \mathbb{E}[\mathbb{1}_{\Lambda_n} Z_n] & \text{if } \mu = 1, \\ \mathbb{E}\left[\mathbb{1}_{\Lambda_n} \frac{P_{\alpha+\beta}(Z_n)}{\mu^{n(\alpha+\beta)}}\right] & \text{if } \mu > 1, \end{cases} \tag{6.13}$$

where $P_{\alpha+\beta}$ *is a polynomial function of degree* $\alpha + \beta$.

In the first both cases, the study of the new probability measure, give us, the behavior of a marked Kesten tree. In the third case, we obtain a marked multi-type tree like in the third case of Theorem 6.2.

Finally, we study the function $\phi(Z_p, M_p) = H_\ell(Z_p)s^{M_p}t^{Z_p}$, with $s, t \in (0,1)$ where H_ℓ is the ℓ-th Hilbert polynomial defined by:

$$H_0(x) = 1 \text{ and } H_\ell(x) = \frac{1}{\ell!}\prod_{j=0}^{\ell-1}(x-j)\,, \text{ for } \ell \geq 1. \tag{6.14}$$

Contrary to the previous study, this function gives weight to trees with a small number of marks.

We obtain the following limits:

Theorem 6.5. *Let* $\mathbf{p}$ *be an offspring distribution satisfying the two last points of* (6.1) *and that admits a moment of order* $\ell \in \mathbb{N}^*$, *and let* $\mathbf{q}$ *be a mark function satisfying* (6.2). *We define*

$$r = \min\{k \in \mathbb{N}, \mathbf{p}(k) > 0\}.$$

Then, for every $s \in (0,1)$, *if* $\kappa(s)$ *denotes the only root in* $[0,1]$ *of the function*

$$f_s(\mathbf{t}) := \mathbb{E}[s^{M_1}t^{Z_1}],$$

we have for every $n \in \mathbb{N}$ *and* $\Lambda_n \in \mathscr{F}_n$,

— *if* $r = 0$, *for every* $t \in [0,1]$,

$$\lim_{p\to+\infty}\frac{\mathbb{E}[\mathbb{1}_{\Lambda_n}H_\ell(Z_{n+p})s^{M_{n+p}}t^{Z_{n+p}}]}{\mathbb{E}[H_\ell(Z_{n+p})s^{M_{n+p}}t^{Z_{n+p}}]} = \begin{cases}\mathbb{E}\left[\mathbb{1}_{\Lambda_n}s^{M_n}\kappa(s)^{Z_n-1}\right] & \text{if } \ell = 0,\\ E\left[\mathbb{1}_{\Lambda_n}\dfrac{s^{M_n}Z_n\kappa(s)^{Z_n-1}}{f_s'(\kappa(s))^n}\right] & \text{if } \ell > 0,\end{cases}$$

— *if* $r = 1$, *for every* $t \in (0,1]$,

$$\lim_{p\to+\infty}\frac{\mathbb{E}[\mathbb{1}_{\Lambda_n}H_\ell(Z_{n+p})s^{M_{n+p}}t^{Z_{n+p}}]}{\mathbb{E}[H_\ell(Z_{n+p})s^{M_{n+p}}t^{Z_{n+p}}]} = \mathbb{E}\left[\mathbb{1}_{\Lambda_n}s^{M_n}\alpha_1(s)^{-n}\mathbb{1}_{Z_n=1}\right],$$

— *if* $r \geq 2$, *for every* $t \in (0,1]$,

$$\lim_{p\to+\infty}\frac{\mathbb{E}[\mathbb{1}_{\Lambda_n}H_\ell(Z_{n+p})s^{M_{n+p}}t^{Z_{n+p}}]}{\mathbb{E}[H_\ell(Z_{n+p})s^{M_{n+p}}t^{Z_{n+p}}]} = \mathbb{E}\left[\mathbb{1}_{\Lambda_n}s^{M_n}\alpha_r(s)^{\frac{-(r^n-1)}{r-1}}\mathbb{1}_{Z_n=r^n}\right],$$

where $\alpha_r(s) = \mathbf{p}(r)(s\mathbf{q}(r) + 1 - \mathbf{q}(r))$.

Let us add that, for sake of completeness, we also study the case $s = 0$ in Theorem 9.16.

We also describe the probability distribution obtained by a change of measure using the martingales obtained in the previous theorem, see Theorem 9.11, Theorem 9.13, Theorem 9.14 and Theorem 9.16.

This part is organized as follows:

In Chapter 7 , we introduce the set of discrete marked trees and define the MGW. We also explain the construction of Kesten's tree and condensation tree.

In Chapter 8 , we give some convergence criterions which is the key to prove the local limits. Then we observe the behavior of the MGW, when we condition with the total number of marks. Section 8.2 is devoted to prove Proposition 6.1 .We explain properties about generic and non-generic distributions in Section 8.3 . In Section 8.4 , we construct a model that permits to prove our results. Section 8.5 is devoted to the proof of Theorem 6.1. We prove in the appendix, Section 8.6 , lemmas that we used in the previous sections.

We study, in Chapter 9 , the penalization in Section 9.1, by M_p in Sub-Section 9.1.2 and general results with , $\ell \in \mathbb{N}^*$, in Sub-Section 9.1.3 , by M_p^ℓ, finding the martingale obtained at the limit and then describing the distribution obtained by a Girsanov transformation using this martingale. We make the same study in Section 9.2 , when the penalization is of the form $H_\ell(Z_p)s^{M_p}t^{Z_p}$.

Chapter 7

Technical background

7.1 The set of discrete trees

Let $\mathbb{N}$ be the set of nonnegative integers and $\mathbb{N}^*$ the set of positive integers. We recall Neveu's formalism [29] for ordered rooted trees. We set

$$\mathcal{U} = \bigcup_{n \geq 0} (\mathbb{N}^*)^n$$

be the set of finite sequences of positive integers with the convention $(\mathbb{N}^*)^0 = \{\emptyset\}$. For $n \geq 0$ and $u = (u_1, ..., u_n) \in \mathcal{U}$, let $|u| = n$ be the length of u and $|u|_\infty = \max\{|u|, u_1, u_2, ..., u_{|u|}\}$ with the convention $|\emptyset| = |\emptyset|_\infty = 0$. We call $|u|_\infty$ the norm of u although it is not a norm since $\mathcal{U}$ is not even a vector space. If u and v are two sequences of $\mathcal{U}$, we denote by uv their concatenation, with the convention that $uv = u$ if $v = \emptyset$ and $uv = v$ if $u = \emptyset$. The set of ancestors of $\emptyset$ is $A_\emptyset = \{\emptyset\}$ and of $u \neq \emptyset$ is:

$$A_u = \{v \in \mathcal{U};\ \text{there exists } w \in \mathcal{U}, w \neq \emptyset,\ \text{such that } u = vw\}. \tag{7.1}$$

The most recent common ancestor of a subset s of $\mathcal{U}$, denoted by $MRCA(s)$, is the unique element v of $\bigcap_{u \in s} A_u$ with maximal length $|v|$. For $u, v \in \mathcal{U}$, we denote by $u < v$, the lexicographic order on $\mathcal{U}$, i.e. $u < v$ if either $u \in A_v$ or, if we set $w = MRCA(\{u, v\})$, then $u = wiu'$ and $v = wjv'$ for some $i, j \in \mathbb{N}^*$ with $i < j$.

A tree $\mathbf{t}$ is a subset of $\mathcal{U}$ that satisfies:

— $\emptyset \in \mathbf{t}$;

— if $u \in \mathbf{t}$, then $A_u \subset \mathbf{t}$;

— for every $u \in \mathbf{t}$, there exists $k_u(\mathbf{t}) \in \mathbb{N} \cup \{+\infty\}$ such that, for every positive integer i, $ui \in \mathbf{t}$ if and only if $1 \leq i \leq k_u(\mathbf{t})$.

The integer $k_u(\mathbf{t})$ represents the number of offsprings of the vertex $u \in \mathbf{t}$. If $k_u(\mathbf{t}) = 0$, the vertex $u \in \mathbf{t}$ is called a leaf and if $k_u(\mathbf{t}) = +\infty$, u is called infinite. By convention, we shall set $k_u(\mathbf{t}) = -1$ if $u \notin \mathbf{t}$. The vertex $\emptyset$ is called the root of $\mathbf{t}$.
Its height and its *norm*, which can be infinite, are respectively defined by:

$$H(\mathbf{t}) = \sup\{|u|,\ u \in \mathbf{t}\};$$
$$H_\infty(\mathbf{t}) = \sup\{|u|_\infty,\ u \in \mathbf{t}\} = \max(H(\mathbf{t}), \sup\{k_u(\mathbf{t}), u \in \mathbf{t}\}).$$

We denote by:

— $\mathbb{T}_\infty$ the set of trees,

— $\mathcal{L}_0(\mathbf{t}) := \{u \in \mathbf{t}, k_u(\mathbf{t}) = 0\}$ the set of leaves of $\mathbf{t} \in \mathbb{T}_\infty$,

— $\mathbb{T}_0 := \{\mathbf{t} \in \mathbb{T}_\infty, |\mathbf{t}| < +\infty\}$ the subset of finite trees, where $|\mathbf{t}|$ is the cardinal of $\mathbf{t}$,

— $\mathbb{T}_\infty^{(h)} := \{\mathbf{t} \in \mathbb{T}_\infty, H_\infty(\mathbf{t}) < h\}$ the subset of finite trees with norm less than $h \in \mathbb{N}$, $H_\infty(\mathbf{t}) = \sup\{|u|_\infty, u \in \mathbf{t}\}$,

— $\mathbb{T} := \{\mathbf{t} \in \mathbb{T}_\infty; k_u(\mathbf{t}) < +\infty \ \forall u \in \mathbf{t}\}$ the subset of trees with no infinite vertex,

— $\mathbb{T}_1 := \{\mathbf{t} \in \mathbb{T}_\infty; \lim\limits_{n\to+\infty} |MRCA(\{u \in \mathbf{t}; |u| = n\})| = +\infty\}$, the subset of trees with a unique infinite spine,

— $\mathbb{T}_2$, the subset of trees with no infinite spine and with exactly one infinite vertex,

— $\mathbf{t} \circledast_x \mathbf{t}' := \mathbf{t} \bigcup \{x\zeta(v, k_x(\mathbf{t})),\ v \in \mathbf{t}' \setminus \{\emptyset\}\}$ the tree obtained by grafting the tree $\mathbf{t}' \in \mathbb{T}_\infty$ at $x \in \mathbf{t}$ on "the right" of $\mathbf{t} \in \mathbb{T}_\infty$, with $\zeta(v, k) := (v_1 + k, v_2, ..., v_n)$ if $v = (v_1, v_2, ..., v_k) \in \mathcal{U}$ with $k \in \mathbb{N}^*$ and $n > 0$,

— $\mathbb{T}(\mathbf{t}, x) := \{\mathbf{t} \circledast_x \mathbf{t}', \mathbf{t}' \in \mathbb{T}_\infty\}$ the set of trees obtained by grafting a tree at $x \in \mathbf{t}$ on "the right" of $\mathbf{t} \in \mathbb{T}_0$,

— $\mathbb{T}_+(\mathbf{t}, x, k) := \{s \in \mathbb{T}(\mathbf{t}, x), k_x(s) \geq k\}$ the subset of $\mathbb{T}(\mathbf{t}, x)$ such that the number of offspring of $x \in s$ is k or more.

Let $\mathbf{t}$ be a tree, for $u \in \mathbf{t}$, we define $\mathcal{S}_u(\mathbf{t}) := \{v \in \mathcal{U};\ uv \in \mathbf{t}\}$ as the sub-tree of $\mathbf{t}$ "above" u. For $u \in \mathbf{t} \setminus \mathcal{L}_0(\mathbf{t})$, we also define $\mathcal{F}_u(\mathbf{t}) := (S_{ui}(\mathbf{t});\ 1 \leq i \leq k_u(\mathbf{t}))$ as the forest of $\mathbf{t}$ "above" u.
For $u \in \mathbf{t} \setminus \{\emptyset\}$, we also define the sub-tree $\mathcal{S}^u(\mathbf{t})$ of $\mathbf{t}$ "below" u as: $\mathcal{S}^u(\mathbf{t}) := \{v \in \mathbf{t};\ u \notin A_v\}$.

For $h \in \mathbb{N}$ the restriction function $r_{h,\infty}$ from $\mathbb{T}_\infty$ to $\mathbb{T}_\infty$ is defined by :

$$r_{h,\infty}(\mathbf{t}) := \{u \in \mathbf{t}, |u|_\infty \leq h\},$$

and the restriction function r_h from $\mathbb{T}$ to $\mathbb{T}$ is defined by :

$$r_h(\mathbf{t}) := \{u \in \mathbf{t}, |u| \leq h\}.$$

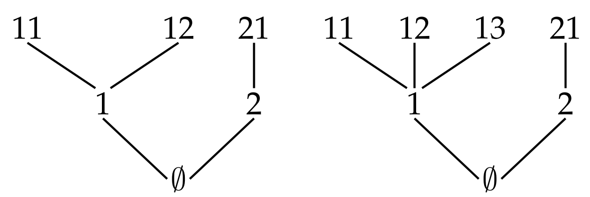


Figure 7.1 – $r_{2,\infty}(\mathbf{t})$ on the left and $r_2(\mathbf{t})$ on the right.

We endow the set $\mathbb{T}_\infty$ (resp. $\mathbb{T}$), with the ultra-metric distance

$$d_\infty(\mathbf{t}, \mathbf{t}') := 2^{-\max\{h \in \mathbb{N}, r_{h,\infty}(\mathbf{t}) = r_{h,\infty}(\mathbf{t}')\}} \ \text{(resp. } d(\mathbf{t}, \mathbf{t}') := 2^{-\max\{h \in \mathbb{N}, r_h(\mathbf{t}) = r_h(\mathbf{t}')\}}\text{)}.$$

A sequence $(\mathbf{t}_n, n \in \mathbb{N})$ of trees converges to a tree $\mathbf{t}$ with respect to the distance d_∞ (resp. d) if and only if, for every $h \in \mathbb{N}$,

$$r_{h,\infty}(\mathbf{t}_n) = r_{h,\infty}(\mathbf{t}) \ \ (\text{ resp. } r_h(\mathbf{t}_n) = r_h(\mathbf{t})) \text{ for } n \text{ large enough.}$$

The Borel σ-field associated with the distance d_∞ (resp. d) is the smallest σ-field containing the singletons for which the restrictions functions $(r_{h,\infty}, h \in \mathbb{N})$ (resp. $(r_h, h \in \mathbb{N})$) are measurable. With this distance, the restriction function are contractant. According to [20], $\mathbb{T}_0$ is dense in $\mathbb{T}_\infty$ (resp. $\mathbb{T}_0 \cap \mathbb{T} \in \mathbb{T}$) and since, $(\mathbb{T}_\infty, d_\infty)$ (resp. $(\mathbb{T}, d)$) is complete, we get that $(\mathbb{T}_\infty, d_\infty)$ (resp. $(\mathbb{T}, d)$) is a Polish metric space. Moreover, according to [20], $(\mathbb{T}_\infty, d_\infty)$ is compact.

7.2 Galton-Watson trees

Let $\mathbf{p} = (\mathbf{p}(n))_{n\in\mathbb{N}}$ be a probability distribution on the set of the non-negative integers satisfying (6.1). Let $g(z) := \mathbb{E}[z^X]$ be the generating function of X, where X is a random variable with distribution $\mathbf{p}$. We denote by $\rho(\mathbf{p})$ the radius of convergence of g and we write ρ when it is clear from the context.

A $\mathbb{T}$-valued random variable $\tau(\mathbf{p})$ (noted τ if it is clear) is a Galton-Watson tree (GW) with offspring distribution $\mathbf{p}$, if the distribution of $k_\emptyset(\tau)$ is $\mathbf{p}$ and for $n \in \mathbb{N}^*$, conditionally on $\{k_\emptyset(\tau) = n\}$, the sub-tree $(\mathcal{S}_1(\tau), \cdots, \mathcal{S}_n(\tau))$ are independent and distributed as the original tree τ. The distribution of a GW is characterized by:

$$\forall h \geq 1, \forall \mathbf{t} \in \mathbb{T}^{(h)},\ \mathbb{P}(r_h(\tau) = \mathbf{t}) = \prod_{u\in\mathbf{t}, |u|<h} \mathbf{p}(k_u(\mathbf{t})). \tag{7.2}$$

In particular,

$$\forall \mathbf{t} \in \mathbb{T}_0,\ \mathbb{P}(\tau = \mathbf{t}) = \prod_{u\in\mathbf{t}} \mathbf{p}(k_u(\mathbf{t})). \tag{7.3}$$

For a tree $\mathbf{t}$, we have:

$$\sum_{u\in\mathbf{t}} k_u(\mathbf{t}) = |\mathbf{t}| - 1. \tag{7.4}$$

Let $X_{i,n}$ be the number of children of i to the generation n, $X_{i,n}$ distributed as X and $(X_{i,n})_{i\in\mathbb{N}^*, n\in\mathbb{N}}$ are independent. Z_n the number of individuals to the generation n.
We have $Z_0 = 1$ and for $n \geq 0$.

$$Z_{n+1} := \begin{cases} \sum\limits_{i=1}^{Z_n} X_{i,n}, \text{ if } Z_n > 0, \\ 0, \text{ if } Z_n = 0. \end{cases}$$

We also have the following equality

$$Z_{n+p} \stackrel{(d)}{=} \sum_{i=1}^{Z_n} \tilde{Z}_{i,p}, \tag{7.5}$$

where $(\tilde{Z}_{i,p})_{i\geq 1}$ is a sequence of i.i.d copies of Z_p and independent of $\mathscr{F}_n$.

We recall that, if:

— $\mu(\mathbf{p}) < 1$ we are in the sub-critical case,

— $\mu(\mathbf{p}) = 1$ we are in the critical case,

— $\mu(\mathbf{p}) > 1$ we are in the super-critical case.

We say that $\mathbf{p}$ is aperiodic if $\{k \in \mathbb{N};\ \mathbf{p}(k) > 0\} \subset d\mathbb{N}$ implies $d = 1$.

Let $\mathbb{P}_k$ be the distribution of the forest $\tau^{(k)} = (\tau_1, \cdots, \tau_k)$ of i.i.d GW with offspring distribution $\mathbf{p}$. We set:

$$\left|\tau^{(k)}\right| = \sum_{j=1}^{k} |\tau_j|, \tag{7.6}$$

with $|\tau_j|$ is the cardinal of the GW τ_j.

7.3 Marking process

7.3.1 The set of marked discrete trees

A marked tree $\mathbf{t}^*$ is defined by a tree $\mathbf{t} \in \mathbb{T}_\infty$ and a mark $\eta_u \in \{0, 1\}$ for every node $u \in \mathbf{t}$, that is

$$\mathbf{t}^* = \big(\mathbf{t}, (\eta_u)_{u\in\mathbf{t}}\big).$$

A node $u \in \mathbf{t}$ is said to be marked if $\eta_u = 1$ and unmarked if $\eta_u = 0$.

Throughout the part the notations defined to $\mathbf{t}$ and use for $\mathbf{t}^*$ are the same. For instance $\mathbb{T}_0^* := \{(\mathbf{t}, (\eta_u(\mathbf{t}))_{u\in\mathbf{t}}), \mathbf{t} \in \mathbb{T}_0\}$. We denote by $\mathbb{T}_\infty^*$ the set of marked trees.

Let $\mathbf{t}^*$, $\mathbf{t}'^*$ be two marked trees and $x \in \mathbf{t}$. If x is marked in $\mathbf{t}^*$, x is marked in $\mathbf{t}^* \circledast_x \mathbf{t}'^*$, we forget the mark on the root of $\mathbf{t}'^*$.
For $x \in \mathbf{t} \setminus \{\emptyset\}$, x is not marked on $\mathcal{S}^x(\mathbf{t}^*)$:

$$\mathcal{S}^x(\mathbf{t}^*) = \{(u, \eta_u(\mathbf{t})), u \in \mathcal{S}^x(\mathbf{t}) \setminus \{x\}\} \bigcup \{(x, 0)\}.$$

We also denote by

$$M(\mathbf{t}^*) := \sum_{u\in\mathbf{t}} \eta_u,$$

the number of marked vertices. Like in Section 7.1, for every $h \in \mathbb{N}$, we define the restriction functions

$$r_h^* : \mathbb{T}^* \longrightarrow \mathbb{T}^*, \qquad \text{and} \qquad r_{h,\infty}^* : \mathbb{T}_\infty^* \longrightarrow \mathbb{T}_\infty^*$$

by, for $\mathbf{t}^* = (\mathbf{t}, (\eta_u)_{u\in\mathbf{t}}) \in \mathbb{T}^*$,

$$r_h^*(\mathbf{t}^*) = \big(r_h(\mathbf{t}), (\eta_u^h)_{u\in r_h(\mathbf{t})}\big), \qquad r_{h,\infty}^*(\mathbf{t}^*) = \big(r_{h,\infty}(\mathbf{t}), (\eta_u^h)_{u\in r_{h,\infty}(\mathbf{t})}\big)$$

with

$$\eta_u^h = \begin{cases} \eta_u & \text{if } |u| \le h-1, \\ 0 & \text{if } |u| = h. \end{cases}$$

Remark 7.1. *Let us stress that the nodes at level h of $r_h^*(\mathbf{t}^*)$ are unmarked so that the tree $r_h^*(\mathbf{t}^*)$ is completely characterized by $\{(k_u(\mathbf{t}), \eta_u), u \in \mathbf{t}, |u| \le h-1\}$.*

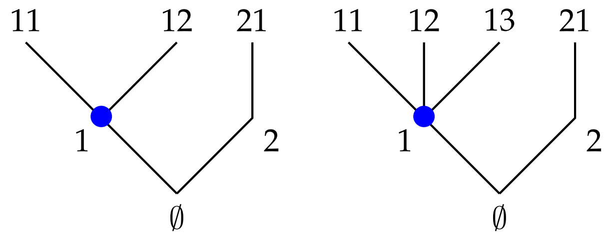


Figure 7.2 – $r^*_{2,\infty}(\mathbf{t}^*)$ on the left and $r^*_2(\mathbf{t}^*)$ on the right.

We can endow the set $\mathbb{T}^*_\infty$ (resp. $\mathbb{T}^*$) with the σ-field $\mathscr{F}$ generated by the family of sets $(\{\mathbf{t}^* \in \mathbb{T}^*_\infty,\ u \in \mathbf{t}\})_{u\in\mathcal{U}}$ $\big($resp. $(\{\mathbf{t}^* \in \mathbb{T}^*,\ u \in \mathbf{t}\})_{u\in\mathcal{U}}\big)$ and hence define probability measures on $(\mathbb{T}^*_\infty, \mathscr{F})$ (resp. $(\mathbb{T}^*, \mathscr{F})$).

We also endow $\mathbb{T}^*_\infty$ (resp. $\mathbb{T}^*$) with the filtration $(\mathscr{F}_n)_{n\geq 0}$ where $\mathscr{F}_n$ is the σ-field generated by the restriction function $r^*_{n,\infty}$ (resp. r^*_n). Notice that $\mathscr{F} = \bigvee_{n\geq 0} \mathscr{F}_n$ as, for every $u \in \mathcal{U}$,

$$\{\mathbf{t}^* \in \mathbb{T}^*_\infty,\ u \in \mathbf{t}\} \in \mathscr{F}_{|u|} \text{ (resp. } \{\mathbf{t}^* \in \mathbb{T}^*,\ u \in \mathbf{t}\} \in \mathscr{F}_{|u|}).$$

Let $\mathbf{t}^* = (\mathbf{t}, (\eta_u)_{u\in\mathbf{t}}) \in \mathbb{T}^*_\infty$ $(resp.\ \mathbb{T}^*)$ be a marked tree. For every $n \in \mathbb{N}$, we define

— $z_n(\mathbf{t}^*)$ the nodes of $\mathbf{t}$ at generation n:

$$z_n(\mathbf{t}^*) = \{u \in \mathbf{t},\ |u| = n\},$$

— $Z_n(\mathbf{t}^*)$ the size of generation n:

$$Z_n(\mathbf{t}^*) = \mathrm{Card}\big(z_n(\mathbf{t}^*)\big),$$

— $M_n(\mathbf{t}^*)$ the total number of marked nodes up to generation $n-1$:

$$M_n(\mathbf{t}^*) = \sum_{u\in\mathbf{t}, |u|\leq n-1} \eta_u \quad \text{for } n \geq 1.$$

Let us point out that the functions $z_n,\ Z_n,\ M_n$ are all $\mathscr{F}_n$-measurable.

7.3.2 Marked Galton-Watson trees

Let $\mathbf{p}_0 = (\mathbf{p}_0(k,\eta))_{k\in\mathbb{N},\eta\in\{0,1\}}$ be a probability distribution on $\mathbb{N} \times \{0,1\}$. There exists a unique probability measure $\mathbb{P}_{\mathbf{p}_0}$ on $(\mathbb{T}^*, \mathscr{F})$ such that, for all $h \in \mathbb{N}^*$ and all $\mathbf{t}^* = (\mathbf{t}, (\eta_u)_{u\in\mathbf{t}}) \in \mathbb{T}^*_\infty$ (resp. $\mathbb{T}^*$), if τ^* denotes the canonical random variable on $\mathbb{T}^*_\infty$ (resp. $\mathbb{T}^*$),

$$\mathbb{P}_{\mathbf{p}_0}(r^*_h(\tau^*) = r^*_h(\mathbf{t}^*)) = \prod_{u\in r_{h-1}(\mathbf{t})} \mathbf{p}_0(k_u(\mathbf{t}), \eta_u), \tag{7.7}$$

We say that the r.w. τ^* under $\mathbb{P}_{\mathbf{p}_0}$ is a marked Galton-Watson tree (MGW) with reproduction-marking law $\mathbf{p}_0$. As $\tau^* \in \mathbb{T}^*_0$, we have for all marked tree $\mathbf{t}^* \in \mathbb{T}^*_0$,

$$\mathbb{P}(\tau^* = \mathbf{t}^*) = \prod_{u\in\mathbf{t}} \mathbf{p}_0(k_u(\mathbf{t}), \eta_u) = \mathbb{P}(\tau = t) \prod_{u\in\mathbf{t},\eta_u=1} \mathbf{q}(k_u(\mathbf{t})) \prod_{u\in\mathbf{t},\eta_u=0} (1 - \mathbf{q}(k_u(\mathbf{t})))\,. \tag{7.8}$$

Equivalently, the distribution of a marked Galton-Watson tree with reproduction-marking law $\mathbf{p}_0$ is also characterized by: an offspring distribution $\mathbf{p}$ and a mark function
$\mathbf{q} : \mathbb{N} \longrightarrow [0,1]$ with

$$\mathbf{p}(k) = \mathbf{p}_0(k,0) + \mathbf{p}_0(k,1), \quad \mathbf{q}(k) = \frac{\mathbf{p}_0(k,1)}{\mathbf{p}(k)} \text{ if } \mathbf{p}(k) \neq 0$$

(the value of $\mathbf{q}(k)$ has no importance if $\mathbf{p}(k) = 0$), or equivalently

$$\mathbf{p}_0(k,1) = \mathbf{p}(k)\mathbf{q}(k), \qquad \mathbf{p}_0(k,0) = \mathbf{p}(k)(1-\mathbf{q}(k)). \tag{7.9}$$

This approach (giving $\mathbf{p}$ and $\mathbf{q}$) consists in first picking a Galton-Watson tree with offspring distribution $\mathbf{p}$ then conditionally on the tree, adding marks on every node, independently of each others, with probabily $\mathbf{q}(k)$ where k is the out-degree of the node.

We will then write $\mathbb{P}_{\mathbf{p}_0}$ or $\mathbb{P}_{\mathbf{p},\mathbf{q}}$ depending on the adopted point of view, or simply $\mathbb{P}$ when the context is clear.

Under $\mathbb{P}_{\mathbf{p},\mathbf{q}}$, the random tree τ^* satisfies the so-called branching property that is

- The random variable $k_\emptyset(\tau^*)$ is distributed according to the probability distribution $\mathbf{p}$;
- Conditionally given $\{k_\emptyset(\tau^*) = j\}$, the root is marked with probability $\mathbf{q}(j)$;
- Conditionally on $\{k_\emptyset(\tau^*) = j\}$, the j sub-trees attached to the root are i.i.d. random marked trees distributed according to the probability $\mathbb{P}_{\mathbf{p},\mathbf{q}}$.

As a consequence, if we set $Z_n := Z_n(\tau^*)$ and $M_n := M_n(\tau^*)$, we have for every $n, p \in \mathbb{N}^*$, the following equality in distribution under $\mathbb{P}_{\mathbf{p},\mathbf{q}}$,

$$M_{n+p} \overset{(d)}{=} M_n + \sum_{i=1}^{Z_n} \tilde{M}_{i,p}, \tag{7.10}$$

where $(\tilde{M}_{i,p})_{i\geq 1}$ is a sequence of i.i.d copies of M_p and independent of $\mathscr{F}_n$.

7.4 Kesten's tree and condensation tree

Let $\tau_l(\mathbf{p})$ denote the random tree defined by:

1. there are two types of nodes: normal and special;
2. the root is special;
3. normal nodes have offspring distribution $\mathbf{p}$;
4. special nodes have for offspring distribution the biased distribution $\check{\mathbf{p}}$ on $\mathbb{N}\bigcup\{+\infty\}$ defined by:
$$\check{\mathbf{p}}(k) := \begin{cases} k\mathbf{p}(k) & \text{if } k \in \mathbb{N}, \\ 1-\mu & \text{if } k = +\infty. \end{cases}$$
5. The offsprings of all the nodes are independent of each others;
6. all the children of a normal node are normal;
7. when a special node gets a finite number of children, one of them is selected uniformly at random and is special while the others are normal.
8. When a special node gets an infinite number of children, all of them are normal.

$\tau_l(\mathbf{p})$ represents a limit tree for the rest of the document.

We can notice that:

— if $\mathbf{p}$ is critical, $\tau_l(\mathbf{p}) = \tau_K(\mathbf{p})$;

— if $\mathbf{p}$ is sub-critical, $\tau_l(\mathbf{p}) = \tau_C(\mathbf{p})$.

We denote by $\mathbf{p}^* := \left(\mathbf{p}^*(n) = \dfrac{n\mathbf{p}(n)}{\mu}\right)_{n\in\mathbb{N}}$ the corresponding size-biased distribution of the Kesten's tree. We define an infinite random tree $\tau_K(\mathbf{p})$ (the Kesten's tree) whose distribution is described in [2] Sub-Section 3.2. This tree has one infinite spine and all its nodes have finite degrees. We denote by $\tau_K^*(\mathbf{p},\mathbf{q})$ the marked Kesten's tree, the mark function is the same since it is independent of the offspring distribution.

We define an infinite random tree $\tau_C(\mathbf{p})$ (the condensation tree) whose described in [6] Sub-Section 3.2. This tree has exactly one node of infinite degree and no infinite spine. This tree has been considered in [19] and [20]. We denote by $\tau_C^*(\mathbf{p},\mathbf{q})$ the marked condensation tree, the mark function is the same since it is independent of the offspring distribution.

Now we consider the marked limit tree. In the both cases, the mark function stay the same as the original MGW with offspring distribution $\mathbf{p}$ and mark function $\mathbf{q}$.

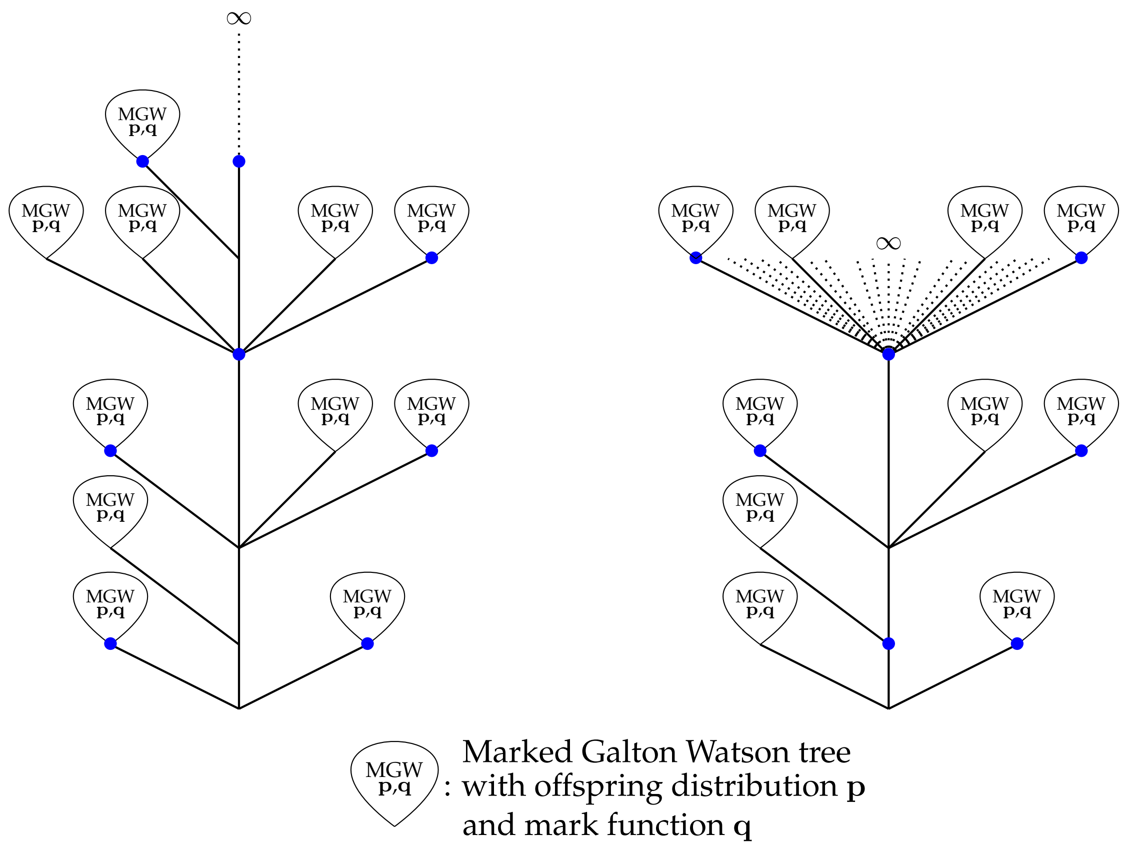


Figure 7.3 – Marked Kesten'tree and condensation tree

7.4.1 Marked Kesten's tree

In this subsection, we write τ_K^* for $\tau_K^*(\mathbf{p},\mathbf{q})$ and τ_K for $\tau_K(\mathbf{p})$.

Proposition 7.1. *Let τ_K^* be a Kesten's tree associated with a critical offspring distribution $\mathbf{p}$ and a mark function $\mathbf{q}$ satisfying* (6.1) *and* (6.2) *respectively. Then we have, for all marked tree $\mathbf{t}^* \in \mathbb{T}_0^*$ and $x \in \mathcal{L}_0(\mathbf{t})$:*

$$\mathbb{P}(\tau_K^* \in \mathbb{T}(\mathbf{t}^*, x)) = \frac{\mathbb{P}(\tau^* = \mathcal{S}^x(\mathbf{t}^*))}{\mathbf{p}(0)(1-\mathbf{q}(0))}\mathbb{E}[X\alpha_{X,x}]. \tag{7.11}$$

Where X is a random variable with distribution $\mathbf{p}$, and $\alpha_{j,x} := \mathbf{q}(j)\eta_x(\mathbf{t}) + (1-\mathbf{q}(j))(1-\eta_x(\mathbf{t}))$ for all $j \in \mathbb{N}$.

Proof. Let $\mathbf{t}^* \in \mathbb{T}_0^*$, $x \in \mathcal{L}_0(\mathbf{t})$, we denote by η_x for $\eta_x(\mathbf{t})$.
First, according to Sub-section 2.4 of [7], we have for $t \in \mathbb{T}_0$, $x \in \mathcal{L}_0(\mathbf{t})$:

$$\mathbb{P}(\tau_K \in \mathbb{T}(\mathbf{t}, x)) = \frac{\mathbb{P}(\tau = \mathbf{t})}{\mathbf{p}(0)}, \tag{7.12}$$

which implies:

$$\mathbb{P}(\tau_K^* \in \mathbb{T}(\mathbf{t}^*, x)) = \frac{\mathbb{P}(\tau = t)}{\mathbf{p}(0)} \prod_{\substack{u\in\mathbf{t},\, u\neq x\\ \eta_u=1}} \mathbf{q}(k_u(\mathbf{t})) \prod_{\substack{u\in\mathbf{t},\, u\neq x\\ \eta_u=0}} 1-\mathbf{q}\left(k_u(\mathbf{t})\right)\sum_{j=0}^{+\infty} j\mathbf{p}(j)\alpha_{j,x}$$

$$= \frac{\mathbb{P}(\tau^* = \mathbf{t}^*)}{\mathbf{p}(0)}\left(\frac{\eta_x}{\mathbf{q}(0)} + \frac{1-\eta_x}{1-\mathbf{q}(0)}\right)\mathbb{E}[X\alpha_{X,x}]$$

And we easily conclude as for $x \in \mathcal{L}_0(\mathbf{t})$, we have:

$$\mathbb{P}(\tau^* = \mathcal{S}^x(\mathbf{t}^*)) = \mathbb{P}(\tau^* = \mathbf{t}^*)\left(1 - \eta_x + \frac{1-\mathbf{q}(0)}{\mathbf{q}(0)}\eta_x\right).$$

That concludes the proof. □

7.4.2 Marked condensation tree

In this subsection, we write τ_C^* for $\tau_C^*(\mathbf{p},\mathbf{q})$. If $\mu(\mathbf{p}) < 1$, for all marked tree $\mathbf{t}^* \in \mathbb{T}_0^*$, $x \in \mathbf{t}$, we set:

$$C(\mathbf{t}^*, x) := \frac{\mathbb{P}\left(\tau^* = \mathcal{S}^x(\mathbf{t}^*)\right)}{\mathbf{p}(0)(1-\mathbf{q}(0))}\mathbb{P}_{k_x(\mathbf{t})}\left(\tau^* = \mathscr{F}_x(\mathbf{t}^*)\right). \tag{7.13}$$

Proposition 7.2. *Let τ_C^* be a condensation tree associated with a sub-critical offspring distribution $\mathbf{p}$ and a mark function $\mathbf{q}$ satisfying* (6.1) *and* (6.2) *respectively. Then, we have for all $\mathbf{t}^* \in \mathbb{T}_0^*$, $x \in \mathbf{t}$, $k \in \mathbb{N}$:*

$$\mathbb{P}\left(\tau_C^* \in \mathbb{T}_+(\mathbf{t}^*, x, k)\right) = C(\mathbf{t}^*, x)\left((1-\mu(\mathbf{p}))\,\alpha_{\infty,x} + \mathbb{E}\left[\left(X - k_x(\mathbf{t})\right)_+ \alpha_{X,x}\mathbb{1}_{\{X\geq k\}}\right]\right). \tag{7.14}$$

with $C(\mathbf{t}^*, x)$ defined in (7.13) *and $z_+ = \max(z, 0)$ for $z \in \mathbb{R}$.*

Proof. Let $\mathbf{t}^* \in \mathbb{T}_0^*$, $x \in \mathbf{t}$, $k \in \mathbb{N}$, we denote by η_x for $\eta_x(\mathbf{t})$, and we have:

$$\mathbb{P}\left(\tau_C^* \in \mathbb{T}_+(\mathbf{t}^*, x, k)\right) = \prod_{\substack{u \in \mathbf{t} \\ u \neq x}} \mathbf{p}(k_u(\mathbf{t})) \prod_{\substack{u \in \mathbf{t},\ u \neq x \\ \eta_u = 1}} \mathbf{q}(k_u(\mathbf{t})) \prod_{\substack{u \in \mathbf{t},\ u \neq x \\ \eta_u = 0}} (1 - \mathbf{q}\left(k_u(\mathbf{t})\right))$$
$$\times \left((1-\mu)\left(\mathbf{q}(\infty)\eta_x + (1-\mathbf{q}(\infty))(1-\eta_x)\right) + \sum_{j \geq 0} \alpha_{k+j,x} \check{\mathbf{p}}(k+j) \frac{k+j-k_x(\mathbf{t})}{k+j} \right).$$

Indeed, x is a special node. Thereby, if x has an infinity of children we obtain the term at the second line, otherwise we have the probability $\mathbf{p}(k+j)(k+j)$ to have $k+j$ children, and we must to choose the special node knowing that it cannot belong to the $k_x(\mathbf{t})$ children from $\mathbf{t}$, which give us the term $\dfrac{k+j-k_x(\mathbf{t})}{k+j}$. Thus we have:

$$\begin{aligned} \frac{\mathbb{P}\left(\tau_C^* \in \mathbb{T}_+(\mathbf{t}^*, x, k)\right)}{C(\mathbf{t}^*, x)} &= (1-\mu)\alpha_{\infty,x} + \sum_{j \geq 0} \mathbf{p}(k+j)\alpha_{k+j,x}(k+j-k_x(\mathbf{t})) \\ &= \left(1-\mu(\mathbf{p})\right)\alpha_{\infty,x} + \mathbb{E}\left[\left(X - k_x(\mathbf{t})\right)_+ \alpha_{X,x} \mathbb{1}_{\{X \geq k\}}\right]. \end{aligned}$$

That concludes the proof. □

Chapter 8

Local Limits of Conditioned Marked Galton Watson Trees

This chapter reflects work that is currently submit in a journal and available in [4]. Plus, we consider $\mathbf{p}$ has a moment of order 2. We begin to give another convergence criterion that permits to have our local limits of conditioned MGW.

8.1 Convergence criterions

We adapt the convergence criterions seen in [7] and [6].

Let $(T_n)_{n\in\mathbb{N}^*}$ and T be $\mathbb{T}_\infty$-valued (resp. $\mathbb{T}$-valued) random variables. We denote by dist(T) the distribution of the random variable T (which is uniquely determined by the sequence of distributions of $r_{h,\infty}(T)$ (resp. $r_h(T)$) for every $h \geq 0$), and we denote :

$$\text{dist}(T_n) \underset{n\to+\infty}{\longrightarrow} \text{dist}(T),$$

for the convergence in distribution of the sequence $(T_n)_{n\in\mathbb{N}^*}$ to T. Notice that this convergence in distribution is equivalent to the finite dimensional convergence in distribution of $(k_u(T_n))_{u\in\mathcal{U}}$ to $(k_u(T))_{u\in\mathcal{U}}$ when n goes to infinity.
We deduce from the portmanteau theorem that the sequence $(T_n)_{n\in\mathbb{N}^*}$ converges in distribution to T if and only if for all $h \in \mathbb{N}$, $\mathbf{t} \in \mathbb{T}_\infty^{(h)}$:

$$\lim_{n\to+\infty} \mathbb{P}(r_{h,\infty}(T_n) = \mathbf{t}) = \mathbb{P}(r_{h,\infty}(T) = \mathbf{t}) \text{ (resp. } \lim_{n\to+\infty} \mathbb{P}(r_h(T_n) = \mathbf{t}) = \mathbb{P}(r_h(T) = \mathbf{t})).$$

From now on, we endow the sets of marked trees, with the distance, for $\mathbf{t}^*, \mathbf{t}'^* \in \mathbb{T}_\infty^*$ (resp. $\mathbb{T}^*$):

$$d_\infty^*(\mathbf{t}^*, \mathbf{t}'^*) := 2^{-\max\{h\in\mathbb{N}, r_{h,\infty}^*(\mathbf{t}^*)=r_{h,\infty}^*(\mathbf{t}'^*)\}} \left(\text{resp. } d^*(\mathbf{t}^*, \mathbf{t}'^*) := 2^{-\max\{h\in\mathbb{N}, r_h^*(\mathbf{t}^*)=r_h^*(\mathbf{t}'^*)\}} \right).$$

The following results stay true when we restrict to $\mathbb{T}^*$.

Proposition 8.1. *The distance d_∞^* is ultra-metric.*

Proof. Let $\mathbf{t}^*, \mathbf{t}'^*, \mathbf{t}''^* \in \mathbb{T}_\infty^*$.

— We have $d^*_\infty(\mathbf{t}^*, \mathbf{t}'^*) \geq 0$ and $d^*_\infty(\mathbf{t}^*, \mathbf{t}^*) = 0$ according to the definition.

— Separation:
According to the definition of $r^*_{h,\infty}$,

$$\begin{aligned} d^*_\infty(\mathbf{t}^*, \mathbf{t}'^*) = 0 &\iff 2^{-\max\{h\in\mathbb{N}, r^*_{h,\infty}(\mathbf{t}^*)=r^*_{h,\infty}(\mathbf{t}'^*)\}} = 0 \\ &\iff \forall h \in \mathbb{N}\; r^*_{h,\infty}(\mathbf{t}^*) = r^*_{h,\infty}(\mathbf{t}'^*) \\ &\iff \mathbf{t}^* = \mathbf{t}'^*. \end{aligned}$$

— The symmetry is obvious.

— To prove ultrametric inequality, we denote by

$$\ell := \max\{h \in \mathbb{N}, r^*_{h,\infty}(\mathbf{t}^*) = r^*_{h,\infty}(\mathbf{t}'^*)\},\; \ell' := \max\{h \in \mathbb{N}, r^*_{h,\infty}(\mathbf{t}''^*) = r^*_{h,\infty}(\mathbf{t}'^*)\}$$
$$\text{and } \ell'' := \max\{h \in \mathbb{N}, r^*_{h,\infty}(\mathbf{t}^*) = r^*_{h,\infty}(\mathbf{t}''^*)\}.$$

If $\ell \geq \ell'$ or $\ell \geq \ell''$, we have:

$$d^*_\infty(\mathbf{t}^*, t'^*) \leq \max\{d^*_\infty(\mathbf{t}''^*, \mathbf{t}'^*), d^*_\infty(\mathbf{t}^*, \mathbf{t}''^*)\}.$$

The case $\ell < \ell'$ and $\ell < \ell''$ is impossible since $\mathbf{t}^*$ and $\mathbf{t}''^*$ are identical on ℓ'' generations and $\mathbf{t}''^*$ and $\mathbf{t}'^*$ are identical on ℓ' generations, thus we must have $\mathbf{t}^*$ and $\mathbf{t}'^*$ identical on $\min\{\ell'', \ell'\}$ generations, but $\min\{\ell'', \ell'\} > \ell$, thus, this case is impossible.

Which proves our result. □

Consider the closed ball $B^*_\infty(\mathbf{t}^*, 2^{-h}) = \{\mathbf{t}'^* \in \mathbb{T}^*_\infty, d^*_\infty(\mathbf{t}^*, \mathbf{t}'^*) \leq 2^{-h}\}$ for some $\mathbf{t}^* \in \mathbb{T}^*_\infty$ and $h \in \mathbb{N}$, we have $B^*_\infty(\mathbf{t}^*, 2^{-h}) = {r^*_{h,\infty}}^{-1}(r^*_{h,\infty}(\mathbf{t}^*))$. Since the distance is ultra-metric, the closed balls are open and the open balls are closed, and the intersection of two balls is either empty or one of them. We deduce that the family $(r^*_{h,\infty}(\mathbf{t}^*), \mathbf{t}^* \in {\mathbb{T}^{(h)}_\infty}^*, h \in \mathbb{N})$ is a π-system, and Theorem 2.3 in [9] implies that this family is convergence determining for the convergence in distribution.

Proposition 8.2. *For all $\mathbf{t}^* \in \mathbb{T}^*_\infty$, $x \in \mathbf{t}$ and $k \in \mathbb{N}$ we have $\mathbb{T}_+(\mathbf{t}^*, x, k)$ is closed and open.*

Proof. — Let $(\mathbf{t}^* \circledast_x s^*_n,\; s^*_n \in \mathbb{T}^*_\infty,\; k_x(\mathbf{t} \circledast_x s_n) \geq k)_{n\in\mathbb{N}^*}$ be a sequence of $\mathbb{T}_+(\mathbf{t}^*, x, k)$ that converges to $z \in (\mathbb{T}^*_\infty, d^*_\infty)$.
For all $n \in \mathbb{N}^*$, $\mathcal{S}^x(\mathbf{t}^* \circledast_x s^*_n) = \mathcal{S}^x(\mathbf{t}^*)$, the forest that contains only the $k_x(\mathbf{t})$ first trees above x on $\mathbf{t}^* \circledast_x s^*_n$ equals $\mathscr{F}_x(\mathbf{t}^*)$ and the marking on x is the same.
Thereby, $z = \mathbf{t}^* \circledast_x s^*$, with s^* the limit of the sequence $(s^*_n, n \in \mathbb{N}^*)$. Thus $z \in \mathbb{T}_+(\mathbf{t}^*, x, k)$.
$\mathbb{T}_+(\mathbf{t}^*, x, k)$ is closed.

— Let $s^* \in \mathbb{T}_+(\mathbf{t}^*, x, k)$ and $c_k = \max(k, H_\infty(\mathbf{t}) - 1)$. We have, according to the definition of s^*:

$$\begin{aligned} B^*_\infty(s^*, 2^{-c_k}) &= {r^*_{c_k,\infty}}^{-1}(r^*_{c_k,\infty}(s^*)) \\ &= {r^*_{c_k,\infty}}^{-1}\big(r_{c_k,\infty}(s), (\eta^{c_k}_u)_{u\in r_{c_k,\infty}(s)}\big) \subset \mathbb{T}_+(\mathbf{t}^*, x, k). \end{aligned}$$

Thus, $\mathbb{T}_+(\mathbf{t}^*, x, k)$ is open.

□

We have the following two lemmas.
According to lemma 2.1 in [7]:

Lemma 8.1. *Let $(T_n^*)_{n\in\mathbb{N}^*}$ and T^* be $\mathbb{T}^*$-valued random variables which belong a.s. to $\mathbb{T}_0^* \cup \mathbb{T}_1^*$. The sequence $(T_n^*)_{n\in\mathbb{N}^*}$ converges in distribution to T^* if and only if for every $\mathbf{t}^* \in \mathbb{T}_0^* \cap \mathbb{T}^*$ and every $x \in \mathcal{L}_0(\mathbf{t})$, we have :*

$$\lim_{n\to+\infty} \mathbb{P}(T_n^* \in \mathbb{T}(\mathbf{t}^*, x)) = \mathbb{P}(T^* \in \mathbb{T}(\mathbf{t}^*, x)) \text{ and } \lim_{n\to+\infty} \mathbb{P}(T_n^* = \mathbf{t}^*) = \mathbb{P}(T^* = \mathbf{t}^*). \tag{8.1}$$

According to lemma 2.2 in [6]:

Lemma 8.2. *Let $(T_n^*)_{n\in\mathbb{N}^*}$ and T^* be $\mathbb{T}_\infty^*$-valued random variables which belong a.s. to $\mathbb{T}_0^* \cup \mathbb{T}_2^*$. The sequence $(T_n^*)_{n\in\mathbb{N}^*}$ converges in distribution to T^* if and only if for every $\mathbf{t}^* \in \mathbb{T}_0^*$, $x \in \mathbf{t}$ and $k \in \mathbb{N}$, we have :*

$$\lim_{n\to+\infty} \mathbb{P}(T_n^* \in \mathbb{T}_+(\mathbf{t}^*, x, k)) = \mathbb{P}(T^* \in \mathbb{T}_+(\mathbf{t}^*, x, k)) \text{ and } \lim_{n\to+\infty} \mathbb{P}(T_n^* = \mathbf{t}^*) = \mathbb{P}(T^* = \mathbf{t}^*). \tag{8.2}$$

We only prove Lemma 8.2 , Lemma 8.1 has a similar proof and similar to the one of Abraham and Delmas in [7].

Proof. The subclass $\mathscr{F} = \{\mathbb{T}_+(\mathbf{t}^*, x, k) \cap (\mathbb{T}_0^* \cup \mathbb{T}_2^*)\,,\ \mathbf{t}^* \in \mathbb{T}_0, x \in \mathbf{t}, k \in \mathbb{N}\} \cup \{\{\mathbf{t}^*\},\ \mathbf{t}^* \in \mathbb{T}_0^*\}$ of Borel sets on $\mathbb{T}_0^* \cup \mathbb{T}_2^*$ forms a π-system, since we have:

— $\mathscr{F} \neq \emptyset$ because $\mathbb{T}_0^* \subset \mathscr{F}$.

— Let $t_1^*, t_2^* \in \mathbb{T}_0^*$, $x_1 \in \mathbf{t}_1$, $x_2 \in \mathbf{t}_2$ and $k_1, k_2 \in \mathbb{N}$,
we denote by $\gamma_{1,2} := \mathbb{T}_+(\mathbf{t}_1^*, x_1, k_1) \cap \mathbb{T}_+(\mathbf{t}_2^*, x_2, k_2)$ and we have:

$$\gamma_{1,2} = \begin{cases} \mathbb{T}_+(\mathbf{t}_i^*, x_i, k_i), & \text{if } \mathbf{t}_i^* \in \mathbb{T}(\mathbf{t}_j^*, x_j) \text{ and } x_j \in A_{x_j}, i \neq j \in \{1,2\} \\ \mathbb{T}_+(\mathbf{t}_i^*, x_i, k_i \vee k_j), & \text{if } \mathbf{t}_i^* \in \mathbb{T}(\mathbf{t}_j^*, x_j) \text{ and } x_i = x_j, i \neq j \in \{1,2\} \\ \{\mathbf{t}_i^*\}, & \text{if } \mathbf{t}_i^* \in \mathbb{T}(\mathbf{t}_j^*, x_j) \text{ and } x_j \notin A_{x_i} \cup \{x_i\}, i \neq j \in \{1,2\} \\ \{\mathbf{t}_2^* \circledast_{x_2} \mathbf{t}''^*\}, & \text{if } \mathbf{t}_1^* = \mathbf{t}^* \circledast_{x_2} \mathbf{t}''^*,\ \mathbf{t}_2^* = \mathbf{t}^* \circledast_{x_1} \mathbf{t}'^*, \\ & k_{x_1}(\mathbf{t}_2) \geq k_1 \text{ and } k_{x_2}(\mathbf{t}_1) \geq k_2, \text{ see Figure 8.1 }, \\ \emptyset, & \text{in the other (non-symetric) cases.} \end{cases}$$

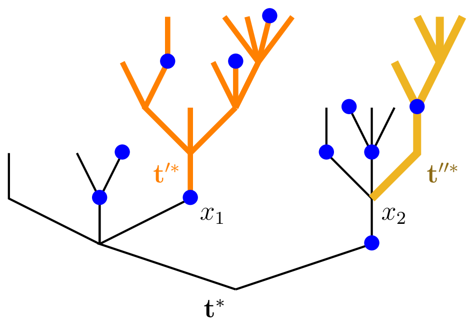


Figure 8.1 – An example for the fourth case.

For every $h \in \mathbb{N}$ and every $\mathbf{t}^* \in \mathbb{T}_\infty^{(h)^*}$, we have that s^* belongs to ${r_{h,\infty}^*}^{-1}(\{\mathbf{t}^*\}) \cap \mathbb{T}_2^*$ if and only if s^* belongs to some $\mathbb{T}_+(\mathbf{t}'^*, x, k) \cap \mathbb{T}_2^*$ with $x \in \mathbf{t}$ such that $|x|_\infty = h$ and t'^* belongs to ${r_{h,\infty}^*}^{-1}(\{\mathbf{t}^*\}) \cap \mathbb{T}_0^*$ with $x \in \mathbf{t}'$. Since $\mathbb{T}_0^*$ is countable, we deduce that $\mathscr{F}$ generates the Borel σ-field on $\mathbb{T}_0^* \cup \mathbb{T}_2^*$. In particular, $\mathscr{F}$ is a separating class in $\mathbb{T}_0^* \cup \mathbb{T}_2^*$.

Since $A \in \mathscr{F}$ is closed and open as well, according to Theorem 2.3 of [9], to prove that the family $\mathscr{F}$ is a convergence determining class, it is enough to check:
for all $\mathbf{t}^* \in \mathbb{T}_0^* \cup \mathbb{T}_2^*$ and $h \in \mathbb{N}$, there exists $A \in \mathscr{F}$ such that $\mathbf{t}^* \in A \subset B_\infty^*(\mathbf{t}^*, 2^{-h})$.

- If $\mathbf{t}^* \in \mathbb{T}_0^*$, this is clear that $\{\mathbf{t}^*\} = {r_{h,\infty}^*}^{-1}(r_{h,\infty}^*(\mathbf{t}^*)) = B_\infty^*(\mathbf{t}^*, 2^{-h})$ for all $h > H_\infty(\mathbf{t})$.
- If $\mathbf{t}^* \in \mathbb{T}_2^*$, for all $s^* \in \mathbb{T}_0^*$ and $x \in s$ such that $\mathbf{t}^* \in \mathbb{T}(s^*, x, k)$, with $k = k_x(s)$, we have $\mathbf{t}^* \in \mathbb{T}_+(s^*, x, k) \subset B_\infty^*(\mathbf{t}^*, 2^{-|x|_\infty})$. Since we can find such a s^* and x such that $|x|_\infty$ is arbitrary large, we deduce that there exists $A \in \mathscr{F}$ such that $\mathbf{t}^* \in A \subset B_\infty^*(\mathbf{t}^*, 2^{-h})$ for all $h \in \mathbb{N}$.

This proves that the family $\mathscr{F}$ is a convergence determining class in $\mathbb{T}_0^* \cup \mathbb{T}_2^*$. Since, for $\mathbf{t}^* \in \mathbb{T}_0^*$, $x \in \mathbf{t}$ and $k \in \mathbb{N}$, the sets $\mathbb{T}_+(\mathbf{t}^*, x, k)$ and $\{\mathbf{t}^*\}$ are open and closed, we deduce from the portmanteau theorem that the equivalence in the lemma holds for every $\mathbf{t}^* \in \mathbb{T}_0^*$, $x \in \mathbf{t}$ and $k \in \mathbb{N}$. $\square$

8.2 Change of offspring distribution and conditioning

Let $\mathbf{p}$ be a distribution on $\mathbb{N}$ satisfying (6.1), X be a random variable with distribution $\mathbf{p}$ and $\mathbf{q}$ be a mark function satisfying (6.2). Let τ^* be a MGW with offspring distribution $\mathbf{p}$ and mark function $\mathbf{q}$ and $\tau^*(\mathbf{p}_\theta, \mathbf{q}_\theta)$ (noted τ_θ^*) a MGW with offspring distribution $\mathbf{p}_\theta$ and mark function $\mathbf{q}_\theta$.
We want to show that the conditional distributions of τ^* given $\{M(\tau^*) = n\}$ is the same as the conditional distribution of τ_θ^* given $\{M(\tau_\theta^*) = n\}$, which allows us to prove the convergence in distribution of τ^* given $\{M(\tau^*) = n\}$ in the sub-critical case, (see Corollary 8.1, Corollary 8.2 and Corollary 8.5).

For every $\theta > 0$, we recall the function $\mathbf{p}_\theta$ (6.3) defined on $\mathbb{N}$ by

$$\forall k \geq 0, \ \mathbf{p}_\theta(k) := \theta^{k-1}\mathbf{p}(k)\left(c_\theta \mathbf{q}(k) + 1 - \mathbf{q}(k)\right),$$

where the normalizing constant c_θ is given by:

$$c_\theta := \frac{1 - \mathbb{E}\left[\theta^{X-1}(1 - \mathbf{q}(X))\right]}{\mathbb{E}\left[\theta^{X-1}\mathbf{q}(X)\right]}.$$

We denote by I the set of θ such that $\mathbf{p}_\theta$ defines a probability distribution on $\mathbb{N}$. We have $\theta \in I$ if and only if $\theta > 0$ and:

$$g(\theta) = \mathbb{E}[\theta^X] < +\infty \text{ and } l(\theta) = \mathbb{E}[\theta^X(1 - \mathbf{q}(X))] < \theta \tag{8.3}$$

Moreover, as $\mathbf{p}$ satisfies (6.1), we can prove that for all $\theta \in I$, $\mathbf{p}_\theta$ is an offspring distribution satisfying the first two conditions of (6.1). Indeed:

- According to (8.3), $c_\theta \geq 0$, thus we have $\mathbf{p}_\theta(0) = \frac{1}{\theta}\mathbf{p}(0)\left(c_\theta \mathbf{q}(0) + 1 - \mathbf{q}(0)\right) > 0$,

- Since $\mathbf{p}$ satisfies (6.1), there exists $i \in \mathbb{N}$ such that $i > 1$ and $\mathbf{p}(i) > 0$. Thereby, $\mathbf{p}_\theta(i) > 0$ and $\mathbf{p}_\theta(0) + \mathbf{p}_\theta(1) < 1$.

The generating function g_θ of $\mathbf{p}_\theta$ is given by:

$$g_\theta(z) := \frac{1}{\theta}\mathbb{E}\left[(z\theta)^X \left(c_\theta \mathbf{q}(X) + 1 - \mathbf{q}(X)\right)\right]. \tag{8.4}$$

We define also a modified mark function $\mathbf{q}_\theta$ on $\mathbb{N}$ by:

$$\mathbf{q}_\theta(k) = \frac{c_\theta \mathbf{q}(k)}{c_\theta \mathbf{q}(k) + 1 - \mathbf{q}(k)}, 1 - \mathbf{q}_\theta(k) = \frac{1 - \mathbf{q}(k)}{c_\theta \mathbf{q}(k) + 1 - \mathbf{q}(k)}, \tag{8.5}$$

and note that $\mathbf{q}_\theta$ satisfies (6.2), and for all $k \in \mathbb{N}$:

$$\mathbf{p}_\theta(k)\mathbf{q}_\theta(k) = c_\theta \theta^{k-1}\mathbf{p}(k)\mathbf{q}(k), \tag{8.6}$$

$$\mathbf{p}_\theta(k)(1 - \mathbf{q}_\theta(k)) = \theta^{k-1}\mathbf{p}(k)(1 - \mathbf{q}(k)). \tag{8.7}$$

Moreover, we have, for all $k \in \mathbb{N}$:

$$\mathbf{p}_\theta(k)\mathbf{q}_\theta(k) > 0 \Leftrightarrow \mathbf{p}(k)\mathbf{q}(k) > 0.$$

Then, throughout the rest of the chapter, $\tau^*(\mathbf{p}_\theta, \mathbf{q}_\theta)$ (noted τ_θ^*) denotes a MGW with offspring distribution $\mathbf{p}_\theta$ and mark function $\mathbf{q}_\theta$.
The introduction of these modified distribution is motivated by the following proposition:

Proposition 8.3. *Let τ^* a sub-critical MGW with offspring distribution $\mathbf{p}$ satisfying* (6.1) *and $\mathbf{q}$ its mark function satisfying* (6.2). *For every $\theta \in I$, let τ_θ^* be a MGW with offspring distribution $\mathbf{p}_\theta$ and $\mathbf{q}_\theta$ its mark function. Then the conditional distributions of τ^* given $\{M(\tau^*) = n\}$ and of τ_θ^* given $\{M(\tau_\theta^*) = n\}$ are the same.*

Remark 8.1. *The proposition allows us to prove the convergence in distribution of τ^* given $\{M(\tau) = n\}$ in the sub-critical case, (see Corollary 8.1, Corollary 8.2 and Corollary 8.5).*

Proof. For all $\mathbf{t}^* \in \mathbb{T}_0^*$, using (7.4), (8.6) and (8.7):

$$\begin{aligned}
\mathbb{P}(\tau_\theta^* = \mathbf{t}^*) &= \prod_{\substack{u\in\mathbf{t},\\ \eta_u=1}} \mathbf{p}_\theta(k_u(\mathbf{t}))\mathbf{q}_\theta(k_u(\mathbf{t})) \prod_{\substack{u\in\mathbf{t},\\ \eta_u=0}} \mathbf{p}_\theta(k_u(\mathbf{t}))(1 - \mathbf{q}_\theta(k_u(\mathbf{t}))) \\
&= \prod_{u\in\mathbf{t}} \theta^{k_u(\mathbf{t})-1}\mathbf{p}(k_u(\mathbf{t})) \prod_{\substack{u\in\mathbf{t},\\ \eta_u=1}} c_\theta \mathbf{q}(k_u(\mathbf{t})) \prod_{\substack{u\in\mathbf{t},\\ \eta_u=0}} (1 - \mathbf{q}(k_u(\mathbf{t}))) \\
&= \theta^{-1}{c_\theta}^{M(\mathbf{t}^*)}\mathbb{P}(\tau^* = \mathbf{t}^*).
\end{aligned}$$

We deduce that

$$\begin{aligned}
\mathbb{P}(M(\tau_\theta^*) = n) &= \sum_{\substack{\mathbf{t}^*\in\mathbb{T}_0^*\\ M(\mathbf{t}^*)=n}} \mathbb{P}(\tau_\theta^* = \mathbf{t}^*) = \sum_{\substack{\mathbf{t}^*\in\mathbb{T}_0^*\\ M(\mathbf{t}^*)=n}} \theta^{-1}{c_\theta}^{M(\mathbf{t}^*)}\mathbb{P}(\tau^* = \mathbf{t}^*) \\
&= \theta^{-1}{c_\theta}^n \sum_{\substack{\mathbf{t}^*\in\mathbb{T}_0^*\\ M(\mathbf{t}^*)=n}} \mathbb{P}(\tau^* = \mathbf{t}^*) = \theta^{-1}{c_\theta}^n\mathbb{P}(M(\tau^*) = n).
\end{aligned}$$

Finally, for every marked tree $\mathbf{t}^* \in \mathbb{T}_0^*$, such that $M(\mathbf{t}^*) = n$, we have

$$\mathbb{P}(\tau_\theta^* = \mathbf{t}^* | M(\tau_\theta^*) = n) = \frac{\mathbb{P}(\tau_\theta^* = \mathbf{t}^*)}{\mathbb{P}(M(\tau_\theta^*) = n)} = \frac{\theta^{-1}{c_\theta}^n\mathbb{P}(\tau^* = \mathbf{t}^*)}{\theta^{-1}{c_\theta}^n\mathbb{P}(M(\tau^*) = n)} = \mathbb{P}(\tau^* = \mathbf{t}^* | M(\tau^*) = n).$$

□

8.3 Generic and non-generic distributions

Here, we consider sub-critical or critical offspring distribution $\mathbf{p}$ and note that the mean of $\mathbf{p}_\theta$ is equal to:

$$\mu(\mathbf{p}_\theta) = \mathbb{E}\left[X\theta^{X-1}\left(c_\theta \mathbf{q}(X) + 1 - \mathbf{q}(X)\right)\right]. \tag{8.8}$$

If there exists $\theta \in I$ such that:

$$\mu(\mathbf{p}_\theta) = 1,$$

$\mathbf{p}$ is said generic, and non-generic otherwise.
The aim of this section is to find criteria for determining whether $\mathbf{p}$ is generic or not.

Remark 8.2. *The results of this section have been obtained in the particular case* $\mathbf{q}(k) = \mathbb{1}_{k\in A}$ *with* $A \subset \mathbb{N}$ *in [6].*

Proposition 8.4. *The following composition rule holds: for all* $\theta \in I$ *and for all* $\ell \in \mathbb{R}$ *such that* $\theta\ell \in I$*, we have :*

$$\mathbf{p}_{\theta\ell} = (\mathbf{p}_\theta)_\ell. \tag{8.9}$$

Proof. Let $\theta \in I$ and $\ell \in \mathbb{R}$ such that $\theta\ell \in I$. We consider X a random variable with distribution $\mathbf{p}$ and its mark function $\mathbf{q}$, and $\bar{X}$ a random variable with distribution $\mathbf{p}_\theta$ and its mark function $\mathbf{q}_\theta$.

For all $k \in \mathbb{N}$ we have:

$$\mathbf{p}_{\theta\ell}(k) = (\theta\ell)^{k-1}\mathbf{p}(k)\left(c_{\theta\ell}\mathbf{q}(k) + 1 - \mathbf{q}(k)\right), \;\; c_{\theta\ell} = \frac{1 - \mathbb{E}\left[(\theta\ell)^{X-1}(1-\mathbf{q}(X))\right]}{\mathbb{E}\left[(\theta\ell)^{X-1}\mathbf{q}(X)\right]},$$

and

$$(\mathbf{p}_\theta)_\ell(k) = \ell^{k-1}\mathbf{p}_\theta(k)\left(\tilde{c}_\ell \mathbf{q}_\theta(k) + 1 - \mathbf{q}_\theta(k)\right), \;\; \tilde{c}_\ell = \frac{1 - \mathbb{E}\left[\ell^{\bar{X}-1}(1-\mathbf{q}_\theta(\bar{X}))\right]}{\mathbb{E}\left[\ell^{\bar{X}-1}\mathbf{q}_\theta(\bar{X})\right]}.$$

Using again (8.6) and (8.7), we have:

$$(\mathbf{p}_\theta)_\ell(k) = (\theta\ell)^{k-1}\mathbf{p}(k)\left(\tilde{c}_\ell c_\theta \mathbf{q}(k) + 1 - \mathbf{q}(k)\right).$$

We have our result if $\tilde{c}_\ell c_\theta = c_{\theta\ell}$, which is true as:

$$\mathbb{E}\left[\ell^{\bar{X}-1}(1-\mathbf{q}_\theta(\bar{X}))\right] = \sum_{k=0}^{+\infty}(\theta\ell)^{k-1}\mathbf{p}(k)(1-\mathbf{q}(k)) = \mathbb{E}\left[(\theta\ell)^{X-1}(1-\mathbf{q}(X))\right],$$

$$\mathbb{E}\left[\ell^{\bar{X}-1}\mathbf{q}_\theta(\bar{X})\right] = c_\theta\sum_{k=0}^{+\infty}(\theta\ell)^{k-1}\mathbf{p}(k)\mathbf{q}(k) = c_\theta\mathbb{E}\left[(\theta\ell)^{X-1}\mathbf{q}(X)\right].$$

Finally, we find $\tilde{c}_\ell c_\theta = c_{\theta\ell}$. □

Notice that I is an interval of $(0, +\infty)$ which contains 1 and let:

$$\theta_s := \sup I \in [1, \rho(\mathbf{p})]. \tag{8.10}$$

with $\rho(\mathbf{p})$ the radius of convergence of g, the generating function in (8.3) and we define the function l by:

$$l(\theta) = \mathbb{E}\left[\theta^X(1-\mathbf{q}(X))\right], \tag{8.11}$$

and note that its radius of convergence $\rho_l(\mathbf{p},\mathbf{q})$ satisfies $\rho(\mathbf{p}) \le \rho_l(\mathbf{p},\mathbf{q})$.

Lemma 8.3. *Let* $\mathbf{p}$ *be a distribution on* $\mathbb{N}$ *satisfying the two first conditions of* (6.1) *and* $\mathbf{q}$ *satisfying* (6.2). *For* $\theta \in \overset{\circ}{I}$, *we have:*

$$\frac{d}{d\theta}\mu(\mathbf{p}_\theta) > 0 \text{ if } \mu(\mathbf{p}_\theta) \leq 1.$$

Proof. Since $\mathbf{p}$ satisfies the two first conditions of (6.1), then $\mathbf{p}_\theta$ satisfies the two first conditions of (6.1) for all $\theta \in I$ such that $\theta < \theta_s$ (see Section 8.2). Thanks to the composition rule (8.9), it is enough to prove that $\frac{\mathrm{d}}{\mathrm{d}\theta}\mu(\mathbf{p}_\theta) > 0$ at $\theta = 1$ if $\mu(\mathbf{p}) \leq 1$, with $\mathbf{p}$ satisfying (6.1) and $\rho(\mathbf{p}) > 1$.

For $\kappa \in I$, we have:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\kappa}\mu(\mathbf{p}_\kappa) &= \lim_{h\to 0}\frac{\mu(\mathbf{p}_{\kappa+h}) - \mu(\mathbf{p}_\kappa)}{h} = \lim_{h\to 0}\frac{\mu(\mathbf{p}_{\kappa(1+\frac{h}{\kappa})}) - \mu(\mathbf{p}_\kappa)}{h} \\ &= \lim_{h\to 0}\frac{\mu((\mathbf{p}_\kappa)_{(1+\frac{h}{\kappa})}) - \mu(\mathbf{p}_\kappa)}{h} = \kappa^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}\eta}\mu\left((\mathbf{p}_\kappa)_\eta\right)\right)_{|\eta=1}.\end{aligned}$$

Therefore, we assume that

$$\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\mu(\mathbf{p}_\theta)\right)_{|\theta=1} > 0 \text{ if } \mu(\mathbf{p}) \leq 1. \tag{8.12}$$

We apply the result in (8.12), with $\theta = \eta$ and $\mathbf{p} = \mathbf{p}_\kappa$.

We recall (6.4) : $c_\theta = \dfrac{1 - \mathbb{E}\left[\theta^{X-1}(1-\mathbf{q}(X))\right]}{\mathbb{E}\left[\theta^{X-1}\mathbf{q}(X)\right]}$.

Let $\theta \in I$. To make the calculations easier, we write the following formula:

$$\mu(\mathbf{p}_\theta) - \mathbb{E}[X] =: \frac{\mathbb{E}\left[X\theta^X\mathbf{q}(X)\right]d(\theta) + \mathbb{E}\left[\theta^X\mathbf{q}(X)\right]h(\theta)}{\mathbb{E}\left[\theta^X\mathbf{q}(X)\right]} =: \frac{f(\theta)}{\mathbb{E}\left[\theta^X\mathbf{q}(X)\right]},$$

with, $d(\theta) = 1 - \mathbb{E}\left[\theta^{X-1}\right]$ and $h(\theta) = \mathbb{E}\left[X\theta^{X-1}\right] - \mathbb{E}[X]$.
Moreover, we have $f(1) = d(1) = h(1) = 0$ and the functions f, d and h are of class $\mathcal{C}^\infty$ on $[0, \rho(\mathbf{p}))$ (we recall that $\rho(\mathbf{p}) < \rho_l(\mathbf{p},\mathbf{q})$). Notice that

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\mu(\mathbf{p}_\theta)\right)_{|\theta=1} &= \frac{f'(1)}{\mathbb{E}[\mathbf{q}(X)]} = \frac{\mathbb{E}[X\mathbf{q}(X)]\,d'(1) + \mathbb{E}[\mathbf{q}(X)]\,h'(1)}{\mathbb{E}[\mathbf{q}(X)]} \\ &= \frac{\mathbb{E}[X\mathbf{q}(X)]}{\mathbb{E}[\mathbf{q}(X)]}(1 - \mathbb{E}[X]) + \mathbb{E}[X(X-1)].\end{aligned}$$

Since $\mathbb{E}[X] \leq 1$ and $\mathbb{E}[X(X-1)] > 0$ by assumption (6.1), we deduce that $f'(1) > 0$, which concludes our demonstration.

□

Lemma 8.4. *Let* $\mathbf{p}$ *be a distribution on* $\mathbb{N}$ *satisfying* (6.1) *and* $\mathbf{q}$ *the associated mark function satisfying* (6.2). *The equation*

$$\gamma(\theta) := \mu(\mathbf{p}_\theta) = 1 \tag{8.13}$$

has at most one solution in I. *If there is no solution to* (8.13), *then we have* $\mu(\mathbf{p}) < 1$, θ_s *belongs to* I *and* $\mu(\mathbf{p}_{\theta_s}) < 1$. *The unique solution of* (8.13) *in* I, *if it exists, is denoted by* θ_c.

Proof. We mimic the proof of Lemma 5.2 in [6].

If $\mu(\mathbf{p}) \leq 1$, we satisfies Lemma 8.3, thus the fact that (8.13) has at most one solution, is a direct consequence of this lemma. Indeed, if it exists another θ satisfying (8.13). Then for all $\varepsilon > 0$ and $\nu \in]\theta - \varepsilon, \theta + \varepsilon[$, $\dfrac{\mathrm{d}}{\mathrm{d}\nu}\mu(\mathbf{p}_\nu) < 0$ and it is a contradiction of Lemma 8.3.

We set $m = \min I \in (0, 1)$. Notice that $l(m) = m$.
Indeed, m satisfies (8.3), thus we have $l(m) \leq m$, with l continuous on I. The function $l(\theta) : \theta \mapsto \mathbb{E}[\theta^X(1 - \mathbf{q}(X))]$ is convex and less than the identity map on $(m, 1]$ since $l(1) < 1$. If $l(m) < m$ by continuity of l, it exists $m' < m$ such that $l(m') < m'$ and $g(m') < +\infty$, which is absurd by definition of m.
Since, $l(m) = m$, we have $c_m = 0$. Thanks to a convexity argument and the fact that the function l is less than the identity map on $(m, 1]$, we can deduce that $\mathbb{E}\left[Xm^{X-1}(1 - \mathbf{q}(X))\right]$ is strictly less than 1. Then use (8.8) to deduce that:

$$\lim_{\theta \to m} \mu(\mathbf{p}_m) = \mathbb{E}\left[Xm^{X-1}(1 - \mathbf{q}(X))\right] = l'(m) \leq l'(1) = \mathbb{E}[X(1 - \mathbf{q}(X))] < 1.$$

To conclude, we deduce that $\inf_{\theta \in I} \mu(\mathbf{p}_\theta) < 1$. Hence, if $\mu(\mathbf{p}) \geq 1$ then (8.13) has at least one solution, since $\mu(\mathbf{p}_1) = \mu(\mathbf{p})$.

Thereby, if there is no solution to (8.13), this implies that $\mu(\mathbf{p}) < 1$ and thus:

$$\mu(\mathbf{p}_\theta) < 1 \text{ for all } \theta \in I. \tag{8.14}$$

Now we want to prove that $\theta_s \in I$ and we only need to consider the case $\theta_s > 1$. Since $\theta_s \leq \rho(\mathbf{p})$, we have $\rho(\mathbf{p}) > 1$. Since $\mu(\mathbf{p}) < 1$, the interval $J = \{\theta;\ g(\theta) < \theta\}$ is non-empty, $1 \in J$. On $J \bigcap I$, we deduce from (6.4) that $c_\theta > 1$ and then from (8.8) that $\mu(\mathbf{p}_\theta) > g'(\theta)$. Thus we have $g'(\theta) < 1$ on $J \bigcap I$, that implies $\theta \mapsto g(\theta) - \theta$ is decreasing and that $I \bigcap (1, +\infty)$ is a subset of $\overline{J}$, since for all $\theta \geq 1$, $g(\theta) - \theta \leq g(1) - 1 < 0$. The properties of g imply that $\overline{J} = \{\theta;\ g(\theta) \leq \theta\}$.
This clearly implies that $g(\theta_s) \leq \theta_s$, since $0 < l(\theta_s) < g(\theta_s)$, then $l(\theta_s) < \theta_s$. Thereby, (8.3) holds for θ_s that is $\theta_s \in I$. Then conclude using (8.14). □

Definition 8.1. *Let* $\mathbf{p}$ *be a sub-critical distribution on* $\mathbb{N}$ *satisfying* (6.1) *and* $\mathbf{q}$ *a mark function satisfying* (6.2). *If* (8.13) *has a (unique) solution in* I, *then* $\mathbf{p}$ *is called generic. If* (8.13) *has no solution, then* $\mathbf{p}$ *is called non-generic.*

In the next lemma we write ρ for $\rho(\mathbf{p})$.

Lemma 8.5. *Let* $\mathbf{p}$ *be a distribution on* $\mathbb{N}$ *satisfying* (6.1), *such that* $\mu(\mathbf{p}) < 1$ *and* $\mathbf{q}$ *a mark function satisfying* (6.2).

1. *If* $\rho = 1$, *then* $\mathbf{p}$ *is non-generic.*
2. *If* $\rho = +\infty$ *or* ($1 < \rho < +\infty$ *and* $l'(\rho) \geq 1$), *then* $\mathbf{p}$ *is generic.*
3. *If* $1 < \rho < +\infty$ *and* $l'(\rho) < 1$, *then,*
   $\mathbf{p}$ *is non-generic if, and only if:*

$$G(\theta_s) < \frac{1 - l'(\theta_s)}{\theta_s - l(\theta_s)}, \text{ with } G(t) := \frac{\mathbb{E}[Xt^{X-1}\mathbf{q}(X)]}{\mathbb{E}[t^X\mathbf{q}(X)]}. \tag{8.15}$$

Moreover, if $g'(\rho) < 1$, *we have* $\theta_s = \rho$.

Proof. Let $\theta \in I$, notice that:

$$\mu(\mathbf{p}_\theta) - 1 = G(\theta)(\theta - l(\theta)) - (1 - l'(\theta)) \,. \tag{8.16}$$

1. If $\rho = 1$, we have $\theta_s = 1 \in I$, and $\mu(\mathbf{p}_1) = \mu(\mathbf{p}) < 1$. Thus, $\mathbf{p}$ is non-generic.
2. If ($\rho = +\infty$ and there exists $k \in \mathbb{N}^*, k \neq 1$, such that $\mathbf{p}(k)(1 - \mathbf{q}(k)) > 0$) or $(1 < \rho < +\infty$ and $l'(\rho) \geq 1)$, since l is convex, l' is non decreasing and then there exists $m \in (1, \rho)$ such that $l'(m) = 1$.
   Thus, we have $l(1) \geq l(m) + l'(m)(1 - m)$, since $l(1) = \mathbb{E}\left[1 - \mathbf{q}(X)\right]$ and $\mathbb{E}\left[\mathbf{q}(X)\right] > 0$ (6.2), we have $l(m) < m$. Since $1 < m < \rho$, $g(m) < +\infty$ which implies that m satisfies (8.3).
   This implies, thanks to (8.16), that $\mu(\mathbf{p}_m) > 1$.
   If $\rho = +\infty$ and for all $k \in \mathbb{N}^*, k \neq 1$, $\mathbf{p}(k)(1 - \mathbf{q}(k)) = 0$. For all $\theta \in I$,
   $$\mu(\mathbf{p}_\theta) \geq 1 \Leftrightarrow (G(\theta)\theta - 1)(1 - l'(\theta)) + G(\theta)(\mathbf{p}(0)(1 - \mathbf{q}(0))) \geq 0,$$
   with $1 - l'(\theta) = 1 - \mathbf{p}(1)(1 - \mathbf{q}(1)) > 0$ and $G(\theta) > 0$. Furthermore,
   $$G(\theta)\theta \geq \frac{\sum\limits_{k>1} \theta^k \mathbf{p}(k) + \theta \mathbf{p}(1)(1 - \mathbf{q}(1))}{\mathbf{p}(0)(1 - \mathbf{q}(0)) + \sum\limits_{k>1} \theta^k \mathbf{p}(k) + \theta \mathbf{p}(1)(1 - \mathbf{q}(1))},$$
   since $\rho = \infty$, the right term tends to 1 when θ tends to infinity, and $\mu(\mathbf{p}_{\theta_s}) \geq 1$. Then there exists $m \leq \theta_s$ such that $\mathbf{p}_m$ is critical.
   Therefore, $\mathbf{p}$ is generic, in any cases.
3. If $1 < \rho < +\infty$ and $l'(\rho) < 1$, $\mathbf{p}$ is non-generic if $\mu(\mathbf{p}_{\theta_s}) < 1$, that is :
   $$G(\theta_s) < \frac{1 - l'(\theta_s)}{\theta_s - l(\theta_s)}.$$
   If $g'(\rho) < 1$, we have $l(\rho) < g(\rho) < \rho$. Thereby, ρ satisfies (8.3) and we have $\theta_s = \rho \in I$.

□

8.4 Marked Tree Decomposition

To prove our results, we need to construct a model adapted to the one introduced by Minami [27], Rizzolo [30] or Abraham and Delmas [6].

In other words, let a marked tree $\mathbf{t}^* \in \mathbb{T}_0^*$. We define a map ϕ from $\{u \in \mathbf{t}, \eta_u = 1\}$ into the set $\bigcup\limits_{n\geq 1} \mathbb{T}_0^n$ of forests of finite trees as follows.

To begin with, for $u \in \mathbf{t}$ we define $\mathcal{S}_u^{\mathbf{q}}(\mathbf{t}^*)$ the sub-tree rooted at u with no marked descendants by

$$\mathcal{S}_u^{\mathbf{q}}(\mathbf{t}^*) := \{w \in u\mathcal{S}_u(\mathbf{t}),\ A_w \cap A_u^c \cap \{v \in \mathbf{t}, \eta_v = 1\} = \emptyset\}.$$

We also define $\mathcal{C}_u^{\mathbf{q}}(\mathbf{t}^*)$, the set of leaves of $\mathcal{S}_u^{\mathbf{q}}(\mathbf{t}^*)$ that belong to $\{u \in \mathbf{t}, \eta_u = 1\}$, in other words

$$\mathcal{C}_u^{\mathbf{q}}(\mathbf{t}^*) := \{v \in \mathcal{S}_u^{\mathbf{q}}(\mathbf{t}),\ k_v(\mathbf{t}) = 0, \eta_v = 1\}.$$

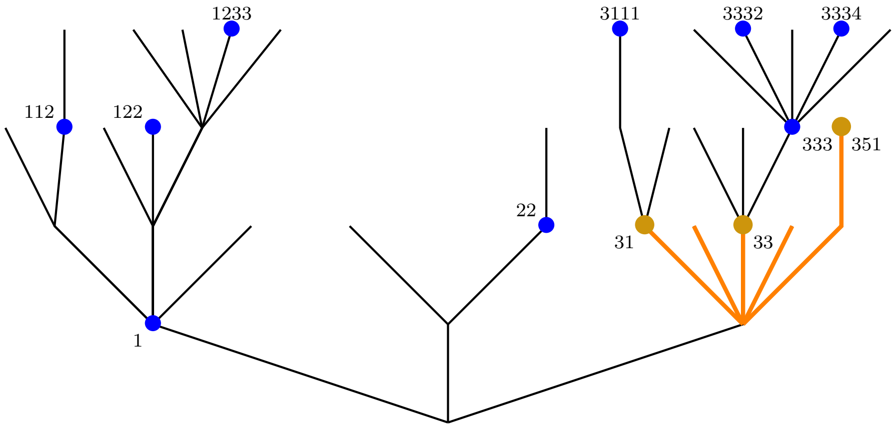


Figure 8.2 – The sub-tree $\mathcal{S}_3^{\mathbf{q}}(\mathbf{t}^*)$ in bold and orange, and the elements of $\mathcal{C}_3^{\mathbf{q}}(\mathbf{t}^*)$ in gold.

We set

$$\tilde{\mathcal{S}}_\emptyset^{\mathbf{q}}(\mathbf{t}^*) := \begin{cases} \mathcal{S}_\emptyset^{\mathbf{q}}(\mathbf{t}^*), & \text{if } \eta_\emptyset = 0 \\ \{\emptyset\}, & \text{if } \eta_\emptyset = 1 \end{cases}$$

and we set $\tilde{\mathcal{C}}_\emptyset^{\mathbf{q}}(\mathbf{t}^*)$, the set of leaves of $\tilde{\mathcal{S}}_\emptyset^{\mathbf{q}}(\mathbf{t}^*)$ that belong to $\{u \in \mathbf{t}, \eta_u = 1\}$.

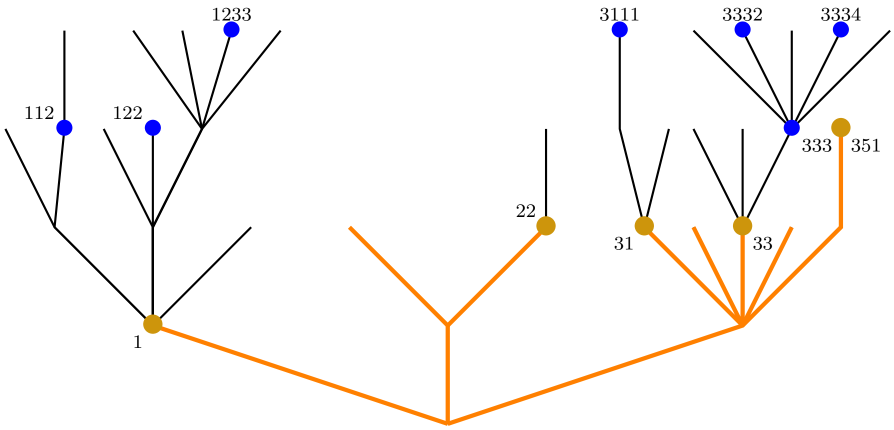


Figure 8.3 – The sub-tree $\tilde{\mathcal{S}}_\emptyset^{\mathbf{q}}(\mathbf{t}^*)$ in bold and orange, and the elements of $\tilde{\mathcal{C}}_\emptyset^{\mathbf{q}}(\mathbf{t}^*)$ in gold.

Let $\tilde{N}_\emptyset(\mathbf{t}^*) := \left|\tilde{\mathcal{C}}_\emptyset^{\mathbf{q}}(\mathbf{t}^*)\right|$. Then the range of ϕ belongs to $\mathbb{T}_0^{\tilde{N}_\emptyset(\mathbf{t}^*)}$. Moreover if $u_1 < u_2 < \cdots < u_{\tilde{N}_\emptyset(\mathbf{t}^*)}$ are the elements of $\tilde{\mathcal{C}}_\emptyset^{\mathbf{q}}(\mathbf{t}^*)$ ranked in lexicographic order, we set for every $1 \leq i \leq \tilde{N}_\emptyset(\mathbf{t}^*)$, $\phi(u_i) := \emptyset^{(i)}$, where $\emptyset^{(i)}$ denotes the root of the i-th tree in $\mathbb{T}_0^{\tilde{N}_\emptyset(\mathbf{t}^*)}$.

We then construct ϕ recursively: if $\eta_u = 1$ and $\phi(u) = v^{(i)}$ (which is an element of the i-th tree), we denote by $u_1 < u_2 < \cdots < u_k$ the elements of $\mathcal{C}_u^{\mathbf{q}}(\mathbf{t}^*)$ ranked in lexicographic order and we set for $1 \leq j \leq k$, $\phi(u_j) = vj^{(i)}$.

Finally, we set $\mathscr{F}_{\mathbf{q}}(\mathbf{t}) = \phi(\mathbf{t})$.

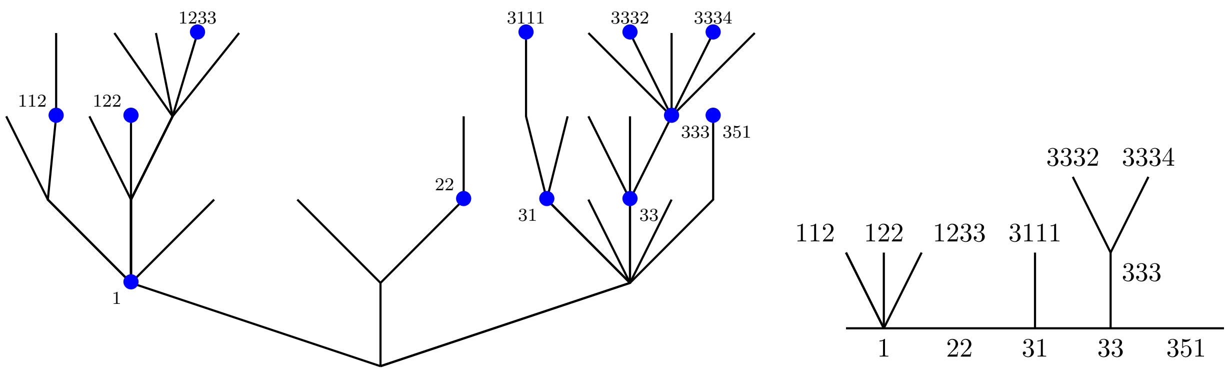


Figure 8.4 – A marked tree $\mathbf{t}^*$ and the forest $\mathscr{F}_{\mathbf{q}}(\mathbf{t})$ with 5 trees.

Let τ^* be a MGW with offspring distribution $\mathbf{p}$ and mark function $\mathbf{q}$. Let us describe the distribution of $\mathscr{F}_{\mathbf{q}}(\tau)$.

We define $\tilde{\mathbf{p}}$ by

$$\tilde{\mathbf{p}}(k) := \begin{cases} \mathbf{p}(k)(1-\mathbf{q}(k)), & \text{if } k \neq 0, \\ \mathbf{p}(0) + \sum\limits_{j=1}^{+\infty} \mathbf{p}(j)\mathbf{q}(j), & \text{otherwise.} \end{cases} \tag{8.17}$$

Note that $\tilde{\mathbf{p}}(0) > 0$ and its mean satisfies $\mu(\tilde{\mathbf{p}}) < +\infty$ as $\mathbf{p}(0) > 0$ and $\mu(\mathbf{p}) < +\infty$.

By construction, $\tilde{\mathcal{S}}^{\mathbf{q}}_{\emptyset}(\tau^*)$ is distributed as a sub-critical MGW with offspring distribution $\tilde{\mathbf{p}}$. On $\tilde{\mathcal{S}}^{\mathbf{q}}_{\emptyset}(\tau^*)$, only leaves can be marked, and any unmarked leaf in this tree was already unmarked in the original tree τ. We denote by L the number of leaves of $\tilde{\mathcal{S}}^{\mathbf{q}}_{\emptyset}(\tau^*)$ and we denote by N the number of marked leaves. Let X be a random variable with distribution $\mathbf{p}$.

Proposition 8.5. *Let* $\mathbf{p}$ *satisfying* (6.1)*, with a moment of order* 2 *and mark function* $\mathbf{q}$ *satisfying* (6.2)*. If* $\mu(\mathbf{p}) < 1$ *assume that* $\mathbf{q}$ *satisfies* (6.7) *and* (6.8)*. Then, L admits a moment of order 2 and:*

$$\mathbb{E}[L] = \frac{\tilde{\mathbf{p}}(0)}{1 - \mathbb{E}[X(1-\mathbf{q}(X))]},$$

$$\mathbb{E}[L^2] = \frac{\tilde{\mathbf{p}}(0) + \mathbb{E}[X(X-1)(1-\mathbf{q}(X))]\mathbb{E}[L]^2}{1 - \mathbb{E}[X(1-\mathbf{q}(X))]}.$$

Proof. Let Y, given X, be a Bernoulli random variable with parameter $\mathbf{q}(X)$. We determine $\mathbb{E}[L^i]$ thanks to the generating function of the random variable L.
Let $s \in [0,1)$, by decomposing according to the number of children of the root:

$$\begin{aligned} \mathbb{E}[s^L] &= \tilde{\mathbf{p}}(0)s + \mathbb{E}\left[s^{\sum_{i=1}^{X} L_i} \mathbb{1}_{\{X>0, Y=0\}} \right] \\ &= \tilde{\mathbf{p}}(0)s + \mathbb{E}\left[\mathbb{E}[s^L]^X (1-\mathbf{q}(X))\mathbb{1}_{\{X>0\}}\right] = \tilde{\mathbf{p}}(0)s + \sum_{i=1}^{+\infty} \mathbf{p}(i)(1-\mathbf{q}(i))\mathbb{E}[s^L]^i, \end{aligned} \tag{8.18}$$

where $(L_i)_{i\geq 1}$ are independent copies of L and independent of (X,Y). L_i represents the number of leaves of the tree rooted in i. Differentiating the equation (8.18), we obtain, for every $s \in [0,1)$:

$$\mathbb{E}[Ls^{L-1}] = \frac{\tilde{\mathbf{p}}(0)}{1 - \sum_{i=1}^{+\infty} \mathbf{p}(i)(1-\mathbf{q}(i))i\mathbb{E}[s^L]^{i-1}}. \tag{8.19}$$

Since we have $\mathbb{E}[X] = 1$ or $\mathbb{E}[X] < 1$ and $\mathbf{q}$ satisfies (6.8), the right member admits a limit when s tends to 1, and consequently:

$$\mathbb{E}[L] = \frac{\tilde{\mathbf{p}}(0)}{1 - \mathbb{E}[X(1-\mathbf{q}(X))]}.$$

we differentiate (8.18) a second time and we have our result as X admits a moment of order 2:

$$\mathbb{E}[L(L-1)s^{L-2}] = \frac{\mathbb{E}[Ls^{L-1}]^2 \sum_{i=1}^{+\infty} \mathbf{p}(i)(1-\mathbf{q}(i))i(i-1)\mathbb{E}[s^L]^{i-2}}{1 - \sum_{i=1}^{+\infty} \mathbf{p}(i)(1-\mathbf{q}(i))i\mathbb{E}[s^L]^{i-1}}. \tag{8.20}$$

Thereby, we have for s tends to 1:

$$\mathbb{E}[L(L-1)] = \frac{\mathbb{E}[L]^2 E[X(X-1)(1-\mathbf{q}(X))]}{1 - \mathbb{E}[X(1-\mathbf{q}(X))]},$$

and we obtain

$$\mathbb{E}[L^2] = \frac{\tilde{\mathbf{p}}(0) + \mathbb{E}[X(X-1)(1-\mathbf{q}(X))]\mathbb{E}[L]^2}{1 - \mathbb{E}[X(1-\mathbf{q}(X))]}.$$

□

Proposition 8.6. *Conditionally on L, the random variable $N := \tilde{N}_\emptyset(\tau^*)$ has a binomial distribution with parameter $\left(L, \dfrac{\mathbb{E}[\mathbf{q}(X)]}{\tilde{\mathbf{p}}(0)}\right)$, that we denote by $\mathbf{p}_N$.*

Proof. For all $k \in [\![0, L]\!]$, $\{N = k\}$ means that $\tilde{\mathcal{S}}^{\mathbf{q}}_\emptyset(\tau)$ has k marked leaves among the L leaves.

So, we have a sum of L independent Bernoulli variables with unknown parameter and we can conclude if we show that this parameter is equal to $\dfrac{\mathbb{E}[\mathbf{q}(X)]}{\tilde{\mathbf{p}}(0)}$:

$$\mathbb{P}(N = 1|L = 1) = \frac{\mathbb{P}(N = 1, L = 1)}{\mathbb{P}(L = 1)} = \frac{\sum_{k\in\mathbb{N}} \mathbf{p}(k)\mathbf{q}(k)}{\tilde{\mathbf{p}}(0)} = \frac{\mathbb{E}[\mathbf{q}(X)]}{\tilde{\mathbf{p}}(0)}.$$

□

Let us define $Z^{(1)}$ by:

$$Z^{(1)} := \sum_{k=1}^{X^{(1)}} N_k, \tag{8.21}$$

where $X^{(1)}$ is distributed as X conditionally to be marked, and $(N_i)_{i\in\mathbb{N}}$ is a sequence of independent copies of N, independent of $X^{(1)}$. Note that:

$$\mathbb{P}(X^{(1)} = k) = \frac{\mathbf{p}(k)\mathbf{q}(k)}{\mathbb{E}[\mathbf{q}(X)]} \text{ and } \mathbb{E}[X^{(1)}] = \frac{\mathbb{E}[X\mathbf{q}(X)]}{\mathbb{E}[\mathbf{q}(X)]}, \tag{8.22}$$

and for typographical simplicity, we denote by $\mathbf{p}^{(1)}$ the distribution of $Z^{(1)}$, and by $\mu^{(1)}$ its mean.

Lemma 8.6. *Assume that* $\mathbf{p}$ *satisfies* (6.1) *with a moment of order* 2 *and* $\mathbf{q}$ *satisfies* (6.2). *We have the three following properties:*

1. *if* $\mu(\mathbf{p}) < 1$*, or* $\mu(\mathbf{p}) = 1$*, then* $\mu^{(1)} < 1$*, or* $\mu^{(1)} = 1$*, respectively;*
2. *there exists* $k \in \mathbb{N}^*$ *such that* $\mathbf{p}^{(1)}(0)\mathbf{p}^{(1)}(k) > 0$*;*
3. *if* $\rho_l(\mathbf{p}, \mathbf{q}) = 1$ *then* $\rho(\mathbf{p}^{(1)}) = 1$.

Proof. 1. Using Proposition 8.5 , Proposition 8.6, and (8.22), we have:

$$\mathbb{E}[Z^{(1)}] = \mathbb{E}[X^{(1)}]\mathbb{E}[L]\frac{\mathbb{E}[\mathbf{q}(X)]}{\tilde{\mathbf{p}}(0)} = \frac{\mathbb{E}[X\mathbf{q}(X)]}{1 - \mathbb{E}[X(1-\mathbf{q}(X))]}.$$

Thus:

$$\mathbb{E}[Z^{(1)}] < 1 \iff \mathbb{E}[X\mathbf{q}(X)] < 1 - \mathbb{E}[X(1-\mathbf{q}(X))] \iff \mathbb{E}[X] < 1,$$
$$\mathbb{E}[Z^{(1)}] = 1 \iff \mathbb{E}[X\mathbf{q}(X)] = 1 - \mathbb{E}[X(1-\mathbf{q}(X))] \iff \mathbb{E}[X] = 1.$$

2. (6.2) implies that there exists $k \in \mathbb{N}^*$ such that $\mathbf{p}(k)\mathbf{q}(k) > 0$ and thus, according to (8.22), $\mathbb{P}(X^{(1)} = k) = \dfrac{\mathbf{p}(k)\mathbf{q}(k)}{\mathbb{E}[\mathbf{q}(X)]} > 0$. Since $\mathbf{p}$ satisfies (6.1) we have $\mathbb{P}(L > 1) > 0$. Consequently:

$$\mathbf{p}^{(1)}(0) = \mathbb{P}\Big(\sum_{i=1}^{X^{(1)}} N_i = 0\Big) \geq \mathbb{P}(X^{(1)} = k)\mathbb{P}(N = 0)^k > 0,$$
$$\mathbf{p}^{(1)}(k) = \mathbb{P}\Big(\sum_{i=1}^{X^{(1)}} N_i = k\Big) \geq \mathbb{P}(X^{(1)} = k)\mathbb{P}(N = 1)^k > 0.$$

3. Let $g^{(1)}$ and g_N be the generating functions of $Z^{(1)}$ and N, respectively. For all $z \geq 0$, as $N \sim \mathscr{B}(L, \frac{\mathbb{E}[\mathbf{q}(X)]}{\tilde{\mathbf{p}}(0)})$, conditionally on L:

$$g^{(1)}(z) = \mathbb{E}\left[g_N(z)^{X^{(1)}}\right] = \frac{\mathbb{E}\left[g_N(z)^X\mathbf{q}(X)\right]}{\mathbb{E}[\mathbf{q}(X)]} = \frac{\mathbb{E}\left[\mathbb{E}\left[\varphi(z)^L\right]^X \mathbf{q}(X)\right]}{\mathbb{E}[\mathbf{q}(X)]},$$

where $\varphi(z) := \dfrac{\mathbf{p}(0)\,(1-\mathbf{q}(0)) + \mathbb{E}[\mathbf{q}(X)]z}{\tilde{\mathbf{p}}(0)}$ and note that if $z > 1$ then $\varphi(z) > 1$. As L is stochastically larger than $X\mathbb{1}_{\{Y=0\}}$, with Y, given X, be a Bernoulli random variable with parameter $\mathbf{q}(X)$, we obtain:

$$g^{(1)}(z) \geq \frac{\mathbb{E}\left[\mathbb{E}\left[\varphi(z)^X(1-\mathbf{q}(X))\right]^X \mathbf{q}(X)\right]}{\mathbb{E}[\mathbf{q}(X)]} = \frac{\mathbb{E}\left[l(\varphi(z))^X\mathbf{q}(X)\right]}{\mathbb{E}[\mathbf{q}(X)]}$$

Since $\rho_l(\mathbf{p},\mathbf{q}) = 1$, if $z > 1$, we have that $l(\varphi(z)) = +\infty$ implying that $g^{(1)}(z) = +\infty$. Consequently, $\rho(\mathbf{p}^{(1)}) = 1$.

$\square$

Note that the forest $\mathscr{F}_{\mathbf{q}}(\tau)$ is distributed as N independent GW with offspring distribution $\mathbf{p}^{(1)}$.

Let $(Z_i^{(1)})_{i\in\mathbb{N}^*}$ be a sequence of independent copies of $Z^{(1)}$, independent of $(N_i)_{i\in\mathbb{N}^*}$. Then, we define $(W_n)_{n\in\mathbb{N}^*}$ and $(S_n)_{n\in\mathbb{N}^*}$ by $W_0 = S_0 = 0$ a.s, and for $n \in \mathbb{N}^*$: $W_n := \sum_{i=1}^{n} N_i$ and $S_n := \sum_{i=1}^{n} Z_i^{(1)}$.

We use the decomposition of the MGW with respect to the descendants of $\emptyset$ in $\mathscr{F}_{\mathbf{q}}(\tau)$ and Dwass formula (see [15]): for every $j \in \mathbb{N}^*$ and every $n \geq j$, we have

$$\mathbb{P}_j\left(\left|\tau^{\mathscr{F}_{\mathbf{q}}}\right| = n\right) = \frac{j}{n}\mathbb{P}(S_n = n-j). \tag{8.23}$$

Thereby, for $n \in \mathbb{N}^*$:

$$\begin{aligned}\mathbb{P}(M(\tau^*) = n) &= \sum_{j\in\mathbb{N}} \mathbf{p}_N(j)\mathbb{P}_j\left(\left|\tau^{\mathscr{F}_{\mathbf{q}}}\right| = n\right) = \sum_{j\in\mathbb{N}} \mathbf{p}_N(j)\frac{j}{n}\mathbb{P}(S_n = n-j) \\ &= \frac{1}{n}\mathbb{E}[N\mathbb{1}_{\{S_n+N=n\}}]. \end{aligned}\tag{8.24}$$

More generally, we have for $j, n \in \mathbb{N}^*$

$$\mathbb{P}_j(M(\tau^*) = n) = \sum_{i\in\mathbb{N}} \mathbb{P}(W_j = i)\mathbb{P}_i(\left|\tau^{\mathscr{F}_{\mathbf{q}}}\right| = n) = \frac{1}{n}\mathbb{E}[W_j\mathbb{1}_{\{S_n+W_j=n\}}] = \frac{j}{n}\mathbb{E}[N\mathbb{1}_{\{S_n+W_{j-1}+N=n\}}], \tag{8.25}$$

with N independent of S_n and W_{j-1}.

We explain the difference between $\mathbb{P}(M(\mathbf{t}^*) = n)$ and $\mathbb{P}_j(M(\mathbf{t}^*) = n)$, with an example, for a marked tree $\mathbf{t}^*$, $n = 6$ and $j = 4$.

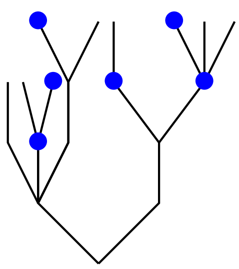


Figure 8.5 – A marked tree $\mathbf{t}^*$, with 6 marks.

When we determine $\mathbb{P}(M(\mathbf{t}^*) = 6)$, we look the whole tree like on the figure 8.5. To determine, $\mathbb{P}_4(M(\mathbf{t}^*) = 6)$, we look at all the decomposition of our tree $\mathbf{t}^*$ in 4 sub-trees such that the sum of the marks on the 4 sub-trees equals 6, it can looks like the figure 8.6.

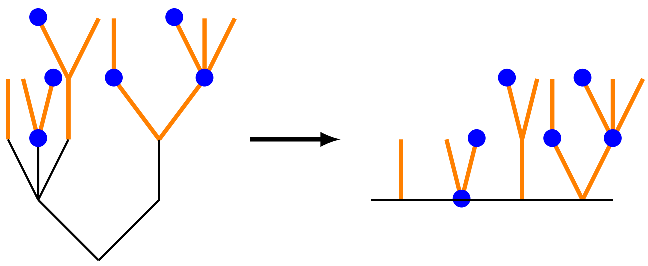


Figure 8.6 – A marked tree $\mathbf{t}^*$, with 6 marks and 4 sub-trees.

8.5 Main results

We want to prove that a MGW τ^* with offspring $\mathbf{p}$ and mark function $\mathbf{q}$ satisfies that, the conditional distribution of τ^* given $\{M(\tau^*) = n\}$ converges in distribution to a Kesten's tree $\tau_K^* = \tau_K^*(\mathbf{p}, \mathbf{q})$, in the critical case. And, in the generic and sub-critical case, the conditional distribution of τ^* given $\{M(\tau^*) = n\}$ converges in distribution to a Kesten's tree $\tau_K^*(\mathbf{p}_{\theta_c}, \mathbf{q}_{\theta_c})$, and converges in distribution to a condensation tree $\tau_C^*(\mathbf{p}, \mathbf{q})$, in the non-generic and sub-critical case.

We use the notations from Section 8.4 for $Z^{(1)}$, $\mathbf{p}^{(1)}$ and N. We assume that $(N_i)_{i\in\mathbb{N}^*}$ are independent random variables distributed as N with distribution $\mathbf{p}_N$, $(Z_i^{(1)})_{i\in\mathbb{N}^*}$ are independent random variables distributed as $Z^{(1)}$, and that two sequences are independent. We recall that $W_0 = 0$, $S_0 = 0$ and for $n \in \mathbb{N}^*$ $W_n = \sum_{i=1}^{n} N_i$ and $S_n = \sum_{i=1}^{n} Z_i^{(1)}$.

8.5.1 Critical case

In this section, we consider trees that belong to $\mathbb{T}^*$.

Theorem 8.1. *Let $\mathbf{p}$ be a critical offspring distribution that satisfies* (6.1) *admitting a moment of order* 2. *Let τ^* be a MGW with offspring distribution $\mathbf{p}$ and mark function $\mathbf{q}$ satisfying* (6.2).

$$\operatorname{dist}(\tau^*|M(\tau^*) = n) \underset{n\to+\infty}{\longrightarrow} \operatorname{dist}(\tau_K^*(\mathbf{p}, \mathbf{q})),$$

where the limit has to be understood along a subsequence for which $\mathbb{P}(M(\tau^*) = n) > 0$.

Proof. According to (8.1), we have $\operatorname{dist}(\tau^*|M(\tau^*) = n) \underset{n\to+\infty}{\longrightarrow} \operatorname{dist}(\tau_K^*(\mathbf{p}, \mathbf{q}))$ if and only if, for every finite marked tree $\mathbf{t}^* \in \mathbb{T}_0^*$ and every leaf $x \in \mathcal{L}_0(\mathbf{t})$,

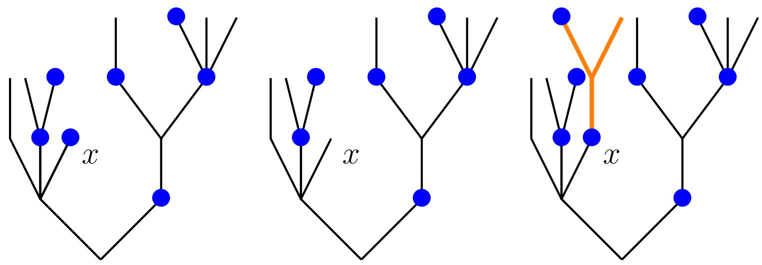
$$\mathbb{P}(\tau^* \in \mathbb{T}(\mathbf{t}^*, x)|M(\tau^*) = n) \underset{n\to+\infty}{\longrightarrow} \mathbb{P}(\tau_K^*(\mathbf{p}, \mathbf{q}) \in \mathbb{T}(\mathbf{t}^*, x)) \text{ and } \mathbb{P}(\tau^* = \mathbf{t}^*|M(\tau^*) = n) \underset{n\to+\infty}{\longrightarrow} 0.$$

Let a marked tree $\mathbf{t}^* \in \mathbb{T}_0^*$ and $x \in \mathcal{L}_0(\mathbf{t})$. The second limit is immediate, since for every $n \geq |\,\mathbf{t}|$:

$$\mathbb{P}(\tau^* = \mathbf{t}^*|M(\tau^*) = n) = 0.$$

We assume that $\mathbf{p}^{(1)}$ is aperiodic. If $\mathbf{p}^{(1)}$ is periodic, the results stay true if we consider a subsequence $(S_{\psi(n)})_{n\in\mathbb{N}^*}$, with ψ a growing, positive function, such that $\mathbb{E}[N\mathbb{1}_{\{S_{\psi(n)}+N=\psi(n)\}}]$ is positive, to apply lemmas obtained in the appendix.

The event "$\tau^* \in \mathbb{T}_+(\mathbf{t}^*, x), M(\tau^*) = n$" is explained as following. We use an example for $n = 8$:



1 2 3

Figure 8.7 – A marked tree $\mathbf{t}^*$ (1), the sub-tree $\mathcal{S}^x(\mathbf{t}^*)$ (2) and the tree obtained by grafting a tree at x on $\mathbf{t}^*$, such that $M(\tau^*) = n$ (3).

Let $m := M(\mathbf{t}^*)$ and $D(\mathbf{t}^*, x) := \dfrac{\mathbb{P}(\tau^* = \mathcal{S}^x(\mathbf{t}^*))}{\mathbf{p}(0)(1-\mathbf{q}(0))}$, thanks to (8.25), we have:

$$\begin{aligned}\mathbb{P}(\tau^* \in \mathbb{T}(\mathbf{t}^*, x), M(\tau^*) = n) &= \frac{\mathbb{P}(\tau^* = \mathcal{S}^x(\mathbf{t}^*))}{\mathbf{p}(0)(1-\mathbf{q}(0))}\sum_{j\in\mathbb{N}} \mathbf{p}(j)\alpha_{j,x}\mathbb{P}_j(M(\tau^*) = n-m) \\ &= D(\mathbf{t}^*, x)\sum_{j\in\mathbb{N}} \mathbf{p}(j)\alpha_{j,x}\frac{j}{n-m}\mathbb{E}[N\mathbb{1}_{\{S_{n+m}+W_{j-1}+N=n-m\}}],\end{aligned}$$

with for all j, $\alpha_{j,x} := \mathbf{q}(j)\eta_x(\mathbf{t}) + (1-\mathbf{q}(j))(1-\eta_x(\mathbf{t}))$.
In particular, according to (8.24), we obtain:

$$\begin{aligned}\mathbb{P}(\tau^* \in \mathbb{T}(\mathbf{t}^*, x)|M(\tau^*) = n) &= D(\mathbf{t}^*, x)\frac{n}{n-m}\sum_{j\in\mathbb{N}} \mathbf{p}(j)\alpha_{j,x}j\frac{\mathbb{E}[N\mathbb{1}_{\{S_{n+m}+W_{j-1}+N=n-m\}}]}{\mathbb{E}\left[N\mathbb{1}_{\{S_n+N=n\}}\right]} \\ &= D(\mathbf{t}^*, x)\delta_{n,m}\sum_{j\in\mathbb{N}} \mathbf{p}(j)\alpha_{j,x}j\frac{\mathbb{E}[N\mathbb{1}_{\{S_{n+m}+W_{j-1}+N=n-m\}}]}{\mathbb{E}\left[N\mathbb{1}_{\{S_{n+m}+N=n-m\}}\right]} \\ &= D(\mathbf{t}^*, x)\delta_{n,m}B_{n-m,0},\end{aligned}$$

with $\delta_{n,m} := \dfrac{n}{n-m}\dfrac{\mathbb{E}[N\mathbb{1}_{\{S_{n+m}+N=n-m\}}]}{\mathbb{E}[N\mathbb{1}_{\{S_n+N=n\}}]}$.

Notice that (8.33), in Lemma 8.9 for $T = N$ and $R = 0$, implies that

$$\lim_{n\to+\infty} \delta_{n,m} = 1.$$

Moreover, according to Lemma 8.18 , for $l = 0$, we have

$$\lim_{n\to+\infty} B_{n,0} = \mathbb{E}[X\mathbf{q}(X)]\eta_x(\mathbf{t}) + \mathbb{E}[X(1-\mathbf{q}(X))](1-\eta_x(\mathbf{t})) = \mathbb{E}[X\alpha_{X,x}].$$

Thereby, according to 7.11 , we get:

$$\lim_{n\to+\infty} \mathbb{P}(\tau^* \in \mathbb{T}(\mathbf{t}^*, x)|M(\tau^*) = n) = D(\mathbf{t}^*, x)\mathbb{E}[X\alpha_{X,x}] = \mathbb{P}(\tau_K^*(\mathbf{p}, \mathbf{q}) \in \mathbb{T}(\mathbf{t}^*, x)).$$

This proves the first limit. □

8.5.2 Sub-critical case

8.5.2.1 Generic case

According to Definition 8.1, it exists θ_c such that $\mathbf{p}_{\theta_c}$ is critical. Thanks to the Proposition 8.3 and Theorem 8.1 , we have:

Corollary 8.1. *Let τ^* be a sub-critical MGW with offspring distribution $\mathbf{p}$ satisfying* (6.1)*, admitting a moment of order 2 and mark function $\mathbf{q}$ satisfying* (6.2)*. If $\mathbf{p}$ is generic and $\mathbf{p}_{\theta_c}$ admits a moment of order 2 (always true for $\theta_c < \rho$), then:*

$$\operatorname{dist}(\tau^*|M(\tau^*) = n) \underset{n\to+\infty}{\longrightarrow} \operatorname{dist}(\tau_K^*(\mathbf{p}_{\theta_c}, \mathbf{q}_{\theta_c})).$$

Proof. As $\operatorname{dist}(\tau^*|M(\tau^*) = n) = \operatorname{dist}(\tau^*_{\theta_c}|M(\tau^*_{\theta_c}) = n)$, according to Proposition 8.5, we have to prove that $\mathbf{p}_{\theta_c}$ and $\mathbf{q}_{\theta_c}$ satisfy respectively (6.1) and (6.2). As $\theta_c \in I$, see the beginning of Section 8.2, it remains to prove that $\sum_{k\geq 0} k^2\mathbf{p}_{\theta_c}(k) < +\infty$, in other words:

$$\sum_{k\geq 0} k^2\theta_c^{k-1}\mathbf{p}(k)(c_{\theta_c}\mathbf{q}(k) + 1 - \mathbf{q}(k)) \leq (c_{\theta_c} + 1)\,\mathbb{E}[X^2\theta_c^{X-1}] < +\infty.$$

Since $\theta_c \in I$, either $\theta_c < \rho$ the convergence radius of g which is $\mathcal{C}^\infty(]-\rho,\rho[)$, thereby $\mathbb{E}[X^2\theta_c^{X-1}] < +\infty$, or $\theta_c = \rho$ and $\mathbf{p}_\rho$ admits a moment of order 2. $\square$

Corollary 8.2. *Let τ be a sub-critical Galton-Watson tree with offspring distribution $\mathbf{p}$ satisfying* (6.1)*, admitting a moment of order 2 and a mark function $\mathbf{q}$ satisfying* (6.2)*. If $\mathbf{p}$ is generic, that is there exists $\theta_c \in I$ such that $\mathbf{p}_{\theta_c}$ is critical, then if $\mathbf{p}_{\theta_c}$ admits a moment of order 2 (always true for $\theta_c < \rho(\mathbf{p})$) we have*

$$\operatorname{dist}(\tau|M(\tau) = n) \underset{n\to+\infty}{\longrightarrow} \operatorname{dist}(\tau_K(\mathbf{p}_{\theta_c}))$$

Proof. According to (8.1), we have $\operatorname{dist}(\tau|M(\tau) = n) \underset{n\to+\infty}{\longrightarrow} \operatorname{dist}(\tau_K(\mathbf{p}_{\theta_c}))$ if and only if, for every finite marked tree $\mathbf{t} \in \mathbb{T}_0$ and every leaf $x \in \mathcal{L}_0(\mathbf{t})$,

$$\mathbb{P}(\tau \in \mathbb{T}(\mathbf{t}, x)|M(\tau) = n) \underset{n\to+\infty}{\longrightarrow} \mathbb{P}(\tau_K(\mathbf{p}_{\theta_c})) \in \mathbb{T}(\mathbf{t}, x)) \text{ and } \mathbb{P}(\tau = \mathbf{t}|M(\tau) = n) \underset{n\to+\infty}{\longrightarrow} 0.$$

Let a tree $\mathbf{t} \in \mathbb{T}_0$ and $x \in \mathcal{L}_0(\mathbf{t})$. The second limit is immediate, since for every $n \geq |\mathbf{t}|$:

$$\mathbb{P}(\tau = \mathbf{t}|M(\tau) = n) = 0.$$

We just have to show that for every finite tree $\mathbf{t} \in \mathbb{T}_0$ and every leaf $x \in \mathcal{L}_0(\mathbf{t})$:

$$\mathbb{P}(\tau \in \mathbb{T}(\mathbf{t}, x)|M(\tau) = n) \underset{n\to+\infty}{\longrightarrow} \mathbb{P}(\tau_K(\mathbf{p}_{\theta_c})) \in \mathbb{T}(\mathbf{t}, x)).$$

First, if $\psi : \mathbb{T}^* \mapsto \mathbb{T}$, is defined by $\psi(\mathbf{t}^*) = \mathbf{t}$, Proposition 8.3 implies:

$$\begin{aligned}\mathbb{P}(\tau \in \mathbb{T}(\mathbf{t}, x)|M(\tau) = n) &= \sum_{\mathbf{t}^*\in\mathbb{T}_0^*, \psi(\mathbf{t}^*)=\mathbf{t}} \mathbb{P}(\tau^* \in \mathbb{T}(\mathbf{t}^*, x)|M(\tau) = n) \\ &= \sum_{\mathbf{t}^*\in\mathbb{T}_0^*, \psi(\mathbf{t}^*)=\mathbf{t}} \mathbb{P}(\tau^*_{\theta_c} \in \mathbb{T}(\mathbf{t}^*, x)|M(\tau_{\theta_c}) = n).\end{aligned}$$

As the previous sum is finite ($|\{\mathbf{t}^* \in \mathbb{T}_0^*, \psi(\mathbf{t}^*) = \mathbf{t}\}| = 2^{|t|}$), Theorem 8.1 implies that:

$$\lim_{n\to+\infty} \mathbb{P}(\tau \in \mathbb{T}(\mathbf{t}, x)|M(\tau) = n) = \sum_{\mathbf{t}^*\in\mathbb{T}_0^*,\psi(\mathbf{t}^*)=\mathbf{t}} \lim_{n\to+\infty} \mathbb{P}(\tau^* \in \mathbb{T}(\mathbf{t}^*, x)|M(\tau) = n)$$

$$= \sum_{\mathbf{t}^*\in\mathbb{T}_0^*,\psi(\mathbf{t}^*)=\mathbf{t}} \mathbb{P}(\tau_K^*(\mathbf{p}_{\theta_c}, \mathbf{q}_{\theta_c}) \in \mathbb{T}(\mathbf{t}^*, x)) = \mathbb{P}(\tau_K(\mathbf{p}_{\theta_c}) \in \mathbb{T}(\mathbf{t}, x)).$$

This proves the first limit. □

8.5.2.2 Non-generic case

We recall that in this case we suppose that $\mathbf{p}$ satisfies (6.6) and $\mathbf{q}$ satisfies (6.2), (6.7) and, if $\ell_{\mathbf{q}} = 1$, (6.8). Thereby, we have that $\rho(\mathbf{p}) = \rho_l(\mathbf{p}, \mathbf{q}) = 1$.
Indeed, we have

$$g(\theta) = \mathbb{E}[\theta^X] = \sum_{k\geq 0} \theta^k \frac{\mathscr{L}(k)}{k^{\alpha+1}},$$

implying again that $\rho(\mathbf{p}) = 1$. Recall that:

$$l(\theta) = \sum_{k\geq 0} \theta^k \mathbf{p}(k)(1 - \mathbf{q}(k)),$$

we have to distinguish two cases:

— $\ell_{\mathbf{q}} \neq 1$ and clearly $\rho(\mathbf{p}) = \rho_l(\mathbf{p}, \mathbf{q})$;

— otherwise, as (6.8), there exists $\beta \geq 2$ such that:

$$\mathbf{p}(k)(1 - \mathbf{q}(k)) = \frac{\mathbf{p}(k)\mathcal{L}(k)}{k^\beta},$$

implying again that $\rho(\mathbf{p}) = \rho_l(\mathbf{p}, \mathbf{q})$.

Theorem 8.2. *Assume that* $\mathbf{p}$ *satisfies* (6.1), (6.6) , $\mu(\mathbf{p}) < 1$, *admitting a moment of order* 2 *and* $\mathbf{q}$ *satisfies* (6.2), (6.7) *and, if* $\ell_{\mathbf{q}} = 1$, (6.8).

$$\mathrm{dist}(\tau^*|M(\tau^*) = n) \underset{n\to+\infty}{\longrightarrow} \mathrm{dist}(\tau_C^*(\mathbf{p}, \mathbf{q})). \tag{8.26}$$

Proof. According to (8.2) it is enough to prove for all marked tree $\mathbf{t}^* \in \mathbb{T}_0^*$, $x \in \mathbf{t}$ and $k \in \mathbb{N}$:

$$\lim_{n\to+\infty} \mathbb{P}(\tau^* \in \mathbb{T}_+(\mathbf{t}^*, x, k)|M(\tau^*) = n) = \mathbb{P}(\tau_C^*(\mathbf{p}, \mathbf{q}) \in \mathbb{T}_+(\mathbf{t}^*, x, k))$$

$$\text{and } \lim_{n\to+\infty} \mathbb{P}(\tau^* = \mathbf{t}^*|M(\tau^*) = n) = 0.$$

Let a marked tree $\mathbf{t}^* \in \mathbb{T}_0^*$, $x \in \mathbf{t}$ and $k \in \mathbb{N}$. The second limit is immediate since, for every $n \geq |\mathbf{t}|$:

$$\mathbb{P}(\tau^* = \mathbf{t}^*|M(\tau^*) = n) = 0.$$

Let $m := M(\mathbf{t}^*)$ and $l := k_x(\mathbf{t})$.

We assume that $\mathbf{p}^{(1)}$ is aperiodic. If $\mathbf{p}^{(1)}$ is periodic, the results stay true if we consider a subsequence $(S_{\psi(n)})_{n\in\mathbb{N}^*}$, with ψ a growing, positive function, such that $\mathbb{E}[N\mathbb{1}_{\{S_{\psi(n)}+N=\psi(n)\}}]$, $C_{\psi(n),0}(0)$ and $C_{\psi(n),l}(-1)$ (see (8.84)) are positive, to apply lemmas obtained in the appendix.

Using Dwass Formula (8.25), we have

$$\begin{aligned}\mathbb{P}(\tau^* \in \mathbb{T}_+(\mathbf{t}^*, x, k), M(\tau^*) = n) &= \sum_{j \geq k-l} C(\mathbf{t}^*, x)\alpha_{j+l,x}\mathbf{p}(j+l)\mathbb{P}_j(M(\tau^*) = n-m)\\ &= C(\mathbf{t}^*, x) \sum_{j \geq \max(l+1,k)} \alpha_{j,x}\mathbf{p}(j)\mathbb{P}_{j-l}(M(\tau^*) = n-m)\\ &= C(\mathbf{t}^*, x) \sum_{j \geq \max(l+1,k)} \mathbf{p}(j)\alpha_{j,x}\frac{j-l}{n-m}\mathbb{E}[N\mathbb{1}_{\{S_{n-m}+W_{j-1-l}+N=n-m\}}],\end{aligned}$$

with for all $j \in \mathbb{N}$, $\alpha_{j,x} := \mathbf{q}(j)\eta_x(\mathbf{t}) + (1-\mathbf{q}(j))(1-\eta_x(\mathbf{t}))$. We recall (7.13) that

$$C(\mathbf{t}^*, x) := \frac{\mathbb{P}(\tau^* = \mathcal{S}^x(\mathbf{t}^*))}{\mathbf{p}(0)(1-\mathbf{q}(0))}\mathbb{P}_{k_x(\mathbf{t})}(\tau^* = \mathscr{F}_x(\mathbf{t}^*)).$$

The event "$\tau^* \in \mathbb{T}_+(\mathbf{t}^*, x, k), M(\tau^*) = n$" is explained as following. In the same way of Figure 8.7, we use an example for $n = 9$ and $k = 3$:

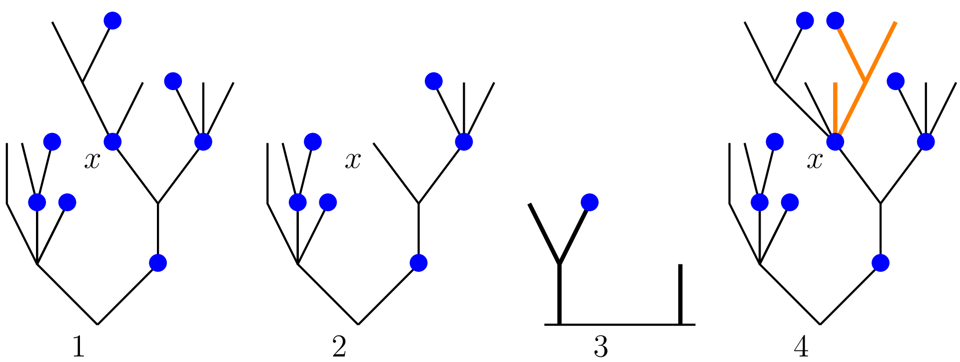


Figure 8.8 – A marked tree $\mathbf{t}^*$ (1), the sub-tree $\mathcal{S}^x(\mathbf{t}^*)$ (2), the forest $\mathscr{F}_x(\mathbf{t}^*)$ (3) and the tree obtained by grafting a tree at x on the right of $\mathbf{t}^*$, such that $M(\tau^*) = n$ and $k_x(t) \geq 3$ (4).

Henceforth, we have:

$$\mathbb{P}(\tau^* \in \mathbb{T}_+(\mathbf{t}^*, x, k)|M(\tau^*) = n) = C(\mathbf{t}^*, x)\delta_{n,m}\left(B_{n-m,l}(x) - \sum_{j=l+1}^{k-1}\mathbf{p}(j)\alpha_{j,x}(j-l)a_{n-m,j}\right), \tag{8.27}$$

with for $i \in \mathbb{Z}$ and $n \in \mathbb{N}^*$:

$$B_{n,i}(x) := \sum_{j>i}\mathbf{p}(j)\alpha_{j,x}(j-i)a_{n,j}, \; a_{n,j} := \frac{\mathbb{E}[N\mathbb{1}_{\{S_n+W_{j-1-i}+N=n\}}]}{\mathbb{E}[N\mathbb{1}_{\{S_n+N=n\}}]}, \tag{8.28}$$

and $\delta_{n,m} := \dfrac{n}{n-m}\dfrac{\mathbb{E}[N\mathbb{1}_{\{S_{n+m}+N=n-m\}}]}{\mathbb{E}[N\mathbb{1}_{\{S_n+N=n\}}]}$.

Notice that Lemma 8.9, more precisely (8.33) for $T = N$ and $R = 0$, implies that

$$\lim_{n\to+\infty}\delta_{n,m} = 1,$$

and Lemma 8.10 implies that
$$\lim_{n\to+\infty} a_{n,j} = 1.$$
Then use Lemma 8.20 to get:
$$\lim_{n\to+\infty} \mathbb{P}(\tau^* \in \mathbb{T}_+(\mathbf{t}^*, x, k)|M(\tau^*) = n)$$
$$= C(\mathbf{t}^*, x)\left((1-\mu(\mathbf{p}))\left(\ell_{\mathbf{q}}\eta_x(\mathbf{t}) + (1-\ell_{\mathbf{q}})(1-\eta_x(\mathbf{t}))\right) + \mathbb{E}[\alpha_{X,x}(X-l)_+\mathbb{1}_{\{X\geq k\}}]\right).$$

□

In the same way of generic case, we are also interesting to the distribution of the GW instead of the MGW, so we have:

Corollary 8.3. *Assume that* $\mathbf{p}$ *satisfies* (6.1), (6.6) , $\mu(\mathbf{p}) < 1$, *non-generic, admitting a moment of order* 2 *and* $\mathbf{q}$ *satisfies* (6.2), (6.7) *and, if* $\ell_{\mathbf{q}} = 1$, (6.8). *We have that:*
$$\operatorname{dist}(\tau|M(\tau) = n) \underset{n\to+\infty}{\longrightarrow} \operatorname{dist}(\tau_C(\mathbf{p})). \tag{8.29}$$

The proof is similar as the one of the Corollary 8.2.

8.6 Appendix

8.6.1 Preliminaries and Key Lemmas

Recall the definition of N in Section 8.4 (see Proposition 8.6), $N = \mathscr{B}\left(L, \frac{\mathbb{E}[\mathbf{q}(X)]}{\tilde{\mathbf{p}}(0)}\right)$, where L is the number of leaves of $\tilde{\mathcal{S}}^{\mathbf{q}}_{\emptyset}(\tau^*)$ with offspring distribution $\tilde{\mathbf{p}}$ (8.17). X be a random variable with distribution $\mathbf{p}$, Y, given X, be a Bernoulli random variable with parameter $\mathbf{q}(X)$, and introduce $Z^{(0)}$ whose definition is very similar to $Z^{(1)}$ (8.21), $Z^{(1)} = \sum_{k=1}^{X^{(1)}} N_k$, where $X^{(1)}$ is distributed as X conditionally to be marked, and $(N_i)_{i\in\mathbb{N}}$ is a sequence of independent copies of N, independent of $X^{(1)}$.
In other words:
$$Z^{(0)} := \sum_{k=1}^{X^{(0)}} N_k, \tag{8.30}$$
where $X^{(0)}$ is distributed as X conditionally not to be marked, and $(N_i)_{i\in\mathbb{N}}$ is a sequence of independent copies of N, independent of $X^{(0)}$.
Moreover, throughout this section, for $j \in \{0,1\}$, $\mathbf{p}^{(j)}$ denotes the distribution of $Z^{(j)}$ and $\mu^{(j)}$, its mean.
Note that the existence of k such that $0 < \mathbf{p}(k)\mathbf{q}(k) < 1$, ensures that $X^{(0)}$ is well defined and we have:
$$\mathbb{P}(X^{(0)} = k) = \frac{\mathbf{p}(k)(1-\mathbf{q}(k))}{1-\mathbb{E}[\mathbf{q}(X)]}.$$

Lemma 8.7. *Assume that* $\mathbf{p}$ *satisfies* (6.1), *with a moment of order* 2 *and* $\mathbf{q}$ (6.2). *If* $\mu(\mathbf{p}) \leq 1$, $Z^{(0)} \in L^2$ *and:*
$$\mathbb{E}[Z^{(0)}] = \frac{\mathbb{E}[X(1-\mathbf{q}(X))]}{1-\mathbb{E}[X(1-\mathbf{q}(X))]}\frac{\mathbb{E}[\mathbf{q}(X)]}{\mathbb{E}[1-\mathbf{q}(X)]}.$$

Proof. According to Lemma 8.6, $N \in L^2$ and (6.1) implies that $X^{(0)}$ is in L^2 too. Consequently, using for instance the generating function, $Z^{(0)} \in L^2$ and we have:

$$\mathbb{E}[Z^{(0)}] = \mathbb{E}[X^{(0)}]\mathbb{E}[N] = \frac{\mathbb{E}[X(1-\mathbf{q}(X))]}{1-\mathbb{E}[\mathbf{q}(X)]}\frac{\mathbb{E}[\mathbf{q}(X)]}{1-\mathbb{E}[X(1-\mathbf{q}(X))]}.$$

□

Lemma 8.8. *Let* $\mathbf{p}$ *satisfying* (6.1) *admitting a moment of order* 2 *and* $\mathbf{q}$ *satisfying* (6.2). *If* $\mu(\mathbf{p}) < 1$ *assume that* $\mathbf{q}$ *satisfies* (6.7) *and* (6.8). *Then* $\mathrm{Var}[Z^{(1)}]$ *is finite.*

Proof. Let's prove that $\mathrm{Var}[Z^{(1)}] < +\infty$. We have, according to Lemma 8.6, $N \in L^2$ and (6.1) implies that $X^{(0)}$ is in L^2 too. Consequently, using for instance the generating function, $Z^{(1)} \in L^2$ and $\mathrm{Var}[Z^{(1)}]$ is finite.

□

Let $(Z_i^{(1)})_{i\in\mathbb{N}^*}$ be a sequence of independent copies of $Z^{(1)}$, independent of $(N_i)_{i\in\mathbb{N}^*}$. Then, we define $(W_n)_{n\in\mathbb{N}^*}$ and $(S_n)_{n\in\mathbb{N}^*}$ by $W_0 = S_0 = 0$ a.s, and for $n \in \mathbb{N}^*$, $W_n = \sum_{i=1}^{n} N_i$ and $S_n = \sum_{i=1}^{n} Z_i^{(1)}$.

We assume that:

$$\mu^{(1)} = 1 \text{ or } (\mu^{(1)} < 1 \text{ and, for all } \theta > 0, \mathbb{E}[e^{\theta Z^{(1)}}] = +\infty). \tag{8.31}$$

and that $\mathbf{p}^{(1)}$ is aperiodic (that is $\mathbb{P}(S_n = n) > 0$ for all n large enough). According to [22] or [28], we have the following strong ratio limit property for all $m, u \in \mathbb{Z}$:

$$\lim_{n\to+\infty} \frac{\mathbb{P}(S_{n+m} = n-u)}{\mathbb{P}(S_n = n)} = 1. \tag{8.32}$$

Proposition 8.7. *If* $\rho(\mathbf{p}^{(1)}) = 1$, *then, for* R *a random variable equals to* $Z^{(1)}$, X, L *or* N, *we have for all* $\theta > 0$, $\mathbb{E}[e^{\theta R}] = +\infty$.

Proof. Since $\rho(\mathbf{p}^{(1)}) = 1$, then $\rho_l(\mathbf{p},\mathbf{q}) = 1$ and $\rho(\mathbf{p}) = 1$ according to (8.11) and Lemma 8.6 . Thus the proposition is true for $Z^{(1)}$ and X. As N, conditionally on L, has a binomial distribution with parameter $\left(L, \dfrac{\mathbb{E}[\mathbf{q}(X)]}{\tilde{\mathbf{p}}(0)}\right)$, if the proposition is true for L, it is also true for N.

We recall the equation for the generating function of L (8.18) for all $s \in \mathbb{R}$:

$$\mathbb{E}[s^L] = \tilde{\mathbf{p}}(0)s + \sum_{i=1}^{+\infty} \mathbf{p}(i)(1-\mathbf{q}(i))\mathbb{E}[s^L]^i.$$

Thereby, $\mathbb{E}[s^L]$ converges if $\sum_{i=1}^{+\infty} \mathbf{p}(i)(1-\mathbf{q}(i))\mathbb{E}[s^L]^i = l(\mathbb{E}[s^L])$ converges. Since, $\rho_l(\mathbf{p},\mathbf{q}) = 1$ and $s \mapsto \mathbb{E}[s^L]$ is increasing, we need to have $\mathbb{E}[s^L] \leq 1$, so $s \leq 1$. Thus, the convergence radius of the generating function of L is 1, the condition is satisfied. □

Lemma 8.9. *Assume that $\mathbf{p}^{(1)}$ is aperiodic and satisfies* (8.31). *Let T and R random variables independents of $Z^{(1)}$ such that $0 < \mathbb{E}[T] < +\infty$. Let $m, k \in \mathbb{Z}$, we have:*

$$\lim_{n\to+\infty} \frac{\mathbb{E}[T\mathbb{1}_{\{S_{n+m}+R+T=n-k\}}]}{\mathbb{E}[T\mathbb{1}_{\{S_n+R+T=n\}}]} = 1, \tag{8.33}$$

If $\mathbf{p}^{(1)}$ is periodic, then (8.33) *still holds along the subsequence for which the denominator is positive.*

Proof. We shall mimic the proof of Lemma 8.3 of [6]. Since $\mathbf{p}^{(1)}$ is aperiodic, the denominator of (8.33) is positive for n large enough and it is enough to prove the result for $m = 1$ and k such that $\mathbf{p}^{(1)}(k) > 0$.
Denote $\mathbf{p}_n^{(1)}(k) := \frac{1}{n}\sum\limits_{i=1}^{n}\mathbb{1}_{\{Z_i^{(1)}=k\}}$, we have:

$$\begin{aligned}
\mathbb{E}[T\mathbf{p}_n^{(1)}(k)\mathbb{1}_{\{S_n+R+T=n\}}] &= \mathbb{E}\left[T\left(\frac{1}{n}\sum_{i=1}^{n}\mathbb{1}_{\{Z_i^{(1)}=k\}}\right)\mathbb{1}_{\{S_n+R+T=n\}}\right] \\
&= \mathbb{E}\left[\mathbb{1}_{\{Z_n^{(1)}=k\}}T\mathbb{1}_{\{S_n+R+T=n\}}\right] \\
&= \mathbf{p}^{(1)}(k)\mathbb{E}\left[T\mathbb{1}_{\{S_{n-1}+R+T=n-k\}}\right].
\end{aligned} \tag{8.34}$$

Thus, we have:

$$\begin{aligned}
\left|\frac{\mathbb{E}[T\mathbb{1}_{\{S_{n+1}+R+T=n-k\}}]}{\mathbb{E}[T\mathbb{1}_{\{S_n+R+T=n\}}]} - 1\right| &= \left|\frac{\mathbb{E}[T\mathbf{p}_n^{(1)}(k)\mathbb{1}_{\{S_n+R+T=n\}}]}{\mathbf{p}^{(1)}(k)\mathbb{E}[T\mathbb{1}_{\{S_n+R+T=n\}}]} - 1\right| \\
&\le \frac{\mathbb{E}\left[T\left|\mathbf{p}_n^{(1)}(k) - \mathbf{p}^{(1)}(k)\right|\mathbb{1}_{\{S_n+R+T=n\}}\right]}{\mathbf{p}^{(1)}(k)\mathbb{E}[T\mathbb{1}_{\{S_n+R+T=n\}}]} \\
&\le \frac{\varepsilon}{\mathbf{p}^{(1)}(k)} + \frac{\mathbb{E}\left[T\mathbb{1}_{\left\{\left|\mathbf{p}_n^{(1)}(k)-\mathbf{p}^{(1)}(k)\right|>\varepsilon\right\}}\mathbb{1}_{\{S_n+R+T=n\}}\right]}{\mathbf{p}^{(1)}(k)\mathbb{E}[T\mathbb{1}_{\{S_n+R+T=n\}}]},
\end{aligned}$$

for every $\varepsilon > 0$.

It remains to prove that:

$$J_n := \frac{\mathbb{E}\left[T\mathbb{1}_{\left\{\left|\mathbf{p}_n^{(1)}(k)-\mathbf{p}^{(1)}(k)\right|>\varepsilon\right\}}\mathbb{1}_{\{S_n+R+T=n\}}\right]}{\mathbb{E}[T\mathbb{1}_{\{S_n+R+T=n\}}]}$$

converges to 0 for all $\varepsilon > 0$. We have:

$$\begin{aligned}
J_n \le \frac{\mathbb{E}\left[T\mathbb{1}_{\left\{\left|\mathbf{p}_n^{(1)}(k)-\mathbf{p}^{(1)}(k)\right|>\varepsilon\right\}}\right]}{\mathbb{E}[T\mathbb{1}_{\{S_n+R+T=n\}}]} &\le \frac{\mathbb{E}\left[T\right]}{\mathbb{E}[T\mathbb{1}_{\{S_n+R+T=n\}}]}\mathbb{P}\left(\left|\mathbf{p}_n^{(1)}(k) - \mathbf{p}^{(1)}(k)\right| > \varepsilon\right) \\
&\le \frac{\mathbb{E}\left[T\right]\mathbb{P}(S_n = n)}{\mathbb{E}[T\mathbb{1}_{\{S_n+R+T=n\}}]}\frac{\mathbb{P}\left(\left|\mathbf{p}_n^{(1)}(k) - \mathbf{p}^{(1)}(k)\right| > \varepsilon\right)}{\mathbb{P}(S_n = n)}.
\end{aligned}$$

According to [28], end of page 2954, we have $\lim_{n\to+\infty} \dfrac{\mathbb{P}\left(\left|\mathbf{p}_n^{(1)}(k)-\mathbf{p}^{(1)}(k)\right|>\varepsilon\right)}{\mathbb{P}(S_n=n)}=0$. We denote $I_n := \dfrac{\mathbb{E}\left[T\right]\mathbb{P}(S_n=n)}{\mathbb{E}[T\mathbb{1}_{\{S_n+R+T=n\}}]}$, for all $n\in\mathbb{N}^*$, by Fatou's lemma and using the strong ratio limit property (8.32), we have:

$$\begin{aligned}
\limsup_{n\to+\infty} I_n &= \limsup_{n\to+\infty} \frac{\mathbb{E}\left[T\right]\mathbb{P}(S_n=n)}{\sum_{i=0}^{+\infty}\mathbb{E}[T\mathbb{1}_{\{R+T=i\}}]\mathbb{P}\left(S_n=n-i\right)\mathbb{1}_{\{i\leq n\}}}\\
&= \limsup_{n\to+\infty} \frac{\mathbb{E}\left[T\right]}{\sum_{i=0}^{+\infty}\mathbb{E}[T\mathbb{1}_{\{R+T=i\}}]\frac{\mathbb{P}(S_n=n-i)}{\mathbb{P}(S_n=n)}\mathbb{1}_{\{i\leq n\}}}\\
&= \frac{\mathbb{E}\left[T\right]}{\liminf_{n\to+\infty}\sum_{i=0}^{+\infty}\mathbb{E}[T\mathbb{1}_{\{R+T=i\}}]\frac{\mathbb{P}(S_n=n-i)}{\mathbb{P}(S_n=n)}\mathbb{1}_{\{i\leq n\}}}\\
&\leq \frac{\mathbb{E}\left[T\right]}{\sum_{i=0}^{+\infty}\mathbb{E}[T\mathbb{1}_{\{R+T=i\}}]\liminf_{n\to+\infty}\frac{\mathbb{P}(S_n=n-i)}{\mathbb{P}(S_n=n)}\mathbb{1}_{\{i\leq n\}}} \leq \frac{\mathbb{E}\left[T\right]}{\mathbb{E}\left[T\right]}=1.
\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, we deduce that $\lim_{n\to+\infty} J_n=0$. $\square$

We recall the Lemma 8.6 of [6]. The proof is same in our case.

Lemma 8.10. *Assume that $\mathbf{p}^{(1)}$ is aperiodic , with $\mu^{(1)}=1$ or ($\mu^{(1)}<1$ and $\rho(\mathbf{p}^{(1)})=1$). For all $m\in\mathbb{N}$, $k\in\mathbb{Z}$, we have:*

$$\lim_{n\to+\infty}\frac{\mathbb{E}[N\mathbb{1}_{\{S_n+W_m+N=n-k\}}]}{\mathbb{E}[N\mathbb{1}_{\{S_n+N=n\}}]}=1. \tag{8.35}$$

If $\mathbf{p}^{(1)}$ is periodic, then (8.35) *still holds along the subsequence for which the denominator is positive.*

Lemma 8.11. *Assume that $\mathbf{p}^{(1)}$ is aperiodic , with $\mu^{(1)}=1$ or ($\mu^{(1)}<1$ and $\rho(\mathbf{p}^{(1)})=1$). For all $m\in\mathbb{N}$, $k\in\mathbb{Z}$, we have:*

$$\lim_{n\to+\infty}\frac{\mathbb{E}\left[\mathbb{1}_{\{S_{n+m}+N=n-k\}}\right]}{\mathbb{P}(S_n=n)}=1. \tag{8.36}$$

If $\mathbf{p}^{(1)}$ is periodic, then (8.36) *still holds along the subsequence for which the denominator is positive.*

Proof. We mimic the proof of the Lemma 8.4 of [6]. Let $m,k\in\mathbb{Z}$. We define for all $j\in\mathbb{N}$:

$$b_n(j)=\mathbf{p}_N(j)\frac{\mathbb{P}(S_{n+m}=n-k-j)}{\mathbb{P}(S_n=n)} \text{ and } d_n(j)=\mathbf{p}^{(1)}(j)\frac{\mathbb{P}(S_{n+m}=n-k-j)}{\mathbb{P}(S_n=n)}.$$

Thanks to the strong ratio limit (8.32) we have $\lim_{n\to+\infty} b_n(j)=\mathbf{p}_N(j)$ and $\lim_{n\to+\infty} d_n(j)=\mathbf{p}^{(1)}(j)$.

Moreover, using 8.34 with $T = k$ and $R = 0$, we have:

$$\sum_{j\in\mathbb{N}} d_n(j) = \frac{\mathbb{P}(S_{n+1+m} = n-k)}{\mathbb{P}(S_n = n)},$$

and $\lim_{n\to+\infty} \sum_{j\in\mathbb{N}} d_n(j) = 1 = \sum_{j\in\mathbb{N}} \mathbf{p}^{(1)}(j) = \sum_{j\in\mathbb{N}} \lim_{n\to+\infty} d_n(j)$.
Then, since it exists $j_0 \in \mathbb{N}^*$ such that $\mathbf{p}(j_0)\mathbf{q}(j_0) > 0$, we have:

$$\begin{aligned}
\mathbf{p}^{(1)}(j) = \mathbb{P}\left(\sum_{i=1}^{X^{(1)}} N_i = j\right) &\geq \mathbb{P}(X^{(1)} = j_0)\mathbb{P}\left(\sum_{i=1}^{j_0} N_i = j\right) \\
&\geq \mathbb{P}(X^{(1)} = j_0)\mathbf{p}_N(j)\mathbf{p}_N(0)^{j_0-1} =: C\mathbf{p}_N(j).
\end{aligned}$$

Thereby, for all $j \in \mathbb{N}$, $b_n(j) \leq C^{-1}d_n(j)$, and the dominated convergence theorem in [21] (Theorem 1.21), implies that

$$\lim_{n\to+\infty} \frac{\mathbb{E}\left[\mathbb{1}_{\{S_{n+m}+N=n-k\}}\right]}{\mathbb{P}(S_n = n)} = \lim_{n\to+\infty} \sum_{j\in\mathbb{N}} b_n(j) = \sum_{j\in\mathbb{N}} \lim_{n\to+\infty} b_n(j) = \sum_{j\in\mathbb{N}} \mathbf{p}_N(j) = 1.$$

□

8.6.2 The sub-critical case: some properties

We add some properties on the random variables in the case where $\mu^{(1)} < 1$.
Let us recall that we assume in this case that:

$$\mathbf{p}(k) = \mathscr{L}(k)k^{-(1+\alpha)}, \tag{8.37}$$

where $\alpha > 2$ and $\mathscr{L}$ is a SV function, and that $\mathbf{q}$ admits a finite limit at infinity (6.7):

$$\lim_{n\to+\infty} \mathbf{q}(n) = \ell_{\mathbf{q}},\ \ell_{\mathbf{q}} \in [0,1]. \tag{8.38}$$

A useful lemma concerning SV functions is the following, Lemma 2 in [16], pp.282:

Lemma 8.12. *If $\mathscr{L}$ is a SV function, for every $0 < a < b < +\infty$, uniformly for $c \in [a,b]$:*

$$\lim_{x\to+\infty} \frac{\mathscr{L}(cx)}{\mathscr{L}(x)} = 1. \tag{8.39}$$

The following corollary is a direct consequence of the previous lemma in the case of regularly varying functions:

Corollary 8.4. *Assume that $f : \mathbb{R}^{+,*} \to \mathbb{R}^{+,*}$ is defined by:*

$$f(x) = \frac{\mathscr{L}(x)}{x^\alpha},$$

where $\alpha > 0$ and $\mathscr{L}$ is a SV function.
Moreover, let us consider two positive functions u, v satisfying:

$$\lim_{x\to+\infty} \frac{u(x)}{x} = A \leq \lim_{x\to+\infty} \frac{v(x)}{x} = B.$$

Then, for any $\varepsilon > 0$, for x large enough and for all $y \in [u(x), v(x)]$:

$$\frac{(1-\varepsilon)}{B^{\alpha}} \leq \frac{f(y)}{f(x)} \leq \frac{(1+\varepsilon)}{A^{\alpha}} \tag{8.40}$$

Remark 8.3. *Note that we can easily adapt the previous result if $\frac{u(x)}{x} \sim A(x)$ when x goes to infinity. In this case, for any $\varepsilon > 0$, for x large enough and for all $y \in [u(x), v(x)]$:*

$$\frac{f(y)}{f(x)} \leq \frac{(1+\varepsilon)}{A(x)^{\alpha}}. \tag{8.41}$$

According to Section 8.4, L is the number of leaves of $\tilde{\mathcal{S}}^{\mathbf{q}}_{\emptyset}(\tau^*)$ with reproduction law (8.17):

$$\tilde{\mathbf{p}}(k) := \begin{cases} \mathbf{p}(k)(1-\mathbf{q}(k)), & \text{if } k \neq 0, \\ \mathbf{p}(0) + \sum\limits_{j=1}^{+\infty} \mathbf{p}(j)\mathbf{q}(j), & \text{otherwise.} \end{cases}$$

We denote by $\tilde{X}$ a random variable with distribution $\tilde{\mathbf{p}}$.

8.6.2.1 Case $\ell_{\mathbf{q}} \in [0, 1)$

According to Remark 4.8 of [7], L is, also, distributed as the total size of a GW with offspring distribution given by the distribution of:

$$\tilde{Y} = \sum_{k=1}^{\mathcal{G}-1} \left(\tilde{X}_{k,+} - 1\right), \tag{8.42}$$

where $\mathcal{G}$ is a geometric law with parameter $\tilde{\mathbf{p}}(0)$ independent of $(\tilde{X}_{k,+})_{k\in\mathbb{N}^*}$, a sequence of i.i.d. random variables, distributed as $\tilde{X}$ given $\{\tilde{X} > 0\}$. Thus, for all $n \in \mathbb{N}^*$:

$$\mathbb{P}(\tilde{X}_+ = n) = \frac{\mathscr{L}(n)(1-\mathbf{q}(n))}{n^{1+\alpha}(1-\tilde{\mathbf{p}}(0))}, \tag{8.43}$$

implying that $\mathbb{P}(\mathcal{G} = n) = o(\mathbb{P}(\tilde{X}_+ = n))$, when n goes to infinity. Thus, applying Theorem 2 of [10] and Lemma 8.12:

$$\mathbb{P}(\tilde{Y} = n) \sim \mathbb{E}\left[\mathcal{G} - 1\right] \mathbb{P}(\tilde{X}_+ - 1 = n) \sim \frac{\mathscr{L}(n)(1-\ell_{\mathbf{q}})}{n^{1+\alpha}\tilde{\mathbf{p}}(0)}.$$

Then, using the representation theory ([16], p.282), we have for all $n \in \mathbb{N}^*$:

$$\mathbb{P}(\tilde{Y} = n) = \frac{\ell_{\tilde{Y}}(n)\mathscr{L}(n)}{n^{1+\alpha}}, \tag{8.44}$$

where $\ell_{\tilde{Y}}(.)$ is a positive function satisfying $\lim\limits_{n\to+\infty} \ell_{\tilde{Y}}(n) = \frac{(1-\ell_{\mathbf{q}})}{\tilde{\mathbf{p}}(0)}$.
Using Dwass formula [15], we have that,

$$\mathbb{P}(L = n) = \frac{\mathbb{P}\left(\tilde{S}_n = n-1\right)}{n},$$

with $\tilde{S}_n = \sum_{k=1}^{n} \tilde{Y}_k$, and $(\tilde{Y}_k)_{k\in\mathbb{N}^*}$ is an i.i.d sequence with distribution $\tilde{Y}$.

As $\alpha > 2$ in (8.44), we can apply Theorem 1 of [12]; as $\mathbb{E}[\tilde{Y}] < 1$, there exists $\varepsilon > 0$ small enough such that $\tilde{\varepsilon} := 1 - \mathbb{E}[\tilde{Y}] - \varepsilon > 0$, and as $n - 1 \geq n\mathbb{E}[\tilde{Y}] + n\tilde{\varepsilon}$, when n goes to infinity:

$$\mathbb{P}\left(\tilde{S}_n = n-1\right) = n\mathbb{P}\left(\tilde{Y} = \left\lfloor n(1-\mathbb{E}[\tilde{Y}])\right\rfloor\right)(1+o(1)).$$

Moreover, (8.42) ensures that:

$$\mathbb{E}[\tilde{Y}] = \mathbb{E}[\mathcal{G}-1]\mathbb{E}[\tilde{X}_+ - 1] = 1 - \mathbb{E}[L]^{-1}.$$

Finally, when n goes to infinity, using Lemma 8.12:

$$\mathbb{P}(L=n) \sim \mathbb{P}\left(\tilde{Y} = \left\lfloor n(1-\mathbb{E}[\tilde{Y}])\right\rfloor\right) \sim \mathbb{P}(\tilde{Y}=n)\mathbb{E}[L]^{1+\alpha};$$

So, we can write:

$$\mathbb{P}(L=n) = \frac{\ell_L(n)\mathscr{L}(n)}{n^{1+\alpha}}, \tag{8.45}$$

where $\ell_L(.)$ is a positive function satisfying $\lim_{n\to+\infty} \ell_L(n) = (1-\ell_{\mathbf{q}})\frac{\mathbb{E}[L]^{1+\alpha}}{\tilde{\mathbf{p}}(0)}$.

We recall, that the random variable N defines the number of marked leaves among the L leaves, thus N, conditionally on L, has a binomial distribution with parameter $\left(L, \dfrac{\mathbb{E}[\mathbf{q}(X)]}{\tilde{\mathbf{p}}(0)}\right)$ and we can consider the i.i.d sequence $(B_i)_{i\in\mathbb{N}^*}$ with distribution law $B \sim \mathscr{B}\left(\dfrac{\mathbb{E}[\mathbf{q}(X)]}{\tilde{\mathbf{p}}(0)}\right)$, such that

$$N = \sum_{i=1}^{L} B_i.$$

We have the following result:

Lemma 8.13. *For all $n \in \mathbb{N}^*$:*

$$\mathbf{p}_N(n) = \mathbb{P}(N=n) = \frac{\ell_N(n)\mathscr{L}(n)}{n^{1+\alpha}}, \tag{8.46}$$

where $\ell_N(.)$ is a positive function satisfying $\lim_{n\to+\infty} \ell_N(n) = (1-\ell_{\mathbf{q}})\frac{\mathbb{E}[N]^{\alpha}}{1-\mathbb{E}[X(1-\mathbf{q}(X))]} =: \mathfrak{c}_N$.

Proof. Since $(B_i)_{i\in\mathbb{N}}$ is a family of i.i.d random variables with Bernoulli distribution $\mathscr{B}(\tilde{c})$, with $\tilde{c} = \dfrac{\mathbb{E}[\mathbf{q}(X)]}{\tilde{\mathbf{p}}(0)}$, and assuming that this sequence is independent of L, $\sum_{i=1}^{L} B_i$ equals N in law and we define for all $k \in \mathbb{N}$, $\tilde{W}_k = \sum_{i=1}^{k} B_i$.

Thus, for $\varepsilon > 0$, we have:

$$\begin{aligned}\mathbb{P}(N=n) = &\sum_{k=n}^{\left\lfloor \frac{n}{\tilde{c}}(1-\varepsilon)\right\rfloor} \mathbb{P}(L=k)\mathbb{P}(\tilde{W}_k=n) + \sum_{k=\left\lfloor \frac{n}{\tilde{c}}(1-\varepsilon)\right\rfloor+1}^{\left\lfloor \frac{n}{\tilde{c}}(1+\varepsilon)\right\rfloor} \mathbb{P}(L=k)\mathbb{P}(\tilde{W}_k=n)\\ &+ \sum_{k>\left\lfloor \frac{n}{\tilde{c}}(1+\varepsilon)\right\rfloor} \mathbb{P}(L=k)\mathbb{P}(\tilde{W}_k=n)\\ &= I_1 + I_2 + I_3.\end{aligned} \tag{8.47}$$

We can easily show that $I_1 \leq \mathbb{P}(\tilde{W}_{\lfloor \frac{n}{\tilde{c}}(1-\varepsilon) \rfloor} \geq n)$, using Hoeffding's inequality, we obtain:

$$I_1 \leq \mathbb{P}\left(\tilde{W}_{\lfloor \frac{n}{\tilde{c}}(1-\varepsilon) \rfloor} \geq n\right) \leq \mathbb{P}\left(\tilde{W}_{\lfloor \frac{n}{\tilde{c}}(1-\varepsilon) \rfloor} - \left\lfloor \frac{n}{\tilde{c}}(1-\varepsilon) \right\rfloor \tilde{c} \geq n\varepsilon\right) \leq e^{-2\tilde{c}n\varepsilon^2},$$

and with a similar reasoning, we can show the existence of a positive constant C such that for n large enough:

$$I_3 \leq \mathbb{P}\left(\tilde{W}_{\lfloor \frac{n}{\tilde{c}}(1+\varepsilon) \rfloor} \left\lfloor \frac{n}{\tilde{c}}(1+\varepsilon) \right\rfloor \tilde{c} \leq -n\varepsilon\right) \leq \mathbb{P}\left(\left|\tilde{W}_{\lfloor \frac{n}{\tilde{c}}(1+\varepsilon) \rfloor} \left\lfloor \frac{n}{\tilde{c}}(1+\varepsilon) \right\rfloor \tilde{c}\right| \geq n\varepsilon\right) \leq Ce^{-2\tilde{c}n\varepsilon^2}.$$

In view of what we want to prove, I_1 and I_3 are negligible.
Corollary 8.4 gives that for n large enough and for all $k \in \llbracket \lfloor \frac{n}{\tilde{c}}(1-\varepsilon) \rfloor + 1, \lfloor \frac{n}{\tilde{c}}(1+\varepsilon) \rfloor \rrbracket$:

$$(1-\varepsilon)\left(\frac{\tilde{c}}{1+\varepsilon}\right)^{1+\alpha} \leq \frac{\mathbb{P}(L=k)}{\mathbb{P}(L=n)} \leq (1+\varepsilon)\left(\frac{\tilde{c}}{1-\varepsilon}\right)^{1+\alpha},$$

and consequently, to study the asymptotic behavior of I_2, we just have to look after:

$$\mathbb{P}(L=n)\tilde{c}^{\alpha+1} \sum_{k=\lfloor \frac{n}{\tilde{c}}(1-\varepsilon) \rfloor + 1}^{\lfloor \frac{n}{\tilde{c}}(1+\varepsilon) \rfloor} \mathbb{P}(\tilde{W}_k = n) =: \mathbb{P}(L=n)\tilde{c}^{\alpha+1}\tilde{I}_2.$$

Moreover, we have that:

$$\sum_{k=\lfloor \frac{n}{\tilde{c}}(1-\varepsilon) \rfloor + 1}^{\lfloor \frac{n}{\tilde{c}}(1+\varepsilon) \rfloor} \mathbb{P}(\tilde{W}_k = n) \leq \sum_{k \geq n} \binom{k}{n} \tilde{c}^n (1-\tilde{c})^{k-n} = \tilde{c}^{-1}, \tag{8.48}$$

and, the proof of the negligibility of I_1 and I_3 implies that:

$$\begin{aligned}
\lim_{n\to+\infty} \tilde{I}_2 &= \lim_{n\to+\infty} \sum_{k\geq n} \mathbb{P}(\tilde{W}_k = n) - \sum_{k=n}^{\lfloor \frac{n}{\tilde{c}}(1-\varepsilon) \rfloor} \mathbb{P}(\tilde{W}_k = n) - \sum_{k > \lfloor \frac{n}{\tilde{c}}(1+\varepsilon) \rfloor} \mathbb{P}(\tilde{W}_k = n) \\
&= \lim_{n\to+\infty} \sum_{k\geq n} \mathbb{P}(\tilde{W}_k = n) = \lim_{n\to+\infty} \sum_{k\geq n} \binom{k}{n} \tilde{c}^n (1-\tilde{c})^{k-n} = \tilde{c}^{-1}.
\end{aligned}$$

Finally, when n goes to infinity

$$\mathbb{P}(N=n) \sim \tilde{c}^{\alpha}\mathbb{P}(L=n),$$

then, we obtain:

$$\mathbf{p}_N(n) = \mathbb{P}(N=n) = \frac{\ell_N(n)\mathscr{L}(n)}{n^{1+\alpha}},$$

where $\ell_N(.) = g(.)\ell_L(.)$, $\ell_L(.)$ is defined in (8.45) and $g(.)$ is a positive function satisfying $\lim_{n\to+\infty} g(n) = \tilde{c}^{\alpha}$. And we obtain our claim result as:

$$\lim_{n\to+\infty} \ell_N(n) = \frac{(1-\ell_{\mathbf{q}})\mathbb{E}[L]^{1+\alpha}\tilde{c}^{\alpha}}{\tilde{\mathbf{p}}(0)} = \frac{(1-\ell_{\mathbf{q}})\mathbb{E}[N]^{\alpha}}{1-\mathbb{E}[X(1-\mathbf{q}(X))]}.$$

□

Now, we can determine the asymptotic behavior of $\mathbf{p}^{(1)}(n) = \mathbb{P}(Z^{(1)} = n)$ where $Z^{(1)} = \sum_{k=1}^{X^{(1)}} N_k$. Recalling that, for all $n \in \mathbb{N}$,

$$\mathbb{P}(X^{(1)} = n) = \frac{\mathscr{L}(n)\mathbf{q}(n)}{\mathbb{E}[\mathbf{q}(X)]n^{1+\alpha}}$$

According to Theorem 1, (iii) of [10], Lemma 8.12 and Lemma 8.13, when $n \to \infty$:

$$\begin{aligned}
\mathbb{P}(Z^{(1)} = n) &\sim \mathbb{E}\left[X^{(1)}\right]\mathbb{P}(N = n) + \mathbb{E}\left[N\right]^{-1}\mathbb{P}\left(X^{(1)} = \left\lfloor \frac{n}{\mathbb{E}\left[N\right]} \right\rfloor\right) \\
&\sim \frac{\mathscr{L}(n)\mathbb{E}[N]^{\alpha}}{n^{1+\alpha}\mathbb{E}[\mathbf{q}[X]]}\left(\frac{\mathbb{E}[X\mathbf{q}(X)]}{1 - \mathbb{E}[X(1-\mathbf{q}(X))]}(1 - \ell_{\mathbf{q}}) + \ell_{\mathbf{q}}\right) \\
&\sim \frac{\mathscr{L}(n)\mathbb{E}[N]^{\alpha}}{n^{1+\alpha}\mathbb{E}[\mathbf{q}[X]]}\frac{\mathbb{E}[X\mathbf{q}(X)] + \ell_{\mathbf{q}}(1 - \mathbb{E}[X])}{1 - \mathbb{E}[X(1-\mathbf{q}(X))]} =: \frac{\mathscr{L}(n)}{n^{1+\alpha}}\aleph
\end{aligned} \tag{8.49}$$

and we have:

$$\mathbf{p}^{(1)}(n) = \mathbb{P}(Z^{(1)} = n) = \frac{\ell_{Z^{(1)}}(n)\mathscr{L}(n)}{n^{1+\alpha}} \tag{8.50}$$

where $\ell_{Z^{(1)}}(.)$ is a positive function satisfying $\lim_{n\to+\infty} \ell_{Z^{(1)}}(n) = \aleph$.

Similarly, recalling that $Z^{(0)} = \sum_{k=1}^{X^{(0)}} N_k$, we have for all $n \in \mathbb{N}^*$,

$$\mathbb{P}(X^{(0)} = n) = \frac{\mathscr{L}(n)(1 - \mathbf{q}(n))}{(1 - \mathbb{E}[\mathbf{q}(X)])n^{1+\alpha}},$$

and according to the point (iii) of Theorem 1 of [10] and Lemma 8.12, we have, when n goes to infinity:

$$\begin{aligned}
\mathbb{P}(Z^{(0)} = n) &\sim \mathbb{E}[X^{(0)}]\mathbb{P}(N = n) + \mathbb{E}\left[N\right]^{-1}\mathbb{P}\left(X^{(0)} = \left\lfloor \frac{n}{\mathbb{E}\left[N\right]} \right\rfloor\right) \\
&\sim \frac{\mathscr{L}(n)}{n^{1+\alpha}}\frac{(1 - \ell_{\mathbf{q}})\mathbb{E}[N]^{\alpha}}{(1 - \mathbb{E}[\mathbf{q}(X)])(1 - \mathbb{E}[X(1-\mathbf{q}(X))])} =: \frac{\mathscr{L}(n)}{n^{1+\alpha}}(1 - \ell_{\mathbf{q}})\mathfrak{c}_{Z^{(0)}},
\end{aligned} \tag{8.51}$$

and we have:

$$\mathbf{p}^{(0)}(n) = \mathbb{P}(Z^{(0)} = n) = \frac{\ell_{Z^{(0)}}(n)\mathscr{L}(n)}{n^{1+\alpha}} \tag{8.52}$$

where $\ell_{Z^{(0)}}(.)$ is a positive function satisfying $\lim_{n\to+\infty} \ell_{Z^{(0)}}(n) = (1 - \ell_{\mathbf{q}})\mathfrak{c}_{Z^{(0)}}$.

8.6.2.2 Case $\ell_{\mathbf{q}} = 1$

In that case, we suppose that, the mark function $\mathbf{q}$ satisfies (6.8) for $k \in \mathbb{N}^*$,

$$1 - \mathbf{q}(k) = k^{-\beta}\mathcal{L}(k) \tag{8.53}$$

with $\beta \geq 2$ and $\mathcal{L}(.)$ is a slowly varying function. According to Remark 4.8 of [7], L is, also, distributed as the total size of a GW with offspring distribution given by the distribution of:

$$\tilde{Y} = \sum_{k=1}^{\mathcal{G}-1}\left(\tilde{X}_{k,+} - 1\right).$$

$\mathcal{G}$ is a geometric law with parameter $\tilde{\mathbf{p}}(0)$ independent of the i.i.d sequence $(\tilde{X}_{k,+})_{k\in\mathbb{N}^*}$ and for all $k,n\in\mathbb{N}^*$, $\mathbb{P}(\tilde{X}_{k,+}=n)=\mathbb{P}(\tilde{X}_+=n)=\mathbb{P}(\tilde{X}=n|\tilde{X}>0)$, so according to (8.17) we have

$$\mathbb{P}(\tilde{X}_+-1=n)=\frac{\mathscr{L}(n+1)\mathcal{L}(n+1)}{(n+1)^{1+\alpha+\beta}(1-\tilde{\mathbf{p}}(0))}$$

and as the product of SV functions is a SV function, all the results obtained in Sub-sub-section 8.6.2.1 can be adapted. Consequently, for all $n\in\mathbb{N}^*$:

$$\mathbb{P}(\tilde{Y}=n)=\frac{\tilde{\ell}_{\tilde{Y}}(n)\mathscr{L}(n)\mathcal{L}(n)}{n^{1+\alpha+\beta}}, \tag{8.54}$$

$$\mathbb{P}(L=n)=\frac{\tilde{\ell}_{L}(n)\mathscr{L}(n)\mathcal{L}(n)}{n^{1+\alpha+\beta}}, \tag{8.55}$$

$$\mathbf{p}_N(n)=\mathbb{P}(N=n)=\frac{\tilde{\ell}_{N}(n)\mathscr{L}(n)\mathcal{L}(n)}{n^{1+\alpha+\beta}}, \tag{8.56}$$

where $\tilde{\ell}_{\tilde{Y}}(.)$, $\tilde{\ell}_L(.)$ and $\tilde{\ell}_N(.)$ are positive functions satisfying $\lim\limits_{n\to+\infty}\tilde{\ell}_{\tilde{Y}}(n)=\frac{1}{\tilde{\mathbf{p}}(0)}$, $\lim\limits_{n\to+\infty}\ell_L(n)=\frac{\mathbb{E}[L]^{1+\alpha+\beta}}{\tilde{\mathbf{p}}(0)}$ and $\lim\limits_{n\to+\infty}\tilde{\ell}_N(n)=\frac{\mathbb{E}[N]^{\alpha+\beta}}{1-\mathbb{E}[X(1-\mathbf{q}(X))]}=:\mathfrak{c}_N$.

To determine the asymptotic behavior of $\mathbf{p}^{(1)}(n)$, when n goes to infinity, note that:

$$\mathbb{P}(X^{(1)}=n)\sim\frac{\mathscr{L}(n)}{n^{1+\alpha}\mathbb{E}[\mathbf{q}(X)]}.$$

Then, we obtain

$$\frac{\mathbb{P}(X^{(1)}=n)}{\mathbb{P}(N=n)}\underset{n\to\infty}{\sim}\frac{\mathscr{L}(n)}{\mathscr{L}(n)}\frac{1}{\mathbb{E}[\mathbf{q}(X)]\tilde{\ell}_{\tilde{Y}}(n)}\frac{n^\beta}{\mathcal{L}(n)},$$

implying that $\mathbb{P}(N=n)\underset{n\to\infty}{=}o\left(\mathbb{P}(X^{(1)}=n)\right)$ and according to Theorem 6 of [10]:

$$\mathbb{P}(Z^{(1)}=n)\sim\mathbb{E}\left[N\right]^{-1}\mathbb{P}\left(X^{(1)}=\left\lfloor\frac{n}{\mathbb{E}\left[N\right]}\right\rfloor\right)\sim\frac{\mathscr{L}(n)}{n^{1+\alpha}}\frac{\mathbb{E}[N]^\alpha}{\mathbb{E}[\mathbf{q}(X)]}.$$

As a result:

$$\mathbf{p}^{(1)}(n)=\mathbb{P}(Z^{(1)}=n)=\frac{\ell_{Z^{(1)}}(n)\mathscr{L}(n)}{n^{1+\alpha}} \tag{8.57}$$

where $\ell_{Z^{(1)}}(.)$ is a positive function satisfying $\lim\limits_{n\to+\infty}\ell_{Z^{(1)}}(n)=\dfrac{\mathbb{E}[N]^\alpha}{\mathbb{E}[\mathbf{q}(X)]}$.

Remark 8.4. *We can remark that we obtain the same limit of* (8.50) *taking* $\ell_{\mathbf{q}}=1$ *in* $\aleph$ *we obtain* $\dfrac{\mathbb{E}[N]^\alpha}{\mathbb{E}[\mathbf{q}(X)]}$.

For $Z^{(0)}$, note that

$$\mathbb{P}(X^{(0)}=n)=\frac{\mathscr{L}(n)\mathcal{L}(n)}{n^{1+\alpha+\beta}(1-\mathbb{E}[\mathbf{q}(X)])},$$

and according to the point (iii) of Theorem 1 of [10] and Lemma 8.12, we have, when n goes to infinity:

$$\begin{aligned}\mathbb{P}(Z^{(0)} = n) &\sim \mathbb{E}[X^{(0)}]\mathbb{P}(N = n) + \mathbb{E}\left[N\right]^{-1}\mathbb{P}(X^{(0)} = \left\lfloor \frac{n}{\mathbb{E}\left[N\right]} \right\rfloor) \\ &\sim \frac{\mathbb{E}[X(1-\mathbf{q}(X))]}{\mathbb{E}[1-\mathbf{q}(X)]}\frac{\tilde{\ell}_N(n)\mathscr{L}(n)\mathcal{L}(n)}{n^{1+\alpha+\beta}} + \frac{\mathbb{E}[N]^{\alpha+\beta}\mathscr{L}(n)\mathcal{L}(n)}{n^{1+\alpha+\beta}(1-\mathbb{E}[\mathbf{q}(X)])},\end{aligned}$$

and we have:

$$\mathbf{p}^{(0)}(n) = \mathbb{P}(Z^{(0)} = n) = \frac{\tilde{\ell}_{Z^{(0)}}(n)\mathcal{L}(n)\mathscr{L}(n)}{n^{1+\alpha+\beta}}, \tag{8.58}$$

where $\tilde{\ell}_{Z^{(0)}}(.)$ is a positive function satisfying $\lim\limits_{n\to+\infty} \tilde{\ell}_{Z^{(0)}}(n) = \frac{\mathbb{E}[N]^{\alpha+\beta}}{(1-\mathbb{E}[\mathbf{q}(X)])(1-\mathbb{E}[X(1-\mathbf{q}(X))])}$.

8.6.2.3 General results

For $S_n = \sum\limits_{i=1}^{n} Z_i^{(1)}$, according to [12], we have for all $\varepsilon > 0$ and uniformly in $k \geq (\mu^{(1)}+\varepsilon)n$:

$$\mathbb{P}(S_n = k) \underset{n\to\infty}{\sim} n\mathbf{p}^{(1)}(\lfloor k - n\mu^{(1)} \rfloor). \tag{8.59}$$

Moreover, according to Theorem 2 in [13], since $\mu^{(1)} < +\infty$, $\sigma^2 := \mathrm{Var}[Z^{(1)}] < +\infty$ (Lemma 8.8) and $\mathbf{p}^{(1)}$ being regularly varying at infinity with index $\kappa < -3$, uniformly in k such that
$\frac{k-n\mu^{(1)}}{\sqrt{n}} \to +\infty$, when n goes to infinity, we have:

$$\mathbb{P}(S_n = k) = \frac{e^{-\frac{(k-n\mu^{(1)})^2}{2n\sigma^2}}}{\sigma\sqrt{2\pi n}}(1+o(1)) + n\mathbf{p}^{(1)}(\lfloor k - n\mu^{(1)} \rfloor)(1+o(1)), \text{ as } n \to \infty, \tag{8.60}$$

with $\sigma^2 = \mathbb{E}[(Z^{(1)} - \mu^{(1)})^2]$.

Lemma 8.14. *Assume that $\mathbf{p}^{(1)}$ is aperiodic , with $\mu^{(1)} < 1$, $\rho(\mathbf{p}^{(1)}) = 1$, $\mathrm{Var}[Z^{(1)}] < +\infty$ and $\mathbf{q}$ satisfies* (6.8) *if $\ell_{\mathbf{q}} = 1$. If T is a nondegenerate positive random variable in L^1, independent of $(S_n)_{n\in\mathbb{N}}$, such that:*

$$\forall k \in \mathbb{N},\ \mathbb{P}(T = k) = \frac{\mathscr{L}(k)}{k^{1+\alpha}}(1-\mathbf{q}(k))\gamma(k),$$

where γ is a positive function such $\lim\limits_{x\to+\infty}\gamma(x) = \mathfrak{c} > 0$, we have

$$\lim_{n\to+\infty} \frac{\mathbb{E}\left[T\mathbb{1}_{\{S_n+T=n\}}\right]}{\mathbb{P}(S_n = n)} = \mathbb{E}[T] + \left(1-\mu^{(1)}\right)\frac{(1-\ell_{\mathbf{q}})\mathfrak{c}}{\aleph}, \tag{8.61}$$

where $\aleph := \frac{\mathbb{E}[N]^{\alpha}}{\mathbb{E}[\mathbf{q}(X)]}\frac{\mathbb{E}[X\mathbf{q}(X)]+\ell_{\mathbf{q}}(1-\mathbb{E}[X])}{1-\mathbb{E}[X(1-\mathbf{q}(X))]}$.

Remark 8.5. *Note that if $\ell_{\mathbf{q}} = 1$, the ratio in* (8.61) *tends to $\mathbb{E}[T]$.*

Proof. Let $0 < \varepsilon < 1 - \mu^{(1)}$, $\frac{2}{3} < d < \frac{3}{4}$, and for n large enough, to determine (8.61), we need to introduce these four intervals:

$$J_1 = \left[0, n(1-\mu^{(1)} - \varepsilon)\right], \ J_2 = \left]n(1-\mu^{(1)}-\varepsilon), n\left(1-\mu^{(1)}\right) - n^d\right],$$
$$J_3 = \left]n\left(1-\mu^{(1)}\right) - n^d, n\left(1-\mu^{(1)}\right) + n^d\right], J_4 = \left]n\left(1-\mu^{(1)}\right) + n^d, n\right],$$

and for typographical simplicity, we introduce for $i \in [\![1,4]\!]$:

$$I_i := \frac{\mathbb{E}\left[T\mathbb{1}_{\{S_n+T=n, T\in J_i\}}\right]}{\mathbb{P}(S_n = n)} = \sum_{k\in J_i} k\mathbb{P}(T=k)\frac{\mathbb{P}(S_n = n-k)}{\mathbb{P}(S_n=n)}.$$

We recall that S_n and T are positive random variable, so, they cannot take values after n.

We can write for all $n \in \mathbb{N}^*$, $n = n\mu^{(1)} + n\left(1-\mu^{(1)}\right)$ with $1-\mu^{(1)} > 0$, so, according to (8.59), we have $\mathbb{P}(S_n = n) \underset{n\to\infty}{\sim} n\mathbf{p}^{(1)}\left(\left\lfloor n\left(1-\mu^{(1)}\right)\right\rfloor\right)$.
On J_1, note that the exponential term is negligible in (8.60), as a result for n large enough for all $k \in I_1$:

$$\frac{\mathbb{P}(S_n = n-k)}{\mathbb{P}(S_n = n)} \le 2\frac{\mathbf{p}^{(1)}\left(\left\lfloor n\left(1-\mu^{(1)}\right) - k\right\rfloor\right)}{\mathbf{p}^{(1)}\left(\left\lfloor n\left(1-\mu^{(1)}\right)\right\rfloor\right)}. \tag{8.62}$$

Thus, using formula (8.50) or (8.57) and Corollary 8.4, for n large enough and all $k \in J_1$:

$$\frac{\mathbb{P}(S_n = n-k)}{\mathbb{P}(S_n = n)} \le 4\left(\frac{1-\mu^{(1)}}{\varepsilon}\right)^{\alpha+1}. \tag{8.63}$$

The strong ratio limit Theorem gives that for fixed k:

$$\lim_{n\to+\infty} k\mathbb{P}(T=k)\frac{\mathbb{P}(S_n=n-k)}{\mathbb{P}(S_n=n)} = k\mathbb{P}(T=k).$$

Consequently, as $T \in L^1$, Lebesgue's dominated convergence theorem implies:

$$\lim_{n\to+\infty} I_1 = \lim_{n\to+\infty}\sum_{k\in J_1} k\mathbb{P}(T=k)\frac{\mathbb{P}(S_n=n-k)}{\mathbb{P}(S_n=n)} = \sum_{k\in\mathbb{N}} k\mathbb{P}(T=k) = \mathbb{E}[T].$$

Note that on J_2, (8.62) remains valid and using Remark 8.3, there exists $C>0$ such that for n large enough and all $k \in J_2$:

$$\frac{\mathbb{P}(S_n = n-k)}{\mathbb{P}(S_n = n)} \le Cn^{(1-d)(\alpha+1)}, \tag{8.64}$$

and as $\mathbb{P}(T=k) = k^{-1-\alpha}\tilde{\mathscr{L}}(k)$ where $\tilde{\mathscr{L}}$ is a SV function, for any $\delta > 0$, for k large enough $\mathbb{P}(T=k) \le k^{\delta-1-\alpha}$, consequently for n large enough and $0<\delta<\alpha$:

$$I_2 \le Cn^{(1-d)(\alpha+1)}\sum_{k\in J_2}\frac{1}{k^{\alpha-\delta}} \le \frac{\varepsilon C n^{2+\delta-d(\alpha+1)}}{(1-\mu^{(1)}-\varepsilon)^{\alpha-\delta}} =: \tilde{C}n^{2+\delta-d(\alpha+1)}.$$

As $\alpha > 2$ and $d > \frac{2}{3}$, we can take δ small enough such that $2+\delta-d(\alpha+1) < 0$ and it implies that

$$\lim_{n\to+\infty} I_2 = 0.$$

In order to determine the limit of I_3, we have to distinguish if $\ell_{\mathbf{q}} = 1$ or not.

Case $\ell_{\mathbf{q}} = 1$:

Recall that when n goes to infinity, there exists $C_0 > 0$ such that $\mathbb{P}(S_n = n) \sim \frac{C_0 \mathscr{L}(n)}{n^\alpha}$. As a result, there exists two positive constants C_1 and C_2, such that for n large enough, using Lemma 8.12, for all $k \in J_3$:

$$\frac{k\mathbb{P}(T=k)}{\mathbb{P}(S_n=n)} \leq \frac{C_1 \mathcal{L}(k)}{k^\beta} \frac{n^\alpha \mathscr{L}(k)}{\mathscr{L}(n) k^\alpha} \leq C_2 \frac{\mathcal{L}(k)}{k^\beta}.$$

Consequently, for n large enough:

$$I_3 \leq \sum_{k\in J_3} \frac{k\mathbb{P}(T=k)}{\mathbb{P}(S_n=n)} \leq C_2 \sum_{k\in J_3} \frac{\mathcal{L}(k)}{k^\beta} =: C_2 R_n,$$

and $\lim\limits_{n\to+\infty} R_n = 0$, $\frac{\mathcal{L}(k)}{k^\beta}$ being the general term of a convergent series as $\mathcal{L}$ is a SV function and $\beta \geq 2$.

Case $\ell_{\mathbf{q}} \neq 1$: We postpone the proof that I_3 has the same limit at infinity as:

$$\tilde{I}_3 := \frac{\mathbb{E}\left[n\left(1-\mu^{(1)}\right)\mathbb{1}_{\{S_n+T=n, T\in J_3\}}\right]}{\mathbb{P}(S_n=n)}.$$

Note that, according to Corollary 8.4 and (8.50), for n large enough and all $k \in J_3$:

$$(1-\varepsilon)\frac{(1-\ell_{\mathbf{q}})\mathfrak{c}}{n\aleph} \leq \frac{\mathbb{P}(T=k)}{\mathbb{P}(S_n=n)} \leq (1+\varepsilon)\frac{(1-\ell_{\mathbf{q}})\mathfrak{c}}{n\aleph}. \tag{8.65}$$

Moreover, as $d > \frac{2}{3}$, thanks to the central limit theorem, when n goes to infinity:

$$\sum_{k\in J_3} \mathbb{P}(S_n = n-k) = \mathbb{P}\left(\frac{S_n - n\mu^{(1)}}{\sqrt{n}} \in \left[-n^{d-\frac{1}{2}}, n^{d-\frac{1}{2}}\right]\right) \to 1. \tag{8.66}$$

This gives an upper bound for $\lim\limits_{n\to+\infty} \tilde{I}_3$. Indeed:

$$\begin{aligned}
\limsup_{n\to+\infty} \tilde{I}_3 &= \limsup_{n\to+\infty} n\left(1-\mu^{(1)}\right) \sum_{k\in J_3} \frac{\mathbb{P}(T=k)}{\mathbb{P}(S_n=n)} \mathbb{P}(S_n = n-k) \\
&\leq (1+\varepsilon)\left(1-\mu^{(1)}\right)\frac{(1-\ell_{\mathbf{q}})\mathfrak{c}}{\aleph} \lim_{n\to+\infty} \mathbb{P}\left(S_n - n\mu^{(1)} \in [-n^d, n^d]\right) \\
&= (1+\varepsilon)\left(1-\mu^{(1)}\right)\frac{(1-\ell_{\mathbf{q}})\mathfrak{c}}{\aleph}
\end{aligned}$$

And we can obtain the lower bound with the same reasoning. As a result:

$$\lim_{n\to+\infty} \tilde{I}_3 = \left(1-\mu^{(1)}\right)\frac{(1-\ell_{\mathbf{q}})\mathfrak{c}}{\aleph}.$$

Now, to prove that I_3 and $\tilde{I}_3$ have the same limit, note that (8.65) ensures that there exists $C > 0$ such that for all $k \in J_3$ and n large enough:

$$\frac{\mathbb{P}(T=k)}{\mathbb{P}(S_n=n)} \leq \frac{C}{n}, \tag{8.67}$$

and as a result, as $d < \frac{3}{4}$ and using (8.66), when n goes to infinity:

$$|I_3 - \tilde{I}_3| \leq \frac{\mathbb{E}\left[\left|T - n\left(1-\mu^{(1)}\right)\right| \mathbb{1}_{\{T\in J_3, S_n+T=n\}}\right]}{\mathbb{P}(S_n = n)}$$
$$\leq \frac{2C}{n^{1-d}}\mathbb{P}\left(S_n - n\mu^{(1)} \in \left[-n^d, n^d\right]\right) \leq \frac{2C}{n^{1-d}} \to 0.$$

It remains to prove that I_4 tends to 0 to obtain our result. Note that for n large enough and $k \in J_4$, (8.67) still true and we have

$$\frac{k\mathbb{P}(T=k)}{\mathbb{P}(S_n = n)} \leq C,$$

and thus, using again the central limit theorem, when n goes to infinity:

$$I_4 \leq C\mathbb{P}\left(S_n - n\mu^{(1)} \leq -n^d\right) \to 0.$$

We obtain (8.61), since

$$\frac{\mathbb{E}\left[T\mathbb{1}_{\{S_n+T=n\}}\right]}{\mathbb{P}(S_n = n)} = I_1 + I_2 + I_3 + I_4 \underset{n\to+\infty}{\longrightarrow} \mathbb{E}[T] + \left(1-\mu^{(1)}\right)\frac{(1-\ell_{\mathbf{q}})\mathfrak{c}}{\aleph}.$$

□

8.6.3 Principal lemmas

In this subsection, we consider the lemmas used in Theorem 8.1 and Theorem 8.2. Recall that $\mathfrak{c}_N$ and $\aleph$ are described in Lemma 8.13 and Assumption (8.49).

Lemma 8.15. *Assume that $\mathbf{p}^{(1)}$ is aperiodic , with $\mu^{(1)} = 1$ or ($\mu^{(1)} < 1$ and $\rho(\mathbf{p}^{(1)}) = 1$) and $\mathrm{Var}[Z^{(1)}] < +\infty$. Let $m \in \mathbb{N}$, $k \in \mathbb{Z}$, we have:*

$$\lim_{n\to+\infty} \frac{\mathbb{E}[N\mathbb{1}_{\{S_n+N=n\}}]}{\mathbb{E}[\mathbb{1}_{\{S_n+W_m+N=n-k\}}]} = \mathbb{E}[N] + \left(1-\mu^{(1)}\right)\frac{(1-\ell_{\mathbf{q}})\mathfrak{c}_N}{\aleph}. \tag{8.68}$$

If $\mathbf{p}^{(1)}$ is periodic, then (8.68) *still holds along the subsequence for which the denominator is positive.*

Remark 8.6. 1. *If $\mu^{(1)} = 1$ or $\ell_{\mathbf{q}} = 1$, the ratio in* (8.68) *tends to $\mathbb{E}[N]$.*

2. *Note that the way the limit is written in formula* (8.68) *helps streamline the proof. Nevertheless, in the remainder of this section, it will be more convenient to simplify the expression further by using that:*

$$\frac{\mathfrak{c}_N}{\aleph} = \frac{\mathbb{E}[\mathbf{q}(X)]}{\mathbb{E}[X\mathbf{q}(X)] + \ell_{\mathbf{q}}(1-\mu(\mathbf{p}))}, \ \mathbb{E}[N] = \frac{\mathbb{E}[\mathbf{q}(X)]}{1-\mu(\mathbf{p}) + \mathbb{E}[X\mathbf{q}(X)]},$$
$$\mu^{(1)} = \frac{\mathbb{E}[X\mathbf{q}(X)]}{1-\mu(\mathbf{p}) + \mathbb{E}[X\mathbf{q}(X)]},$$

and we obtain:

$$\lim_{n\to+\infty} \frac{\mathbb{E}[N\mathbb{1}_{\{S_n+N=n\}}]}{\mathbb{E}[\mathbb{1}_{\{S_n+W_m+N=n-k\}}]} = \frac{\mathbb{E}[\mathbf{q}(X)]}{\mathbb{E}[X\mathbf{q}(X)] + \ell_{\mathbf{q}}(1-\mu(\mathbf{p}))}. \tag{8.69}$$

Proof. First, assume that there exists $a > 0$, such that for all k, m in $\mathbb{Z}$:

$$\lim_{n\to+\infty} \frac{\mathbb{E}\left[\mathbb{1}_{\{S_{n+m}+N=n-k\}}\right]}{\mathbb{E}\left[N\mathbb{1}_{\{S_n+N=n\}}\right]} = a. \tag{8.70}$$

To mimic the proof of Lemma 8.10 (see Lemma 8.6 of [6]), we introduce for all $j \in \mathbb{Z}$:

$$c_{n,j} := \frac{\mathbb{E}\left[\mathbb{1}_{\{S_n+N=n-k-j\}}\right]}{\mathbb{E}\left[N\mathbb{1}_{\{S_n+N=n\}}\right]}, \tag{8.71}$$

and denote by $t := (t(j))_{j\in\mathbb{N}}$ and $r := (r(j))_{j\in\mathbb{N}}$ respectively the distributions of W_m and S_m. Thanks to (8.70), $\lim\limits_{n\to+\infty} c_{n,j} = a$ and consequently:

$$\lim_{n\to+\infty} \sum_{j\in\mathbb{N}} r(j)c_{n,j} = \lim_{n\to+\infty} \frac{\mathbb{E}\left[\mathbb{1}_{\{S_{n+m}+N=n-k\}}\right]}{\mathbb{E}\left[N\mathbb{1}_{\{S_n+N=n\}}\right]} = a = \sum_{j\in\mathbb{N}} r(j) \lim_{n\to+\infty} c_{n,j}.$$

According to (6.2), there exists $j_0 \in \mathbb{N}^*$ such that $\mathbf{p}(j_0)\mathbf{q}(j_0) > 0$ implying that $\mathbb{P}(X^{(1)} = j_0) > 0$. As $Z_1^{(1)} = \sum_{i=1}^{X_1^{(1)}} N_i$, for $j \in \mathbb{N}$:

$$\begin{aligned} r(j) = \mathbb{P}(S_m = j) &\geq \mathbb{P}\left(\sum_{i=1}^{m} X_i^{(1)} = mj_0, \sum_{i=1}^{mj_0} N_i = j\right) \\ &\geq \mathbb{P}\left(\sum_{i=1}^{m} X_i^{(1)} = mj_0, W_m = j, \sum_{i=m+1}^{mj_0} N_i = 0\right) \geq Ct(j), \end{aligned}$$

with $C > 0$, independent of j. Thus, as $t(j) \leq \frac{r(j)}{C}$ for all $j \in \mathbb{N}$, by dominated convergence theorem in [21] (Theorem 1.21), we deduce that

$$\lim_{n\to+\infty} \frac{\mathbb{E}\left[\mathbb{1}_{\{S_n+W_m+N=n-k\}}\right]}{\mathbb{E}\left[N\mathbb{1}_{\{S_n+N=n\}}\right]} = \lim_{n\to+\infty} \sum_{j\in\mathbb{N}} t(j)c_{n,j} = \sum_{j\in\mathbb{N}} t(j) \lim_{n\to+\infty} c_{n,j} = a.$$

As a result, in order to prove (8.68), we have to obtain (8.70) but, according to Lemma 8.10:

$$\begin{aligned} \lim_{n\to+\infty} \frac{\mathbb{E}\left[N\mathbb{1}_{\{S_n+N=n\}}\right]}{\mathbb{E}\left[\mathbb{1}_{\{S_{n+m}+N=n-k\}}\right]} &= \lim_{n\to+\infty} \frac{\mathbb{E}\left[N\mathbb{1}_{\{S_n+N=n\}}\right]}{\mathbb{P}(S_n = n)} \frac{\mathbb{P}(S_n = n)}{\mathbb{E}\left[\mathbb{1}_{\{S_{n+m}+N=n-k\}}\right]} \\ &= \lim_{n\to+\infty} \frac{\mathbb{E}\left[N\mathbb{1}_{\{S_n+N=n\}}\right]}{\mathbb{P}(S_n = n)} =: \lim_{n\to+\infty} R_n, \end{aligned}$$

and it suffices to study the behavior at infinity of R_n.
The case $\mu^{(1)} < 1$ is straightforward by applying Lemma 8.14 with $T = N$.

$$\lim_{n\to+\infty} \frac{\mathbb{E}[N\mathbb{1}_{\{S_n+N=n\}}]}{\mathbb{E}[\mathbb{1}_{\{S_n+W_m+N=n-k\}}]} = \mathbb{E}[N] + \left(1 - \mu^{(1)}\right) \frac{(1-\ell_{\mathbf{q}})\mathfrak{c}_N}{\aleph}.$$

To obtain the case $\mu^{(1)} = 1$, we denote by $Z_n := S_n - n = \sum_{i=1}^{n}(Z_i^{(1)} - 1)$, a centered random walk such that $\mathbb{E}[Z_n] = 0$. We have:

$$R_n = \frac{\mathbb{E}\left[N\mathbb{1}_{\{Z_n=-N\}}\right]}{\mathbb{P}(Z_n=0)} = \sum_{k\geq 0} k\mathbf{p}_N(k)\frac{\mathbb{P}(Z_n=-k)}{\mathbb{P}(Z_n=0)}.$$

According to [14] (Theorem 5.2 p.132),

$$\sup_{k\in\mathbb{Z}}\left|\sqrt{n}\mathbb{P}(Z_n=k) - \frac{e^{-\frac{k^2}{2n\sigma^2}}}{\sigma\sqrt{2\pi}}\right|$$

tends to zero when n goes to infinity. Thus for all $k \in \mathbb{Z}$ we have:

$$\mathbb{P}(Z_n=k) = \frac{e^{-\frac{k^2}{2n\sigma^2}}}{\sigma\sqrt{2\pi n}} + o\left(\frac{1}{\sqrt{n}}\right), \tag{8.72}$$

with a uniform error term for all $k \in \mathbb{Z}$. Consequently, there exists a positive constant C such that:

$$\sup_{k\in\mathbb{Z}}\frac{\mathbb{P}(Z_n=k)}{\mathbb{P}(Z_n=0)} \leq C \text{ and } \lim_{n\to+\infty}\frac{\mathbb{P}(Z_n=k)}{\mathbb{P}(Z_n=0)} = 1 \tag{8.73}$$

for all $k \in \mathbb{Z}$, and with Lebesgue's dominated convergence theorem:

$$\lim_{n\to+\infty} R_n = \sum_{k\geq 0} k\mathbf{p}_N(k)\lim_{n\to+\infty}\frac{\mathbb{P}(Z_n=-k)}{\mathbb{P}(Z_n=0)} = \mathbb{E}[N].$$

$\square$

Lemma 8.16. *Assume that $\mathbf{p}^{(1)}$ is aperiodic, with $\mu^{(1)} = 1$ or ($\mu^{(1)} < 1$ and $\rho(\mathbf{p}^{(1)}) = 1$) and $\mathrm{Var}[Z^{(1)}] < +\infty$. Let $m \in \mathbb{N}$, $k \in \mathbb{Z}$, we have:*

$$\lim_{n\to+\infty}\frac{\mathbb{E}[N\mathbb{1}_{\{S_n+W_m+Z^{(0)}+N=n-k\}}]}{\mathbb{E}[N\mathbb{1}_{\{S_n+N=n\}}]} = 1 \tag{8.74}$$

If $\mathbf{p}^{(1)}$ is periodic, then (8.74) *still holds along the subsequence for which the denominator is positive.*

Proof. First, writing the ratio from (8.74) in the form:

$$\frac{\mathbb{E}[N\mathbb{1}_{\{S_n+W_m+Z^{(0)}+N=n-k\}}]}{\mathbb{P}(S_n=n)}\frac{\mathbb{P}(S_n=n)}{\mathbb{E}\left[N\mathbb{1}_{\{S_n+N=n\}}\right]} =: v_n w_n,$$

one can note that Lemma 8.15 gives that w_n tends to an explicit constant w, as a result we just have to show that $\lim_{n\to+\infty} v_n = w^{-1}$.

Case $\mu^{(1)} = 1$:

Here, it suffices to prove that $\lim_{n\to+\infty} v_n = \mathbb{E}[N]$. Using the same notation as Lemma 8.15 for Z_n, we have:

$$v_n = \frac{\mathbb{E}[N\mathbb{1}_{\{Z_n+W_m+Z^{(0)}+N=-k\}}]}{\mathbb{P}(Z_n=0)} = \sum_{i\in\mathbb{Z}} \mathbb{E}[N\mathbb{1}_{\{W_m+Z^{(0)}+N=i-k\}}]\frac{\mathbb{P}(Z_n=-i)}{\mathbb{P}(Z_n=0)}.$$

Consequently, (8.73) permits us to apply Lebesgue's dominated convergence theorem, so:

$$\lim_{n\to+\infty} v_n = \sum_{i\in\mathbb{Z}} \mathbb{E}[N\mathbb{1}_{\{W_m+Z^{(0)}+N=i-k\}}] = \mathbb{E}[N].$$

Case $\mu^{(1)} < 1$:

Like in Lemma 8.15, we can prove that for all $k, m \in \mathbb{Z}$:

$$\lim_{n\to+\infty} v_n = \lim_{n\to+\infty} \frac{\mathbb{E}[N\mathbb{1}_{\{S_{n+m}+Z^{(0)}+N=n-k\}}]}{\mathbb{P}(S_n=n)} =: \lim_{n\to+\infty} z_n,$$

and we write:

$$z_n = \frac{\mathbb{E}[N\mathbb{1}_{\{S_{n+m}+Z^{(0)}+N=n-k\}}]}{\mathbb{E}[N\mathbb{1}_{\{S_n+Z^{(0)}+N=n\}}]}\frac{\mathbb{E}[N\mathbb{1}_{\{S_n+Z^{(0)}+N=n\}}]}{\mathbb{P}(S_n=n)} =: a_n b_n.$$

According to Lemma 8.9, with $T = N$ and $R = Z^{(0)}$, $\lim_{n\to+\infty} a_n = 1$ and as a result, we need to determine the limit of b_n and distinguish between the cases depending on whether $\ell_{\mathbf{q}} = 0$ or not.

Case $\ell_{\mathbf{q}} \neq 0$:

Our strategy here is to mimic the proof of Lemma 8.10, in other words, we search $(\beta_j)_{j\in\mathbb{N}}$ such that for all $j \in \mathbb{N}$:

$$\mathbb{P}(Z^{(0)} = j) \le \beta_j,$$

and satisfying:

$$\lim_{n\to+\infty} \sum_{j\in\mathbb{N}} \beta_j c_{n,j} = \sum_{j\in\mathbb{N}} \beta_j \lim_{n\to+\infty} c_{n,j}, \tag{8.75}$$

where:

$$c_{n,j} = \frac{\mathbb{E}\left[N\mathbb{1}_{\{S_n+N=n-j\}}\right]}{\mathbb{P}(S_n=n)}.$$

Thanks to Lemma 8.14 with $T = N$, for all $j \in \mathbb{N}$, $\lim_{n\to+\infty} c_{n,j} = w^{-1}$, thus, by applying the dominated convergence theorem in [21] (Theorem 1.21), we obtain the claimed result as:

$$\lim_{n\to+\infty} b_n = \lim_{n\to+\infty} \sum_{j\in\mathbb{N}} \mathbb{P}(Z^{(0)}=j)c_{n,j} = \sum_{j\in\mathbb{N}} \mathbb{P}(Z^{(0)}=j) \lim_{n\to+\infty} c_{n,j} = w^{-1}.$$

To find $(\beta_j)_{j\in\mathbb{N}}$, first note that since $\lim_{n\to+\infty} \mathbf{q}(n) = \ell_{\mathbf{q}} > 0$, there exists k_0 such that for all $k \ge k_0$, $\mathbf{q}(k) \neq 0$, and c defined by:

$$c = \frac{\mathbb{E}[\mathbf{q}(X)]}{1-\mathbb{E}[\mathbf{q}(X)]} \max_{k\in\mathbb{N},\mathbf{q}(k)>0} \left\{\frac{1-\mathbf{q}(k)}{\mathbf{q}(k)}\right\},$$

is finite, and one can write for all $k \in \mathbb{N},\ \mathbf{q}(k) > 0$:

$$\mathbb{P}(X^{(0)} = k) = \frac{\mathbf{p}(k)(1-\mathbf{q}(k))}{1-\mathbb{E}[\mathbf{q}(X)]} \leq c\frac{\mathbf{p}(k)\mathbf{q}(k)}{\mathbb{E}[\mathbf{q}(X)]} = c\mathbb{P}(X^{(1)} = k).$$

Consequently, for all $j \in \mathbb{N}$:

$$\begin{aligned}\mathbb{P}(Z^{(0)} = j) = \mathbb{P}\left(\sum_{i=1}^{X^{(0)}} N_i = j\right) &= \sum_{k=1}^{+\infty} \mathbb{P}\left(W_k = j\right)\mathbb{P}(X^{(0)} = k) \\ &\leq \sum_{k=1,\mathbf{q}(k)=0}^{k_0} \mathbb{P}\left(W_k = j\right)\frac{\mathbf{p}(k)}{1-\mathbb{E}[\mathbf{q}(X)]} + c\mathbb{P}(Z^{(1)} = j) \\ &\leq \frac{1}{1-\mathbb{E}[\mathbf{q}(X)]}\sum_{k=1,\mathbf{q}(k)=0}^{k_0} \mathbb{P}\left(W_k = j\right) + c\mathbf{p}^{(1)}(j) =: \beta_j.\end{aligned}$$

In one hand, thanks to Lemma 8.9 and Lemma 8.14 with $T = N$ and $R = 0$, we have for all $j \in \mathbb{N}$:

$$\lim_{n\to+\infty}\sum_{j\in\mathbb{N}} \mathbf{p}^{(1)}(j)c_{n,j} = \lim_{n\to+\infty}\frac{\mathbb{E}\left[N\mathbb{1}_{\{S_{n+1}+N=n\}}\right]}{\mathbb{P}(S_n = n)} = w^{-1},$$

implying that, as $\lim\limits_{n\to+\infty} c_{n,j} = w^{-1}$:

$$\lim_{n\to+\infty}\sum_{j\in\mathbb{N}} \mathbf{p}^{(1)}(j)c_{n,j} = w^{-1} = \sum_{j\in\mathbb{N}} \mathbf{p}^{(1)}(j)\lim_{n\to+\infty} c_{n,j}. \tag{8.76}$$

On the other hand, for all $k \in [\![0, k_0]\!]$ such that $\mathbf{q}(k) = 0$, we write:

$$\begin{aligned}\sum_{j\in\mathbb{N}} \mathbb{P}\left(W_k = j\right)c_{n,j} &= \frac{\mathbb{E}\left[N\mathbb{1}_{\{S_n+W_k+N=n\}}\right]}{\mathbb{P}(S_n = n)} \\ &= \frac{\mathbb{E}\left[N\mathbb{1}_{\{S_n+W_k+N=n\}}\right]}{\mathbb{E}\left[N\mathbb{1}_{\{S_n+N=n\}}\right]}\frac{\mathbb{E}\left[N\mathbb{1}_{\{S_n+N=n\}}\right]}{\mathbb{P}(S_n = n)} =: r_n s_n.\end{aligned}$$

Thanks to Lemma 8.10, $\lim\limits_{n\to+\infty} r_n = 1$ and according to Lemma 8.14 with $T = N$, $\lim\limits_{n\to+\infty} s_n = w^{-1}$. Then:

$$\lim_{n\to+\infty}\sum_{j\in\mathbb{N}} \mathbb{P}\left(W_k = j\right)c_{n,j} = w^{-1} = \sum_{j\in\mathbb{N}} \mathbb{P}\left(W_k = j\right)\lim_{n\to+\infty} c_{n,j}. \tag{8.77}$$

(8.76) and (8.77) imply that $(\beta_j)_{j\in\mathbb{N}}$ satisfies (8.75) and we can conclude.

Case $\ell_{\mathbf{q}} = 0$:

As in Lemma 8.15, to obtain $\lim\limits_{n\to+\infty} b_n$, we need to distinguish between several cases. To that end, we introduce $0 < \varepsilon < 1-\mu^{(1)}$, $\max\left(\frac{2}{3}, \frac{1+\alpha}{2\alpha}\right) < d < \frac{3}{4}$, and for n large enough, we introduce the same four intervals:

$$\begin{aligned}&J_1 = \left[0, n(1-\mu^{(1)}-\varepsilon)\right],\ J_2 = \left]n(1-\mu^{(1)}-\varepsilon), n\left(1-\mu^{(1)}\right)-n^d\right], \\ &J_3 = \left]n\left(1-\mu^{(1)}\right)-n^d, n\left(1-\mu^{(1)}\right)+n^d\right], J_4 = \left]n\left(1-\mu^{(1)}\right)+n^d, n\right],\end{aligned}$$

and for typographical simplicity, we introduce for $i \in [\![1,4]\!]$:

$$I_i := \frac{\mathbb{E}\left[N\mathbb{1}_{\{S_n+Z^{(0)}+N=n, Z^{(0)}+N\in J_i\}}\right]}{\mathbb{P}(S_n=n)}.$$

Writing the equality in distribution

$$N+Z^{(0)} = \sum_{i=1}^{X^{(0)}+1} N_i,$$

we can apply the same method as (8.52), and we obtain:

$$\mathbb{P}(N+Z^{(0)}=n) = \frac{\ell_{N+Z^{(0)}}(n)\mathscr{L}(n)}{n^{1+\alpha}}, \tag{8.78}$$

where $\ell_{N+Z^{(0)}}(.)$ is a positive function such that

$$\lim_{n\to+\infty} \ell_{N+Z^{(0)}}(n) = \frac{\mathbb{E}[N]^\alpha(2-\mathbb{E}[\mathbf{q}(X)])}{(1-\mathbb{E}[\mathbf{q}(X)])(1-\mathbb{E}[X(1-\mathbf{q}(X))])}.$$

Then, with a similar reasoning as the one of the proof of Lemma 8.14 with $T = N+Z^{(0)}$, we obtain for $i \in \{2,4\}$:

$$I_i \le \frac{\mathbb{E}[\left(N+Z^{(0)}\right)\mathbb{1}_{\{S_n+Z^{(0)}+N=n, Z^{(0)}+N\in J_i\}}]}{\mathbb{P}(S_n=n)} \underset{n\to+\infty}{\longrightarrow} 0,$$

and as $N \in L^1$, (8.63) and the strong ratio limit Theorem allow us to apply Lebesgue's dominated convergence theorem and:

$$\begin{aligned}\lim_{n\to+\infty} I_1 &= \sum_{k\in J_1} \mathbb{E}[N\mathbb{1}_{\{Z^{(0)}+N=k\}}]\frac{\mathbb{P}(S_n=n-k)}{\mathbb{P}(S_n=n)}\\ &= \sum_{k\in\mathbb{N}} \mathbb{E}[N\mathbb{1}_{\{Z^{(0)}+N=k\}}] = \mathbb{E}[N].\end{aligned}$$

To study the limit of I_3, we need more information on N and, for this purpose, we introduce three new intervals:

$$\begin{aligned}&K_1 = \left[0, n^d\right], K_2 = \left[n^d+1, n\left(1-\mu^{(1)}\right)-n^d-1\right],\\ &K_3 = \left[n\left(1-\mu^{(1)}\right)-n^d, n\left(1-\mu^{(1)}\right)+n^d\right],\end{aligned}$$

and we define for all $i \in \{1,2,3\}$:

$$I_{3,i} = \frac{\mathbb{E}\left[N\mathbb{1}_{\{S_n+Z^{(0)}+N=n, Z^{(0)}+N\in J_3, N\in K_i\}}\right]}{\mathbb{P}(S_n=n)}.$$

Recalling (8.52)

$$\mathbf{p}^{(0)}(n) = \mathbb{P}(Z^{(0)}=n) = \frac{\ell_{Z^{(0)}}(n)\mathscr{L}(n)}{n^{1+\alpha}} \tag{8.79}$$

where $\ell_{Z^{(0)}}(.)$ is a positive function satisfying $\lim_{n\to+\infty} \ell_{Z^{(0)}}(n) = (1-\ell_{\mathbf{q}})\mathfrak{c}_{Z^{(0)}}$. If $N \leq n^d$, according to Corollary 8.4, there exists $C > 0$, such that for n large enouch and for all $k \in J_3 - N$:

$$\frac{\mathbb{P}(Z^{(0)} = k)}{\mathbb{P}(S_n = n)} \leq \frac{C}{n}.$$

As a result, for n large enough:

$$I_{3,1} \leq \frac{C}{n} \sum_{j \leq n^d} j\mathbb{P}(N = j) \sum_{k \in J_3 - j} \mathbb{P}(S_n = n - k - j) \leq \frac{C}{n}\mathbb{E}[N] \to 0.$$

To study $I_{3,2}$, we still need further refinement, in other words we decompose J_3 in three intervals:

$$\begin{aligned}
J_3^1 &= \left[n\left(1-\mu^{(1)}\right) - n^d, n\left(1-\mu^{(1)}\right) - \varepsilon n^d\right] \\
J_3^2 &= \left]n\left(1-\mu^{(1)}\right) - \varepsilon n^d, n\left(1-\mu^{(1)}\right) + \varepsilon n^d\right] \\
J_3^3 &= \left]n\left(1-\mu^{(1)}\right) + \varepsilon n^d, n\left(1-\mu^{(1)}\right) + n^d\right]
\end{aligned}$$

and we define for all $i \in \{1, 2, 3\}$:

$$I_{3,2}^i = \frac{\mathbb{E}\left[N\mathbb{1}_{\{S_n + Z^{(0)} + N = n, Z^{(0)} + N \in J_3^i, N \in K_2\}}\right]}{\mathbb{P}(S_n = n)}.$$

First, note that if $k \in J_3^1$ then $n - k \in [n\mu^{(1)} + \varepsilon n^d, n\mu^{(1)} + n^d]$, and as result for any $\delta > 0$, according to Corollary 8.4, there exists $C > 0$ such that for n large enough and all $k \in J_3^1$:

$$\frac{\mathbb{P}(S_n = n - k)}{\mathbb{P}(S_n = n)} \leq \frac{C}{n^{(d-1)(\alpha+1+\delta)}}.$$

As a result, for n large enough:

$$\begin{aligned}
I_{3,2}^1 &\leq \frac{C}{n^{(d-1)(\alpha+1+\delta)}} \sum_{k \in J_3^1} \mathbb{E}\left[N\mathbb{1}_{\{Z^{(0)}+N=k, N\in K_2\}}\right] \leq \frac{C}{n^{(d-1)(\alpha+1+\delta)}} \sum_{j \in K_2} j\mathbf{p}_N(j) \\
&\leq \frac{C}{n^{(d-1)(\alpha+1+\delta)}} \sum_{j \geq n^d} \frac{\mathscr{L}(j)}{j^\alpha} \leq \frac{C}{n^{(d-1)(\alpha+1+\delta)}} \sum_{j \geq n^d} \frac{1}{j^{\alpha-\delta}} \\
&\leq \frac{C}{n^{(d-1)(\alpha+1+\delta)}} \times \frac{1}{n^{d(\alpha-\delta-1)}} \leq \frac{C}{n^{\alpha(2d-1)-1-\delta}} \underset{n\to+\infty}{\to} 0,
\end{aligned}$$

for δ small enough, as $\alpha(2d-1) - 1 > 0$ since $\alpha > 2$ and $\max\left(\frac{2}{3}, \frac{1+\alpha}{2\alpha}\right) < d$.
In the second case, the reasoning is very similar; indeed for any $\delta > 0$, there exists $C > 0$ such that for n large enough and all $k \in \left[n^d(1-\varepsilon), n\left(1-\mu^{(1)}\right) - n^d(1-\varepsilon)\right]$:

$$\frac{\mathbf{p}^{(0)}(k)}{\mathbb{P}(S_n = n)} \leq \frac{C}{n^{d(\alpha+1)-\alpha+\delta(d-1)}}.$$

Then, as $n^d(1-\varepsilon) \leq Z^{(0)} \leq n\left(1-\mu^{(1)}\right) - n^d(1-\varepsilon)$ a.s., we obtain:

$$I_{3,2}^2 \leq \frac{C}{n^{d(\alpha+1)-\alpha+\delta(d-1)}} \sum_{j\in K_2} j\mathbf{p}_N(j) \leq \frac{C}{n^{\alpha(2d-1)-\delta}} \underset{n\to+\infty}{\to} 0.$$

In the last case, note that for a n large enough, for all k satisfying $-n^d \leq k - n\mu^{(1)} \leq -\varepsilon n^d$, according to (8.60)

$$\mathbb{P}(S_n = k) \leq C\frac{e^{-\frac{\varepsilon^2 n^{2d-1}}{2\sigma^2}}}{\sqrt{n}}.$$

Consequently, as $-n^d \leq S_n - n\mu^{(1)} \leq -\varepsilon n^d$ in $I_{3,2}^3$, we have:

$$I_{3,2}^3 \leq C\frac{e^{-\frac{\varepsilon^2 n^{2d-1}}{2\sigma^2}}}{\sqrt{n}\mathbb{P}(S_n = n)}\mathbb{E}[N] \underset{n\to+\infty}{\to} 0.$$

Finally, with $I_{3,3}$, we use the same method as Lemma 8.14 for J_3 in the case where $\ell_{\mathbf{q}} \neq 1$. We first study the limit of:

$$\tilde{I}_{3,3} := \frac{\mathbb{E}[n\left(1-\mu^{(1)}\right)\mathbb{1}_{\{S_n+Z^{(0)}+N=n, Z^{(0)}+N\in J_3, N\in K_3\}}]}{\mathbb{P}(S_n = n)}.$$

Let $\varepsilon > 0$, according to Lemma 8.13 and Corollary 8.4, for n large enough and every $k \in K_3$:

$$(1-\varepsilon)\frac{\mathfrak{c}_N}{n\aleph} \leq \frac{\mathbb{P}(N = k)}{\mathbb{P}(S_n = n)} \leq (1+\varepsilon)\frac{\mathfrak{c}_N}{n\aleph}. \tag{8.80}$$

Thus, for n large enough:

$$\begin{aligned}
\tilde{I}_{3,3} &\leq (1+\varepsilon)\frac{(1-\mu^{(1)})\mathfrak{c}_N}{n\aleph}\sum_{j=0}^{2n^d}\mathbb{P}(Z^{(0)} = j)\sum_{k\in K_3}\mathbb{P}(S_n = n-k-j) \\
&\leq (1+\varepsilon)\frac{(1-\mu^{(1)})\mathfrak{c}_N}{n\aleph}\mathbb{P}\left(Z^{(0)} \in [0, 2n^d]\right)\mathbb{P}\left(S_n - n\mu^{(1)} \in [-3n^d, n^d]\right)
\end{aligned}$$

and:

$$\begin{aligned}
\tilde{I}_{3,3} &\geq (1-\varepsilon)\frac{(1-\mu^{(1)})\mathfrak{c}_N}{n\aleph}\sum_{j=0}^{\frac{n^d}{2}}\mathbb{P}(Z^{(0)} = j)\sum_{k\in K_3}\mathbb{P}(S_n = n-k-j) \\
&\geq (1-\varepsilon)\frac{(1-\mu^{(1)})\mathfrak{c}_N}{n\aleph}\mathbb{P}\left(Z^{(0)} \in \left[0, \frac{n^d}{2}\right]\right)\mathbb{P}\left(S_n - n\mu^{(1)} \in \left[-n^d, \frac{n^d}{2}\right]\right).
\end{aligned}$$

As our upper and lower bounds are true for every $\varepsilon > 0$, using again the central limit Theorem, we obtain:

$$\lim_{n\to+\infty}\tilde{I}_{3,3} = \left(1-\mu^{(1)}\right)\frac{\mathfrak{c}_N}{\aleph}.$$

We can conclude as $\lim_{n\to+\infty} I_{3,3} = \lim_{n\to+\infty}\tilde{I}_{3,3}$, following the same reasoning for I_3 in Lemma 8.14, thanks to the upper bound in (8.80).
Finally when $\ell_{\mathbf{q}} = 0$:

$$\lim_{n\to+\infty}\frac{\mathbb{E}[N\mathbb{1}_{\{S_n+Z^{(0)}+N=n\}}]}{\mathbb{P}(S_n = n)} = \mathbb{E}[N] + \left(1-\mu^{(1)}\right)\frac{\mathfrak{c}_N}{\aleph}.$$

We can write for all $\ell_{\mathbf{q}} \in [0,1]$:

$$\lim_{n\to+\infty} \frac{\mathbb{E}[N\mathbb{1}_{\{S_n+Z^{(0)}+N=n\}}]}{\mathbb{P}(S_n=n)} = \mathbb{E}[N] + \left(1-\mu^{(1)}\right)\frac{(1-\ell_{\mathbf{q}})\mathfrak{c}_N}{\aleph}.$$

That concludes the proof of the lemma in the second case. □

Recall that $\mathfrak{c}_{Z^{(0)}}$ is explicit and given in (8.51).

Lemma 8.17. *Assume that $\mathbf{p}^{(1)}$ is aperiodic , with $\mu^{(1)} = 1$ or ($\mu^{(1)} < 1$ and $\rho(\mathbf{p}^{(1)}) = 1$) and $\mathrm{Var}[Z^{(1)}] < +\infty$. Let $m \in \mathbb{N}$, $k \in \mathbb{Z}$, we have:*

$$\lim_{n\to+\infty} \frac{\mathbb{E}[Z^{(0)}\mathbb{1}_{\{S_n+W_m+Z^{(0)}+N=n-k\}}]}{\mathbb{E}[N\mathbb{1}_{\{S_n+N=n\}}]} = \frac{\mathbb{E}[Z^{(0)}] + \left(1-\mu^{(1)}\right)(1-\ell_{\mathbf{q}})\frac{\mathfrak{c}_{Z^{(0)}}}{\aleph}}{\mathbb{E}[N] + (1-\mu^{(1)})\,(1-\ell_{\mathbf{q}})\frac{\mathfrak{c}_N}{\aleph}}. \tag{8.81}$$

If $\mathbf{p}^{(1)}$ is periodic, then (8.81) *still holds along the subsequence for which the denominator is positive.*

Remark 8.7. *1. If $\mu^{(1)} = 1$ or $\ell_{\mathbf{q}} = 1$, the ratio in* (8.81) *tends to $\frac{\mathbb{E}[Z^{(0)}]}{\mathbb{E}[N]}$.*

2. As in Remark 8.6, we can considerably simplify (8.81) *by noting that:*

$$\frac{\mathfrak{c}_{Z^{(0)}}}{\aleph} = \frac{1}{1-\mathbb{E}[\mathbf{q}(X)]},\ \mathbb{E}[Z^{(0)}] = \frac{\mathbb{E}[X(1-\mathbf{q}(X))]}{1-\mathbb{E}[\mathbf{q}(X)]}\mathbb{E}[N],$$

and we obtain:

$$\lim_{n\to+\infty} \frac{\mathbb{E}[Z^{(0)}\mathbb{1}_{\{S_n+W_m+Z^{(0)}+N=n-k\}}]}{\mathbb{E}[N\mathbb{1}_{\{S_n+N=n\}}]} = \frac{\mathbb{E}[X(1-\mathbf{q}(X))] + (1-\ell_{\mathbf{q}})\,(1-\mu(\mathbf{p}))}{1-\mathbb{E}[\mathbf{q}(X)]}. \tag{8.82}$$

Proof. This proof is very similar to those of Lemma 8.15 and Lemma 8.16, and as a result, we begin by writing the ratio on the left-hand side of (8.81) in the form:

$$\frac{\mathbb{E}[Z^{(0)}\mathbb{1}_{\{S_n+W_m+Z^{(0)}+N=n-k\}}]}{\mathbb{P}(S_n=n)}\frac{\mathbb{P}(S_n=n)}{\mathbb{E}\left[N\mathbb{1}_{\{S_n+N=n\}}\right]} =: v_n w_n.$$

Lemma 8.15 gives that w_n tends to the denominator of the right-hand side of the equation (8.81), and consequently, we just have to show that v_n tends to the numerator that we denote by $\mathscr{N}$ for typographical simplicity.

The roles of N and $Z^{(0)}$ are completely symmetric, thus following the proof of Lemma 8.16, we easily obtain that $\lim_{n\to+\infty} v_n = \mathbb{E}[Z^{(0)}]$, if $\mu^{(1)} = 1$, which completes this case, and if $\mu^{(1)} < 1$, for all $k, m \in \mathbb{Z}$, we have:

$$\begin{aligned}\lim_{n\to+\infty} v_n &= \lim_{n\to+\infty} \frac{\mathbb{E}[Z^{(0)}\mathbb{1}_{\{S_n+Z^{(0)}+N=n\}}]}{\mathbb{P}(S_n=n)} \\ &= \lim_{n\to+\infty}\sum_{j\in\mathbb{N}} \mathbf{p}_N(j)\frac{\mathbb{E}[Z^{(0)}\mathbb{1}_{\{S_n+Z^{(0)}=n-j\}}]}{\mathbb{P}(S_n=n)} = \lim_{n\to+\infty}\sum_{j\in\mathbb{N}}\mathbf{p}_N(j)d_{n,j}.\end{aligned}$$

Thanks to (8.33) and (8.61), with $T = Z^{(0)}$ and $R = 0$, we have $\lim_{n\to+\infty} d_{n,j} = \mathscr{N}$ and:

$$\lim_{n\to+\infty}\sum_{j\in\mathbb{N}}\mathbf{p}^{(1)}(j)d_{n,j} = \lim_{n\to+\infty}\frac{\mathbb{E}[Z^{(0)}\mathbb{1}_{\{S_{n+1}+Z^{(0)}=n\}}]}{\mathbb{P}(S_n=n)} = \mathscr{N} = \sum_{j\in\mathbb{N}}\mathbf{p}^{(1)}(j)\lim_{n\to+\infty} d_{n,j}.$$

And we have already proved in Lemma 8.10, that there exists $C > 0$ such that for all $j \in \mathbb{N}$, $C\mathbf{p}_N(j) \leq \mathbf{p}^{(1)}(j)$.

Using again Theorem 1.21 in [21]:

$$\lim_{n\to+\infty} \sum_{j\in\mathbb{N}} \mathbf{p}_N(j) d_{n,j} = \sum_{j\in\mathbb{N}} \mathbf{p}_N(j) \lim_{n\to+\infty} d_{n,j} = \mathscr{N}.$$

Thereby, for all $m \in \mathbb{N}$, $k \in \mathbb{Z}$:

$$\lim_{n\to+\infty} \frac{\mathbb{E}[Z^{(0)} \mathbb{1}_{\{S_n+W_m+Z^{(0)}+N=n-k\}}]}{\mathbb{E}[N\mathbb{1}_{\{S_n+N=n\}}]} = \frac{\mathbb{E}[Z^{(0)}] + \left(1-\mu^{(1)}\right)(1-\ell_{\mathbf{q}})\frac{\mathfrak{c}_{Z^{(0)}}}{\aleph}}{\mathbb{E}[N] + (1-\mu^{(1)})(1-\ell_{\mathbf{q}})\frac{\mathfrak{c}_N}{\aleph}}.$$

That concludes the proof in the case $\mu^{(1)} < 1$. □

Let a marked tree $\mathbf{t}^* \in \mathbb{T}_0^*$, $x \in \mathbf{t}$ and recall the expression (8.28), for all $i \in \mathbb{Z}$ and $n \in \mathbb{N}^*$:

$$B_{n,i}(x) = \sum_{j>i} \mathbf{p}(j) \alpha_{j,x}(j-i) a_{n,j},$$

where :

$$\alpha_{j,x} = \mathbf{q}(j)\eta_x(\mathbf{t}) + (1-\mathbf{q}(j))(1-\eta_x(\mathbf{t})) \text{ and } a_{n,j} = \frac{\mathbb{E}[N\mathbb{1}_{\{S_n+W_{j-1-i}+N=n\}}]}{\mathbb{E}[N\mathbb{1}_{\{S_n+N=n\}}]}.$$

Lemma 8.18. *Assume that $\mathbf{p}^{(1)}$ is aperiodic , with $\mu^{(1)} = 1$ or ($\mu^{(1)} < 1$ and $\rho(\mathbf{p}^{(1)}) = 1$) and $\mathrm{Var}[Z^{(1)}] < +\infty$. For all $l \leq 0$, we have*

$$\lim_{n\to+\infty} B_{n,l}(x) = \mathbb{E}[(X-l)\alpha_{X,x}] + (1-\mu(\mathbf{p}))\left(\ell_{\mathbf{q}}\eta_x(\mathbf{t}) + (1-\ell_{\mathbf{q}})(1-\eta_x(\mathbf{t}))\right).$$

Proof. Recall that for $j, n \in \mathbb{N}^*$, $j \leq n$:

$$\mathbb{P}\left(M(\tau^*) = n\right) = \frac{1}{n}\mathbb{E}[N\mathbb{1}_{\{S_n+N=n\}}],\ \mathbb{P}_j(M(\tau^*) = n) = \frac{j}{n}\mathbb{E}[N\mathbb{1}_{\{S_n+W_{j-1}+N=n\}}].$$

For $i \geq 0$:

$$\begin{aligned} B_{n,-i}(x) &= \sum_{j>-i} \mathbf{p}(j) \frac{\mathbb{E}[N\mathbb{1}_{\{S_n+W_{j-1+i}+N=n\}}]}{\mathbb{E}[N\mathbb{1}_{\{S_n+N=n\}}]}(j+i)\alpha_{j,x} \\ &= \frac{n}{\mathbb{E}[N\mathbb{1}_{\{S_n+N=n\}}]} \sum_{j>-i} \mathbf{p}(j)\alpha_{j,x}\mathbb{P}_{j+i}(M(\tau^*) = n). \end{aligned}$$

We suppose that x is marked on τ, i.e. $\eta_x = 1$. By decomposing τ under $\mathbb{P}_{i+1}$ with respect to the number of children of the root of the first tree in the forest ($\emptyset_1$). We consider that this root is associated to x, we get :

$$\mathbb{P}_{i+1}\left(M(\tau^*) = n+1, \eta_{\emptyset_1} = 1\right) = \sum_{j\in\mathbb{N}} \mathbf{p}(j)\mathbf{q}(j)\mathbb{P}_{j+i}(M(\tau^*) = n). \tag{8.83}$$

We explain this result with an example, for $j = 5$, and $n = 8$.

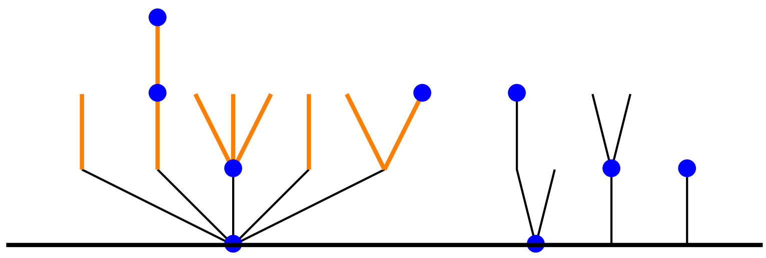


Figure 8.9 – The forest of τ by decomposing under $\mathbb{P}_{i+1}$.

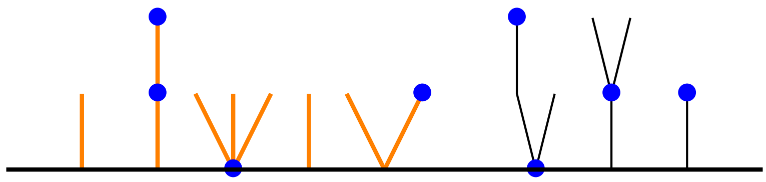


Figure 8.10 – The forest of τ by decomposing under $\mathbb{P}_{i+j}$, with the first j trees represent the trees resulting from the children of the root of the first tree on the figure 8.9 .

On the figure 8.9 , the first tree represents the tree with root x on τ. Figure 8.10 allows to explain our formula (8.83). In the sum we have the term $\mathbf{q}(j)$ because we know x is marked on τ, since there are n marks on the $j+i$ trees obtained (as the figure 8.10), we have the term $\mathbb{P}_{j+i}(M(\tau^*)=n)$.

Thereby, if $\eta_x = 1$, we have:

$$B_{n,-i}(x) = \frac{n\mathbb{P}_{i+1}(M(\tau^*) = n+1, \eta_{\emptyset_1} = 1)}{\mathbb{E}[N\mathbb{1}_{\{S_n+N=n\}}]}.$$

Moreover, using Dwass formula like in (8.25) and the fact that $\eta_{\emptyset_1}$ is independent of $(W_n)_{n\in\mathbb{N}}$ and $(S_n)_{n\in\mathbb{N}}$, we have:

$$\begin{aligned}
\mathbb{P}_{i+1}(M(\tau^*) = n+1, \eta_{\emptyset_1} = 1) &= \sum_{j\in\mathbb{N}} \mathbb{P}(W_i = j)\mathbb{P}_{j+1}\left(\left|\tau^{\mathscr{F}_q}\right| = n+1\right)\mathbb{P}(\eta_{\emptyset_1} = 1) \\
&= \mathbb{E}[\mathbf{q}(X)] \sum_{j\in\mathbb{N}} \mathbb{P}(W_i = j)\frac{j+1}{n+1}\mathbb{P}(S_{n+1} = n-j) \\
&= \frac{\mathbb{E}[\mathbf{q}(X)]}{n+1}\left(i\mathbb{E}[N\mathbb{1}_{\{S_{n+1}+W_{i-1}+N=n\}}] + \mathbb{E}[\mathbb{1}_{\{S_{n+1}+W_{i-1}+N=n\}}]\right).
\end{aligned}$$

Then, thanks to Lemma 8.9 , for $T = N$ and $R = 0$, Lemma 8.10 and Lemma 8.15 , we obtain:

$$\begin{aligned}
&\lim_{n\to+\infty} \frac{\mathbb{E}[N\mathbb{1}_{\{S_{n+1}+W_{i-1}+N=n\}}]}{\mathbb{E}[N\mathbb{1}_{\{S_n+N=n\}}]} = 1 \\
&\lim_{n\to+\infty} \frac{\mathbb{E}[\mathbb{1}_{\{S_{n+1}+W_{i-1}+N=n\}}]}{\mathbb{E}[N\mathbb{1}_{\{S_n+N=n\}}]} = \frac{\mathbb{E}[X\mathbf{q}(X)] + \ell_{\mathbf{q}}(1-\mu(\mathbf{p}))}{\mathbb{E}[\mathbf{q}(X)]}.
\end{aligned}$$

and as a result:

$$\lim_{n\to+\infty} B_{n,-i}(x) = i\mathbb{E}[\mathbf{q}(X)] + \mathbb{E}[X\mathbf{q}(X)] + \ell_{\mathbf{q}}\left(1-\mu(\mathbf{p})\right).$$

Now, we suppose that x is not marked on τ, i.e. $\eta_x = 0$. In the same way of (8.83), we obtain:

$$\mathbb{P}_{i+1}(M(\tau^*) = n, \eta_{\emptyset_1} = 0) = \sum_{j\in\mathbb{N}} \mathbf{p}(j)(1-\mathbf{q}(j))\mathbb{P}_{j+i}(M(\tau^*) = n),$$

and we have:

$$\begin{aligned}
\mathbb{P}_{i+1}(M(\tau^*) = n, \eta_{\emptyset_1} = 0) &= \sum_{j\in\mathbb{N}} \mathbb{P}(W_i + Z^{(0)} = j)\mathbb{P}_j(\left|\tau^{\mathscr{F}_q}\right| = n)\mathbb{P}(\eta_{\emptyset_1} = 0) \\
&= (1-\mathbb{E}[\mathbf{q}(X)]) \sum_{j\in\mathbb{N}} \mathbb{P}(W_i + Z^{(0)} = j)\frac{j}{n}\mathbb{P}(S_n = n-j) \\
&= \frac{1-\mathbb{E}[\mathbf{q}(X)]}{n}\mathbb{E}[(W_i + Z^{(0)})\mathbb{1}_{\{S_n+W_i+Z^{(0)}=n\}}] \\
&= \frac{1-\mathbb{E}[\mathbf{q}(X)]}{n}\left(i\mathbb{E}[N\mathbb{1}_{\{S_n+W_{i-1}+Z^{(0)}+N=n\}}] + \mathbb{E}[Z^{(0)}\mathbb{1}_{\{S_n+W_i+Z^{(0)}=n\}}]\right).
\end{aligned}$$

Thereby, thanks to Lemma 8.16 and 8.82 , we have:

$$\begin{aligned}
&\lim_{n\to+\infty} \frac{\mathbb{E}[N\mathbb{1}_{\{S_n+W_{i-1}+Z^{(0)}+N=n\}}]}{\mathbb{E}[N\mathbb{1}_{\{S_n+N=n\}}]} = 1 \\
&\lim_{n\to+\infty} \frac{\mathbb{E}[Z^{(0)}\mathbb{1}_{\{S_n+W_{i-1}+Z^{(0)}+N=n\}}]}{\mathbb{E}[N\mathbb{1}_{\{S_n+N=n\}}]} = \frac{\mathbb{E}[X(1-\mathbf{q}(X))] + (1-\ell_{\mathbf{q}})(1-\mu(\mathbf{p}))}{1-\mathbb{E}[\mathbf{q}(X)]}.
\end{aligned}$$

Implying:

$$\lim_{n\to+\infty} B_{n,-i}(x) = i(1-\mathbb{E}[\mathbf{q}(X)]) + \mathbb{E}[X(1-\mathbf{q}(X))] + (1-\ell_{\mathbf{q}})(1-\mu(\mathbf{p})).$$

When we regroup these two results, we obtain the lemma. □

If $\mathbf{p}^{(1)}$ is periodic, then Lemma 8.18 still holds along the subsequence for which the denominator is positive when we use Lemma 8.10 .

We adapt the lemmas 8.8 and 8.9 of [6].

In order to extend Lemma 8.18 for $l > 0$, we give a preliminary lemma and introduce for $l, k \in \mathbb{Z}$, such that $l \geq k$:

$$C_{n,l,x}(k) = \mathbb{E}\left[\alpha_{X,x} N(X-l)_+ \mathbb{1}_{\{S_n+W_{X-1-k}+N=n\}}\right].$$

We recall $z_+ = \max(z, 0)$. Notice that for $l \in \mathbb{Z}$:

$$C_{n,l,x}(l) = nB_{n,l}(x)\mathbb{P}(M(\tau^*) = n) = B_{n,l}(x)\mathbb{E}[N\mathbb{1}_{\{S_n+N=n\}}], \tag{8.84}$$

thanks to Dwass Formula, (8.24).

Lemma 8.19. *Assume $\mathbf{p}^{(1)}$ is aperiodic, with $\mu^{(1)} < 1$, $\rho(\mathbf{p}^{(1)}) = 1$. We have for $k, l \in \mathbb{Z}$ such that $k \leq l$:*

$$\lim_{n\to+\infty} \frac{C_{n,l,x}(k)}{C_{n,l,x}(l)} = 1 \tag{8.85}$$

If $\mathbf{p}^{(1)}$ is periodic, then (8.85) *still holds along the subsequence for which the denominator is positive.*

Proof. Notice that $\alpha_{X,x}N(X-l)_+$ is integrable, consequently, mimicking the proof of Lemma 8.9, we prove for $m, u \in \mathbb{Z}$:

$$\lim_{n\to+\infty} \frac{\mathbb{E}\left[\alpha_{X,x}N(X-l)_+\mathbb{1}_{\{S_{n+m}+W_{X-1-l}+N=n-u\}}\right]}{\mathbb{E}\left[\alpha_{X,x}N(X-l)_+\mathbb{1}_{\{S_n+W_{X-1-l}+N=n\}}\right]} = 1. \tag{8.86}$$

Since $\mathbf{p}^{(1)}$ is aperiodic, the denominator of (8.86) is positive for n large enough and it is enough to prove the result for $m = 1$ and u such that $\mathbf{p}^{(1)}(u) > 0$.

Denote $\mathbf{p}_n^{(1)}(u) := \frac{1}{n}\sum_{i=1}^{n}\mathbb{1}_{\{Z_i^{(1)}=u\}}$ and $\gamma_n := \mathbb{E}\left[\alpha_{X,x}N(X-l)_+\mathbf{p}_n^{(1)}(u)\mathbb{1}_{\{S_n+W_{X-1-l}+N=n\}}\right]$, we have:

$$\begin{aligned}\gamma_n &= \mathbb{E}\left[\alpha_{X,x}N(X-l)_+\left(\frac{1}{n}\sum_{i=1}^{n}\mathbb{1}_{\{Z_i^{(1)}=u\}}\right)\mathbb{1}_{\{S_n+W_{X-1-l}+N=n\}}\right]\\ &= \mathbf{p}^{(1)}(u)\mathbb{E}\left[\alpha_{X,x}N(X-l)_+\mathbb{1}_{\{S_{n-1}+W_{X-1-l}+N=n-u\}}\right].\end{aligned}$$

We also denote by $\xi_n := \frac{\mathbb{E}\left[\alpha_{X,x}N(X-l)_+\mathbb{1}_{\{S_{n+1}+W_{X-1-l}+N=n-u\}}\right]}{\mathbb{E}\left[\alpha_{X,x}N(X-l)_+\mathbb{1}_{\{S_n+W_{X-1-l}+N=n\}}\right]} - 1$, thus, we have:

$$\begin{aligned}\xi_n &= \left|\frac{\gamma_n}{\mathbf{p}^{(1)}(u)\mathbb{E}\left[\alpha_{X,x}N(X-l)_+\mathbb{1}_{\{S_n+W_{X-1-l}+N=n\}}\right]} - 1\right|\\ &\le \frac{\mathbb{E}\left[\alpha_{X,x}N(X-l)_+\left|\mathbf{p}_n^{(1)}(u)-\mathbf{p}^{(1)}(u)\right|\mathbb{1}_{\{S_n+W_{X-1-l}+N=n\}}\right]}{\mathbf{p}^{(1)}(u)\mathbb{E}\left[\alpha_{X,x}N(X-l)_+\mathbb{1}_{\{S_n+W_{X-1-l}+N=n\}}\right]}\\ &\le \frac{\varepsilon}{\mathbf{p}^{(1)}(u)} + \frac{\mathbb{E}\left[\alpha_{X,x}N(X-l)_+\mathbb{1}_{\{\left|\mathbf{p}_n^{(1)}(u)-\mathbf{p}^{(1)}(u)\right|>\varepsilon\}}\mathbb{1}_{\{S_n+W_{X-1-l}+N=n\}}\right]}{\mathbf{p}^{(1)}(u)\mathbb{E}\left[\alpha_{X,x}N(X-l)_+\mathbb{1}_{\{S_n+W_{X-1-l}+N=n\}}\right]},\end{aligned}$$

for every $\varepsilon > 0$.

It remains to prove that:

$$J_n := \frac{\mathbb{E}\left[\alpha_{X,x}N(X-l)_+\mathbb{1}_{\{\left|\mathbf{p}_n^{(1)}(u)-\mathbf{p}^{(1)}(u)\right|>\varepsilon\}}\mathbb{1}_{\{S_n+W_{X-1-l}+N=n\}}\right]}{\mathbb{E}\left[\alpha_{X,x}N(X-l)_+\mathbb{1}_{\{S_n+W_{X-1-l}+N=n\}}\right]}$$

converges to 0 for all $\varepsilon > 0$. We have:

$$\begin{aligned}J_n &\le \frac{\mathbb{E}\left[\alpha_{X,x}N(X-l)_+\mathbb{1}_{\{\left|\mathbf{p}_n^{(1)}(u)-\mathbf{p}^{(1)}(u)\right|>\varepsilon\}}\right]}{\mathbb{E}\left[\alpha_{X,x}N(X-l)_+\mathbb{1}_{\{S_n+W_{X-1-l}+N=n\}}\right]}\\ &\le \frac{\mathbb{E}\left[\alpha_{X,x}N(X-l)_+\right]}{\mathbb{E}\left[\alpha_{X,x}N(X-l)_+\mathbb{1}_{\{S_n+W_{X-1-l}+N=n\}}\right]}\mathbb{P}\left(\left|\mathbf{p}_n^{(1)}(u)-\mathbf{p}^{(1)}(u)\right|>\varepsilon\right)\\ &\le \frac{\mathbb{E}\left[\alpha_{X,x}N(X-l)_+\right]\mathbb{P}(S_n=n)}{\mathbb{E}\left[\alpha_{X,x}N(X-l)_+\mathbb{1}_{\{S_n+W_{X-1-l}+N=n\}}\right]}\frac{\mathbb{P}\left(\left|\mathbf{p}_n^{(1)}(u)-\mathbf{p}^{(1)}(u)\right|>\varepsilon\right)}{\mathbb{P}(S_n=n)}.\end{aligned}$$

According to [28], end of page 2954, we have $\lim_{n\to+\infty} \frac{\mathbb{P}\left(\left|\mathbf{p}_n^{(1)}(u)-\mathbf{p}^{(1)}(u)\right|>\varepsilon\right)}{\mathbb{P}(S_n=n)}=0$. We denote $I_n := \frac{\mathbb{E}\left[\alpha_{X,x}N(X-l)_+\right]\mathbb{P}(S_n=n)}{\mathbb{E}\left[\alpha_{X,x}N(X-l)_+\mathbb{1}_{\{S_n+W_{X-1-l}+N=n\}}\right]}$, for all $n\in\mathbb{N}^*$, by Fatou and using the strong ratio limit property (8.32), we have:

$$\begin{aligned}
\limsup_{n\to+\infty} I_n &= \limsup_{n\to+\infty} \frac{\mathbb{E}\left[\alpha_{X,x}N(X-l)_+\right]\mathbb{P}(S_n=n)}{\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}\mathbf{p}(j)\mathbf{p}_N(i)\alpha_{j,x}i(j-l)_+\mathbb{P}\left(S_n=n-i-W_{j-1-l}\right)}\\
&= \frac{\mathbb{E}\left[\alpha_{X,x}N(X-l)_+\right]}{\liminf_{n\to+\infty}\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}\mathbf{p}(j)\mathbf{p}_N(i)\alpha_{j,x}i(j-l)_+\frac{\mathbb{P}\left(S_n=n-i-W_{j-1-l}\right)}{\mathbb{P}(S_n=n)}}\\
&\le \frac{\mathbb{E}\left[\alpha_{X,x}N(X-l)_+\right]}{\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}\liminf_{n\to+\infty}\mathbf{p}(j)\mathbf{p}_N(i)\alpha_{j,x}i(j-l)_+\frac{\mathbb{P}\left(S_n=n-i-W_{j-1-l}\right)}{\mathbb{P}(S_n=n)}}\\
&\le \frac{\mathbb{E}\left[\alpha_{X,x}N(X-l)_+\right]}{\mathbb{E}\left[\alpha_{X,x}N(X-l)_+\right]}=1,
\end{aligned}$$

with,

$$\begin{aligned}
\liminf_{n\to+\infty}\frac{\mathbb{P}\left(S_n=n-i-W_{j-1-l}\right)}{\mathbb{P}(S_n=n)} &= \liminf_{n\to+\infty}\frac{\mathbb{E}\left[\mathbb{1}_{\{S_n=n-W_{j-1-l}-i\}}\right]}{\mathbb{P}(S_n=n)}\\
&= \liminf_{n\to+\infty}\sum_{h=0}^{+\infty}\mathbb{P}(W_{j-1-l}=h)\frac{\mathbb{P}\left(S_n=n-i-h\right)}{\mathbb{P}(S_n=n)}\\
&\ge \sum_{h=0}^{+\infty}\mathbb{P}(W_{j-1-l}=h)\liminf_{n\to+\infty}\frac{\mathbb{P}\left(S_n=n-i-h\right)}{\mathbb{P}(S_n=n)}\ge 1,
\end{aligned}$$

for all $i,j\in\mathbb{N}$.
Since $\varepsilon>0$ is arbitrary, we deduce that $\lim_{n\to+\infty} J_n=0$.

We proved the limit (8.86), now we prove a second limit, to proof (8.85). For $m,u\in\mathbb{Z}$:

$$\lim_{n\to+\infty}\frac{\mathbb{E}\left[\alpha_{X,x}N(X-l)_+\mathbb{1}_{\{S_n+W_{X-1-l+m}+N=n-u\}}\right]}{\mathbb{E}\left[\alpha_{X,x}N(X-l)_+\mathbb{1}_{\{S_n+W_{X-1-l}+N=n\}}\right]}=1. \tag{8.87}$$

The proof of (8.87) is the same as that of the second part of the proof of Lemma 8.15 . In this proof we use

$$c_{n,j}=\frac{\mathbb{E}\left[\alpha_{X,x}N(X-l)_+\mathbb{1}_{\{S_n+W_{X-1-l}+N=n-u-j\}}\right]}{\mathbb{E}\left[\alpha_{X,x}N(X-l)_+\mathbb{1}_{\{S_n+W_{X-1-l}+N=n\}}\right]},$$

in (8.71) and the demonstration happens in the same way. That concludes the proof of the second limit.

Finally, with $m=l-k$ and $u=0$ in (8.87), we get the result. □

Lemma 8.20. *Assume that* $\mathbf{p}^{(1)}$ *is aperiodic , with* $\mu^{(1)} < 1$, $\rho(\mathbf{p}^{(1)}) = 1$ *and* $\mathrm{Var}[Z^{(1)}] < +\infty$*. For* $l > 0$*, we have*

$$\lim_{n\mapsto+\infty} B_{n,l} = \mathbb{E}\left[\alpha_{X,x}(X-l)_+\right] + (1-\mu(\mathbf{p}))\left(\ell_{\mathbf{q}}\eta_x(\mathbf{t}) + (1-\ell_{\mathbf{q}})\left(1-\eta_x(\mathbf{t})\right)\right).$$

Proof. Let $l \geq -1$, We have:

$$\begin{aligned}
C_{n,l,x}(-1) &= \mathbb{E}\left[\alpha_{X,x}(X-l)_+ N\mathbb{1}_{\{S_n+W_X+N=n\}}\right] = \sum_{j\geq l}^{+\infty}\mathbf{p}(j)\alpha_{j,x}(j-l)\mathbb{E}\left[N\mathbb{1}_{\{S_n+W_j+N=n\}}\right] \\
&= C_{n,0,x}(-1) - \sum_{j=0}^{l-1}\mathbf{p}(j)\alpha_{j,x}(j-l)\mathbb{E}\left[N\mathbb{1}_{\{S_n+W_j+N=n\}}\right] - l\mathbb{E}\left[\alpha_{X,x}N\mathbb{1}_{\{S_n+W_X+N=n\}}\right],
\end{aligned} \tag{8.88}$$

with the convention that $\sum_{\emptyset} = 0$. According to Lemma 8.18 , we have

$$\begin{aligned}
&\lim_{n\to+\infty} B_{n,0}(x) = \mathbb{E}[X\alpha_{X,x}] + (1-\mu(\mathbf{p}))\left(\ell_{\mathbf{q}}\eta_x(\mathbf{t}) + (1-\ell_{\mathbf{q}})\left(1-\eta_x(\mathbf{t})\right)\right) =: \xi_{X,x} \\
&\lim_{n\to+\infty} B_{n,-1}(x) = \xi_{X,x} + \mathbb{E}[\alpha_{X,x}]
\end{aligned}$$

Using (8.84), we have:

$$C_{n,-1,x}(-1) = B_{n,-1}(x)\mathbb{E}\left[N\mathbb{1}_{\{S_n+N=n\}}\right] \underset{n\to+\infty}{\sim} (\xi_{X,x} + \mathbb{E}[\alpha_{X,x}])\mathbb{E}\left[N\mathbb{1}_{\{S_n+N=n\}}\right] \tag{8.89}$$

$$C_{n,0,x}(0) \underset{n\to+\infty}{\sim} \xi_{X,x}\mathbb{E}\left[N\mathbb{1}_{\{S_n+N=n\}}\right]. \tag{8.90}$$

Thereby, using (8.90) and Lemma 8.19:

$$\lim_{n\to+\infty}\frac{C_{n,0,x}(-1)}{\mathbb{E}\left[N\mathbb{1}_{\{S_n+N=n\}}\right]} = \lim_{n\to+\infty}\xi_{X,x}\frac{C_{n,0,x}(-1)}{C_{n,0,x}(0)} = \xi_{X,x}. \tag{8.91}$$

Using (8.88) for $l = 0$, $C_{n,-1,x}(-1) = C_{n,0,x}(-1) + \mathbb{E}\left[\alpha_{X,x}N\mathbb{1}_{\{S_n+W_X+N=n\}}\right]$, thereby, with (8.89) and (8.91), we deduce that:

$$\begin{aligned}
\lim_{n\to+\infty}\frac{\mathbb{E}\left[\alpha_{X,x}N\mathbb{1}_{\{S_n+W_X+N=n\}}\right]}{\mathbb{E}\left[N\mathbb{1}_{\{S_n+N=n\}}\right]} &= \lim_{n\to+\infty}\frac{C_{n,-1,x}(-1) - C_{n,0,x}(-1)}{\mathbb{E}\left[N\mathbb{1}_{\{S_n+N=n\}}\right]} \\
&= (\xi_{X,x} + \mathbb{E}[\alpha_{X,x}]) - \xi_{X,x} = \mathbb{E}[\alpha_{X,x}].
\end{aligned} \tag{8.92}$$

Let $l \geq 1$, according to Lemma 8.10 and the equations (8.88), (8.91) and(8.92) we obtain:

$$\begin{aligned}
\frac{C_{n,l,x}(-1)}{\mathbb{E}\left[N\mathbb{1}_{\{S_n+N=n\}}\right]} &= \frac{C_{n,0,x}(-1)}{\mathbb{E}\left[N\mathbb{1}_{\{S_n+N=n\}}\right]} - \sum_{j=0}^{l-1}\frac{p(j)\alpha_{j,x}(j-l)\mathbb{E}\left[N\mathbb{1}_{\{S_n+W_j+N=n\}}\right]}{\mathbb{E}\left[N\mathbb{1}_{\{S_n+N=n\}}\right]} \\
&\quad - l\frac{\mathbb{E}\left[\alpha_{X,x}N\mathbb{1}_{\{S_n+W_X+N=n\}}\right]}{\mathbb{E}\left[N\mathbb{1}_{\{S_n+N=n\}}\right]} \\
&\underset{n\to+\infty}{\longrightarrow} \xi_{X,x} - \sum_{j=0}^{l-1}\mathbf{p}(j)\alpha_{j,x}(j-l) - l\mathbb{E}[\alpha_{X,x}].
\end{aligned} \tag{8.93}$$

Now, using successively (8.84), Lemma 8.19 and (8.93), we obtain:

$$\begin{aligned}\lim_{n\to+\infty} B_{n,l}(x) &= \lim_{n\to+\infty} \frac{C_{n,l,x}(-1)}{\mathbb{E}\left[N\mathbb{1}_{\{S_n+N=n\}}\right]}\frac{C_{n,l,x}(l)}{C_{n,l,x}(-1)} \\ &= \mathbb{E}[(X-l)\alpha_{X,x}] + (1-\mu(\mathbf{p}))\left(\ell_{\mathbf{q}}\eta_x(\mathbf{t}) + (1-\ell_{\mathbf{q}})\left(1-\eta_x(\mathbf{t})\right)\right) \\ &- \sum_{j=0}^{l-1}\mathbf{p}(j)\alpha_{j,x}(j-l) \\ &= \mathbb{E}\left[\alpha_{X,x}(X-l)_+\right] + (1-\mu(\mathbf{p}))\left(\ell_{\mathbf{q}}\eta_x(\mathbf{t}) + (1-\ell_{\mathbf{q}})\left(1-\eta_x(\mathbf{t})\right)\right).\end{aligned}$$

□

If $\mathbf{p}^{(1)}$ is periodic, then Lemma 8.20 still holds along the subsequence for which the denominators are positive when we use Lemma 8.10 and the expression (8.85).

8.7 Conjecture

We recall that if $\rho_l(\mathbf{p},\mathbf{q}) = 1$ then we have $\rho(\mathbf{p}) = 1$, since, $\rho(\mathbf{p}) \le \rho_l(\mathbf{p},\mathbf{q})$ and $g(1) = 1$. Thereby, in the general case we just need to suppose the first one condition.
Unfortunately, we do not find a result in the case where $\mathbf{p}$ is sub-critical and non-generic. However, in this one, we conjecture the following results:

Conjecture 8.1. *Let τ^* be a sub-critical non-generic MGW with offspring distribution $\mathbf{p}$ satisfying* (6.1)*, $\rho_l(\mathbf{p},\mathbf{q}) = 1$, and mark function $\mathbf{q}$ satisfying* (6.2)*. We have that:*

$$\operatorname{dist}(\tau^*|M(\tau^*)=n) \underset{n\to+\infty}{\longrightarrow} \operatorname{dist}(\tau_C^*(\mathbf{p},\mathbf{q})). \tag{8.94}$$

To obtain this result we need to have the equivalents for the randomly stopped sum like in [10] but in a general case. Thanks to this result we obtain:

Corollary 8.5. *Assume that $\mathbf{p}$ satisfies* (6.1)*, $\mu(\mathbf{p}) < 1$, is non-generic, $1 \le \rho(\mathbf{p}) = \rho_l(\mathbf{p},\mathbf{q}) = \theta_s$ satisfies* (8.15)*, $\mathbf{p}_{\rho(\mathbf{p})}$ has a moment of order 2, and $\mathbf{q}$ satisfies* (6.2)*. We have:*

$$\lim_{n\to+\infty} \operatorname{dist}(\tau^*|M(\tau^*)=n) = \operatorname{dist}(\tau_C^*(\mathbf{p}_{\rho(\mathbf{p})},\mathbf{q}_{\rho(\mathbf{p})})),$$

$$\lim_{n\to+\infty} \operatorname{dist}(\tau|M(\tau)=n) = \operatorname{dist}(\tau_C(\mathbf{p}_{\rho(\mathbf{p})})).$$

Remark 8.8. *Recall that:*

$$l(\theta) = \sum_{k\ge0}\theta^k\mathbf{p}(k)(1-\mathbf{q}(k)),$$

and if $(\mathbf{q}(k))_{k\ge0}$ admits a finite limit $\ell_{\mathbf{q}}$, we have to distinguish two cases:

- *$\ell_{\mathbf{q}} \ne 1$ and clearly $\rho(\mathbf{p}) = \rho_l(\mathbf{p},\mathbf{q})$;*
- *otherwise, as* (6.8)*, there exists $\beta \ge 2$ such that:*

$$\mathbf{p}(k)(1-\mathbf{q}(k)) = \frac{\mathbf{p}(k)\mathcal{L}(k)}{k^\beta},$$

implying again that $\rho(\mathbf{p}) = \rho_l(\mathbf{p},\mathbf{q})$.

Proof. We consider the case $1 < \rho(\mathbf{p}) < +\infty$.

Thanks to Proposition 8.3, $\mathrm{dist}(\tau^*|M(\tau^*) = n) = \mathrm{dist}(\tau^*_{\theta_s}|M(\tau^*_{\theta_s}) = n)$, then, in order to apply Conjecture 8.1 to $(\mathbf{p}_{\theta_s}, \mathbf{q}_{\theta_s})$ we have to check if we have $\rho_l(\mathbf{p}_{\theta_s}, \mathbf{q}_{\theta_s}) = 1$ and $\mathbb{E}[X^2\rho(\mathbf{p})^X]$ is finite.

For $x \geq 0$, we have:

$$\begin{aligned} l_{\mathbf{p}_{\theta_s},\mathbf{q}_{\theta_s}}(x) &= \sum_{j\geq 0} x^j \mathbf{p}_{\theta_s}(j)(1-\mathbf{q}_\theta(j)) = \sum_{j\geq 0} x^j {\theta_s}^{j-1}\mathbf{p}(j)(1-\mathbf{q}(j)) \\ &= \frac{1}{\theta_s}\mathbb{E}\left[(x\theta_s)^X(1-\mathbf{q}(X))\right]. \end{aligned}$$

Consequently, $\rho_l(\mathbf{p}_{\theta_s}, \mathbf{q}_{\theta_s}) = \dfrac{\rho_l(\mathbf{p},\mathbf{q})}{\theta_s} = 1.$ $\square$

Chapter 9

Penalization of Galton–Watson Trees with Marked Vertices

Another way of getting a tree with an "abnormal" number of marked vertices is to make a change of measure via a Girsanov transformation with a martingale which gives more weight to trees with a large number of marks. In order to find such a martingale, a general method called penalization has been introduced by Roynette, Vallois and Yor [31, 32, 33] in the case of the one-dimensional Brownian motion to generate new martingales and to define, by change of measure, Brownian-like process conditioned on some specific zero-probability events.

We generalize the conditioning on having a large number of marked vertices, in the critical and sub-critical cases. We also decided to look the Galton-Watson process's behavior when we penalize by the total number of mark in the super-critical case. In 2020, Abraham and Debs [5], obtained some results about penalization without marks, we completed this research with marks in Section 9.2 .

A part of this chapter was published in the article "Penalization of Galton–Watson Trees with Marked Vertices" [3], in August 2024, in "Journal of Theoretical Probability".

Moreover, in all this chapter we do not have condensation phenomenon, thereby we do not need to consider $|.|_\infty$, $r_{n,\infty}(.)$ and $r^*_{n,\infty}(.)$. We recall that saying a probability measure $\mathbb{Q}$ on $(\Omega, \mathscr{F})$, where $\mathscr{F} = \bigvee_{n\geq 0} \mathscr{F}_n$ is the filtration generated by the restriction functions $r^*_n(.)$, is the distribution of $\tilde{\tau}^*$ defined on $(\Omega, \mathscr{A}, P)$, is equivalent to the following: as $\tau^* \in \mathbb{T}^*$, for every marked tree $\mathbf{t}^* \in \mathbb{T}^*$ and every $n \in \mathbb{N}$,

$$\mathbb{Q}\left(r^*_n(\tau^*) = r^*_n(\mathbf{t}^*)\right) = P\left(r^*_n(\tilde{\tau}^*) = r^*_n(t^*)\right). \tag{9.1}$$

9.1 Polynomial penalization

9.1.1 Masses

Let $\mathbf{t}^* = (\mathbf{t}, (\eta_u)_{u\in\mathbf{t}}) \in \mathbb{T}^*$ be a marked tree. We then define for every node $u \in \mathbf{t}$ a new quantity called the mass of the node u and denoted by m_u. We set $m_\emptyset = 0$ and the masses are then defined recursively as follows.

Let $n \in \mathbb{N}$ and let us suppose that the masses $(m_u, |u| \leq n)$ are constructed. Let $v \in z_{n+1}(\mathbf{t}^*)$ that we write $v = uj$ with $u \in \mathbf{t}$ and $1 \leq j \leq k_u(\mathbf{t})$. Then, we set

$$m_{uj} = \frac{m_u + \eta_u}{k_u(\mathbf{t})} + \frac{\sum\limits_{w \in z_n(\mathbf{t}^*), k_w(\mathbf{t})=0} (m_w + \eta_w(\mathbf{t}))}{Z_{n+1}(\mathbf{t})}. \tag{9.2}$$

This formula can be interpreted as follows:

— The first term means that u transmits its mass, plus one if it is marked, uniformly to all its children;

— The second term means that each childless node at generation n transmits its mass, plus one if it is marked, uniformly to all nodes of generation $n+1$.

We give an example of a tree with this construction.

Example 9.1. *We consider a tree with 4 generations.*

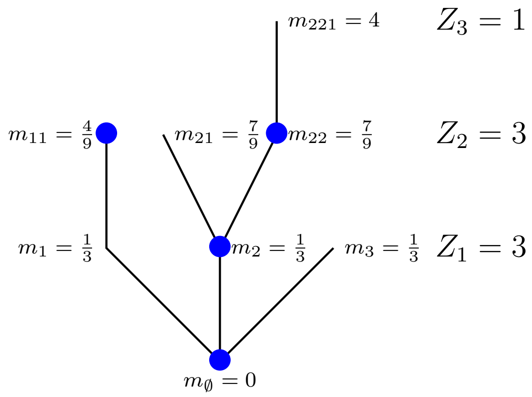


Figure 9.1 – Example of a marked tree and its associated masses.

For instance to obtain m_{21}:

$$m_{21} = \frac{m_2 + \eta_2}{2} + \frac{m_3 + \eta_3}{Z_2} = \frac{\frac{1}{3} + 1}{2} + \frac{\frac{1}{3}}{3} = \frac{4}{6} + \frac{1}{9} = \frac{7}{9}.$$

Notice that the map $\mathbf{t}^* \mapsto (m_u)_{u \in z_n(\mathbf{t}^*)}$ is $\mathscr{F}_n$-measurable.
Moreover, by construction, we have the following link between the masses and the number of marked nodes up to some level:

Proposition 9.1. *Let $n \in \mathbb{N}$ and $\mathbf{t}^* \in \mathbb{T}_0^*$. Conditionally on $\mathscr{F}_n$, we have*

$$\sum_{u \in z_n(\mathbf{t}^*)} m_u = M_n(\mathbf{t}^*). \tag{9.3}$$

Proof. (9.3) is satisfied for $n = 0$ as by convention $m_\emptyset = 0$ and $M_0(\mathbf{t}^*) = 0$.
We suppose that there exists $n \in \mathbb{N}$ such that (9.3) is true. Using (9.2), we obtain:

$$\begin{aligned}
\sum_{v \in z_{n+1}(\mathbf{t}^*)} m_v &= \sum_{u \in z_n(\mathbf{t}^*), k_u(\mathbf{t}) \neq 0} k_u(\mathbf{t}) \left(\frac{m_u + \eta_u(\mathbf{t})}{k_u(\mathbf{t})} + \frac{\sum_{j \in z_n(\mathbf{t}^*), k_j(\mathbf{t}) = 0} (m_j + \eta_j(\mathbf{t}))}{Z_{n+1}(\mathbf{t})} \right) \\
&= \sum_{u \in z_n(\mathbf{t}^*), k_u(\mathbf{t}) \neq 0} \left(m_u + \eta_u(\mathbf{t}) + \frac{k_u(\mathbf{t})}{Z_{n+1}(\mathbf{t})} \sum_{j \in z_n(\mathbf{t}^*), k_j(\mathbf{t}) = 0} (m_j + \eta_j(\mathbf{t})) \right) \\
&= \sum_{u \in z_n(\mathbf{t}^*), k_u(\mathbf{t}) \neq 0} m_u + \eta_u(\mathbf{t}) + \sum_{j \in z_n(\mathbf{t}^*), k_j(\mathbf{t}) = 0} m_j + \eta_j(\mathbf{t}) \\
&= \sum_{u \in z_n(\mathbf{t}^*)} m_u + \sum_{j \in z_n(\mathbf{t}^*)} \eta_j(\mathbf{t}) = M_n(\mathbf{t}^*) + \sum_{j \in z_n(\mathbf{t}^*)} \eta_j(\mathbf{t}) = M_{n+1}(\mathbf{t}^*).
\end{aligned}$$

The property is satisfied for $n + 1$. $\square$

9.1.2 Penalization by M_p

The aim of this Sub-section is to determine the limit when p goes to infinity of $\frac{\mathbb{E}[\mathbb{1}_{\Lambda_n} M_{n+p}]}{\mathbb{E}[M_{n+p}]}$, with $n \in \mathbb{N}$ and $\Lambda_n \in \mathscr{F}_n$ and to determine Theorem 6.2 in the case $k = 1$. That permits to introduce and understand the tools and the objects that we use in Sub-section 9.1.3.
We have for all $p > n \in \mathbb{N}$, according to (7.10):

$$\mathbb{E}\left[M_{n+p} | \mathscr{F}_n\right] = \mathbb{E}\left[M_n + \sum_{i=1}^{Z_n} \tilde{M}_{i,p} \middle| \mathscr{F}_n \right] = M_n + Z_n \mathbb{E}[M_p]. \tag{9.4}$$

Moreover we have the following expression for $\mathbb{E}[M_p]$.

Proposition 9.2. *For all $p \in \mathbb{N}$*

$$\mathbb{E}[M_p] = \begin{cases} \mathbb{E}[M_1] \dfrac{1 - \mu^p}{1 - \mu}, & \text{if } \mu \neq 1, \\ p\mathbb{E}[M_1], & \text{otherwise.} \end{cases}, \tag{9.5}$$

with $\mathbb{E}[M_1] = \mathbb{E}[\mathbf{q}(X)]$.

Proof. M_1 represents the number of marks at generation 0, thereby M_1 equals 1 if the root is marked otherwise M_1 equals 0. We recall $Y_\emptyset$ is a Bernoulli random variable of parameter $\mathbf{q}(k_\emptyset)$ and equals 1 if $\emptyset$ has a mark. $k_\emptyset$ represents the number of children of $\emptyset$. We have:

$$\mathbb{E}[M_1] = \mathbb{P}(Y_\emptyset = 1) = \sum_{j \geq 0} \mathbf{p}(j)\mathbf{q}(j) = \mathbb{E}[\mathbf{q}(X)].$$

Let $p \in \mathbb{N}^*$,

$$\mathbb{E}[M_{p+1}] = \mathbb{E}[M_p] + \mu^p \mathbb{E}[M_1].$$

Thus we obtain,

$$\mathbb{E}[M_{p+1}] = \mathbb{E}[M_1]\sum_{i=0}^{p}\mu^i = \begin{cases} \mathbb{E}[M_1]\dfrac{1-\mu^{p+1}}{1-\mu}, & \text{if } \mu \neq 1, \\ (p+1)\mathbb{E}[M_1], & \text{otherwise.} \end{cases}.$$

□

9.1.2.1 The critical and super-critical cases: $\mu \geq 1$

Theorem 9.1. *Let $n \in \mathbb{N}$, $\Lambda_n \in \mathscr{F}_n$ and $\mathbf{p}$ an offspring distribution satisfing* (6.1) *such that $\mu \geq 1$ and with its mark function $\mathbf{q}$ satisfying* (6.2)*, we have*

$$\frac{\mathbb{E}\left[\mathbb{1}_{\Lambda_n} M_{n+p}\right]}{\mathbb{E}\left[M_{n+p}\right]} \underset{p\to+\infty}{\longrightarrow} \mathbb{E}\left[\mathbb{1}_{\Lambda_n}\frac{Z_n}{\mu^n}\right]. \tag{9.6}$$

Proof. Let $p > n \in \mathbb{N}$. We have:

$$\frac{\mathbb{E}\left[\mathbb{1}_{\Lambda_n} M_{n+p}\right]}{\mathbb{E}\left[M_{n+p}\right]} = \frac{\mathbb{E}\left[\mathbb{E}[\mathbb{1}_{\Lambda_n} M_{n+p}|\,\mathscr{F}_n]\right]}{\mathbb{E}\left[M_{n+p}\right]} = \frac{\mathbb{E}\left[\mathbb{1}_{\Lambda_n}\mathbb{E}[M_{n+p}|\mathscr{F}_n]\right]}{\mathbb{E}\left[M_{n+p}\right]}.$$

We suppose $\mu > 1$, the other case is similar. According to (9.4) and (9.5), we have

$$\frac{\mathbb{E}[M_{n+p}|\mathscr{F}_n]}{\mathbb{E}\left[M_{n+p}\right]} = \left(\frac{M_n}{\mathbb{E}[\mathbf{q}(X)]}\frac{1-\mu}{1-\mu^{n+p}} + Z_n\frac{1-\mu^p}{1-\mu^{n+p}}\right) \tag{9.7}$$

The limit of this term when p goes to infinity is $\dfrac{Z_n}{\mu^n}$. Moreover we have

$$\frac{M_n}{\mathbb{E}[\mathbf{q}(X)]}\frac{1-\mu}{1-\mu^{n+p}} + Z_n\frac{1-\mu^p}{1-\mu^{n+p}} \leq \frac{M_n}{\mathbb{E}[M_1]} + \frac{Z_n}{\mu^n}.$$

M_n and Z_n being in L^1, the dominated convergence theorem gives the claimed result. □

Remark 9.1. *In these cases the martingale is a known one and the associated random marked tree is a size-biased tree (i.e. Kesten's tree, see [24]) associated with the offspring distribution $\mathbf{p}$ and with the mark function $\mathbf{q}$ being unchanged.*

9.1.2.2 The sub-critical case: $\mu < 1$

Theorem 9.2. *Let $n \in \mathbb{N}$, $\Lambda_n \in \mathscr{F}_n$ and $\mathbf{p}$ an offspring distribution satisfying* (6.1) *such that $\mu < 1$ and with its mark function $\mathbf{q}$ satisfying* (6.2)*, we have*

$$\frac{\mathbb{E}\left[\mathbb{1}_{\Lambda_n} M_{n+p}\right]}{\mathbb{E}\left[M_{n+p}\right]} \underset{p\to+\infty}{\longrightarrow} \mathbb{E}[\mathbb{1}_{\Lambda_n} B_{n,1}] \tag{9.8}$$

with $B_{n,1} = \dfrac{1}{\xi_1}\left(M_n + \xi_1 Z_n\right)$ is a positive $\mathscr{F}_n$-martingale such that $B_{0,1} = 1$ and $\xi_1 := \dfrac{\mathbb{E}[\mathbf{q}(X)]}{1-\mu}$.

Proof. Let $p > n \in \mathbb{N}$. We consider the expression (9.7). Since $\mu < 1$, the limit when p goes to infinity is $\dfrac{1-\mu}{\mathbb{E}[\mathbf{q}(X)]} M_n + Z_n$. Moreover, we have

$$\frac{M_n}{\mathbb{E}[\mathbf{q}(X)]}\,(1-\mu) + Z_n\,(1-\mu^p) \leq \frac{1-\mu}{\mathbb{E}[\mathbf{q}(X)]} M_n + Z_n.$$

M_n and Z_n being L^1, the dominated convergence theorem gives (9.8).

As M_n and Z_n are $\mathscr{F}_n$-measurable and integrable, it is also the case for $B_{n,1}$. Thanks to 6.10, we already know that $B_{n,1}$ is a positive $\mathscr{F}_n$-martingale. The mean of $B_{n,1}$ equals 1, as $B_{0,1} = \dfrac{1}{\xi_1}\,(M_0 + \xi_1 Z_0) = 1$. □

We can describe a new probability denoted by $\mathbb{Q}_1$, such that for all $\Lambda_n \in \mathscr{F}_n$, we have

$$\mathbb{Q}_1(\Lambda_n) = \mathbb{E}[\mathbb{1}_{\Lambda_n} B_{n,1}] = \mathbb{E}\left[\mathbb{1}_{\Lambda_n} \frac{1}{\xi_1}\,(M_n + \xi_1 Z_n)\right].$$

Let τ^* be a MGW with offspring distribution $\mathbf{p}$ and mark function $\mathbf{q}$, and $\mathbf{t}^* \in \mathbb{T}_0^*$. To describe this probability measure $\mathbb{Q}_1$, let us define a $\mathbb{T}^*$-valued random tree $\tau_1(\mathbf{p},\mathbf{q})$ defined on some probability space $(\Omega, \mathscr{A}, P)$.

τ_1 is called the *weighted* tree of type 1. On this tree, each individual $u \in \tau_1$ has a mass denoted by m_u, defined in Sub-section 9.1.1. At each generation we have a special node and only one, the other nodes are normal. Thereby the root is special.

From now on, we present the new offspring distribution associated to this random tree. On each generation, we choose u as the special node with the probability

$$\frac{\dfrac{\mathbb{E}\left[\mathbf{q}(X)\right]}{1-\mu} + m_u}{M_n + \dfrac{\mathbb{E}\left[\mathbf{q}(X)\right]}{1-\mu} Z_n}.$$

Let u an individual of special type, with mass m_u, he has $k \in \mathbb{N}$ children and has mark $\eta \in \{0,1\}$ with probability:

$$\mathbf{p}_1(k,\eta) := \frac{\mathbf{p}_0(k,\eta)\left(\dfrac{\mathbb{E}\left[\mathbf{q}(X)\right]}{1-\mu} k + (m_u + \eta)\right)}{\dfrac{\mathbb{E}\left[\mathbf{q}(X)\right]}{1-\mu} + m_u}$$

Now we can state the Theorem 6.3 for the case $k = 1$ thanks to the characterization (9.1):

Theorem 9.3. *Let τ^* be a MGW with offspring distribution $\mathbf{p}$ satisfying* (6.1) *such that $\mu < 1$, its mark function $\mathbf{q}$ satisfying* (6.2)*, and $\mathbf{t}^* \in \mathbb{T}_0^*$. For all $n \in \mathbb{N}$ we have*

$$\mathbb{Q}_1(r_n^*(\tau^*) = r_n^*(\mathbf{t}^*)) = P(r_n^*(\tau_1) = r_n^*(\mathbf{t}^*)). \tag{9.9}$$

Proof. Let $\mathbf{t}^* \in \mathbb{T}_0^*$, according to the definition of $\mathbb{Q}_1$, for $n \in \mathbb{N}$ we have:

$$\begin{aligned}\mathbb{Q}_1(r_n^*(\tau^*) = r_n^*(\mathbf{t}^*)) &= \mathbb{E}\left[\mathbb{1}_{\{r_n^*(\tau^*)=r_n^*(\mathbf{t}^*)\}}\frac{1}{\xi_1}\left(M_n(\mathbf{t}) + \frac{\mathbb{E}[\mathbf{q}(X)]}{1-\mu}Z_n(\mathbf{t})\right)\right] \\ &= \frac{1}{\xi_1}\left(M_n(\mathbf{t}) + \frac{\mathbb{E}[\mathbf{q}(X)]}{1-\mu}Z_n(\mathbf{t})\right)\mathbb{P}(r_n^*(\tau^*) = r_n^*(\mathbf{t}^*)).\end{aligned}$$

We prove (9.9) with an induction reasoning.
In the case $n = 0$, we have $\mathbb{P}(r_0^*(\tau^*) = r_0^*(\mathbf{t}^*)) = 1 = P(r_0^*(\tau_1) = r_0^*(\mathbf{t}^*))$, $Z_0(\mathbf{t}) = 1$ and $M_0(\mathbf{t}) = 0$. Moreover,

$$\mathbb{Q}_1\left(r_0^*(\tau^*) = r_0^*(\mathbf{t}^*)\right) = \frac{1}{\xi_1}\left(\frac{\mathbb{E}\left[\mathbf{q}(X)\right]}{1-\mu}Z_0(\mathbf{t}) + M_0(\mathbf{t})\right)\mathbb{P}(r_0^*(\tau^*) = r_0^*(\mathbf{t}^*)) = 1,$$

thereby (9.9) is checked.
Assume that (9.9) is true for $n \in \mathbb{N}$. Then, using successively the reproduction law of τ_1, formula (7.7) and the definition of $\mathbb{Q}_1$, we have:

$$\begin{aligned}&P(r_{n+1}^*(\tau_1) = r_{n+1}^*(\mathbf{t}^*)) = P(r_n^*(\tau_1) = r_n^*(\mathbf{t}^*))\prod_{u\in z_n(\mathbf{t}^*)}\mathbf{p}_0(k_u(\mathbf{t}),\eta_u(\mathbf{t})) \\ &\times \sum_{\substack{u\in\mathbf{t}\\ |u|_\infty=n}}\frac{\dfrac{\mathbb{E}\left[\mathbf{q}(X)\right]}{1-\mu}+m_u}{M_n(\mathbf{t}) + \dfrac{\mathbb{E}\left[\mathbf{q}(X)\right]}{1-\mu}Z_n(\mathbf{t})}\frac{\frac{\mathbb{E}[\mathbf{q}(X)]}{1-\mu}k_u(\mathbf{t}) + (m_u+\eta_u(\mathbf{t}))}{\frac{\mathbb{E}[\mathbf{q}(X)]}{1-\mu}+m_u} \\ &= \frac{\mathbb{Q}_1(r_n^*(\tau^*) = r_n^*(\mathbf{t}^*))}{M_n(\mathbf{t}) + \dfrac{\mathbb{E}\left[\mathbf{q}(X)\right]}{1-\mu}Z_n(\mathbf{t})}\prod_{\substack{u\in\mathbf{t}\\ |u|_\infty=n}}\mathbf{p}_0(k_u(\mathbf{t}),\eta_u(\mathbf{t}))\sum_{\substack{u\in\mathbf{t}\\ |u|_\infty=n}}\frac{\mathbb{E}[\mathbf{q}(X)]}{1-\mu}k_u(\mathbf{t}) + (m_u+\eta_u(\mathbf{t})) \\ &= \frac{1}{\xi_1}\mathbb{P}(r_n^*(\tau^*) = r_n^*(\mathbf{t}^*))\prod_{\substack{u\in\mathbf{t}\\ |u|_\infty=n}}\mathbf{p}_0(k_u(\mathbf{t}),\eta_u(\mathbf{t}))\left(\frac{\mathbb{E}[\mathbf{q}(X)]}{1-\mu}Z_{n+1}(\mathbf{t}) + M_{n+1}(\mathbf{t})\right) \\ &= \frac{1}{\xi_1}\mathbb{P}(r_{n+1}^*(\tau^*) = r_{n+1}^*(\mathbf{t}^*))\left(\frac{\mathbb{E}[\mathbf{q}(X)]}{1-\mu}Z_{n+1}(\mathbf{t}) + M_{n+1}(\mathbf{t})\right) \\ &= \mathbb{Q}_1(r_{n+1}^*(\tau^*) = r_{n+1}^*(\mathbf{t}^*)),\end{aligned}$$

we obtain our result. □

9.1.3 Penalization by M_p^ℓ

In this section, we consider an offspring distribution $\mathbf{p}$ satisfying (6.1) and that admits a moment of order $\ell \in \mathbb{N}^*$, and a mark function $\mathbf{q}$ satisfying (6.2). We denote by μ the mean of $\mathbf{p}$. We study the penalization by M_{n+p}^ℓ, i.e. we look for the limit for all $n \in \mathbb{N}$ and $\Lambda_n \in \mathscr{F}_n$:

$$\lim_{p\to+\infty}\frac{\mathbb{E}[\mathbb{1}_{\Lambda_n}M_{n+p}^\ell]}{\mathbb{E}[M_{n+p}^\ell]} = \lim_{p\to+\infty}\frac{\mathbb{E}[\mathbb{1}_{\Lambda_n}\mathbb{E}[M_{n+p}^\ell|\mathscr{F}_n]]}{\mathbb{E}[M_{n+p}^\ell]}.$$

To begin with, let us prove that this expression makes sense. Remark first that (6.2) implies that $\mathbb{E}[M_{n+p}^\ell] \neq 0$ for $p \geq 1$.

Lemma 9.1. *Let* $\mathbf{p}$ *be an offspring distribution that admits a moment of order* $\ell \in \mathbb{N}^*$ *and let* $\mathbf{q}$ *be a mark function. Then, under* $\mathbb{P}$*, for every* $p \in \mathbb{N}$*,* Z_p *and* M_p *have a moment of order* ℓ*.*

Proof. We begin to prove that Z_p has a moment of order ℓ by induction on p. The result is clear for $p = 0$, as $Z_0 = 1$.

Let $p \in \mathbb{N}$ be such that Z_j has a moment of order ℓ for all $j \leq p$. Let X, $(X_{i,p})_{i\geq 1}$ be i.i.d. random variables with distribution $\mathbf{p}$, independent of $\mathscr{F}_p$. We have:

$$\begin{aligned}\mathbb{E}\left[Z_{p+1}^{\ell}\right] &= \mathbb{E}\left[\mathbb{E}\left[\left(\sum_{i=1}^{Z_p} X_{i,p}\right)^{\ell}\middle|\mathscr{F}_p\right]\right] = \mathbb{E}\left[\sum_{t_1+\ldots+t_{Z_p}=\ell}\begin{pmatrix} k \\ t_1 \cdots t_{Z_p}\end{pmatrix}\prod_{k=1}^{Z_p}\mathbb{E}\left[X^{t_k}\right]\right] \\ &\leq \mathbb{E}\left[\sum_{t_1+\ldots+t_{Z_p}=\ell}\begin{pmatrix} k \\ t_1 \cdots t_{Z_p}\end{pmatrix}\mathbb{E}\left[X^{\ell}\right]^{\ell}\right] = \mathbb{E}\left[Z_p^{\ell}\right]\mathbb{E}\left[X^{\ell}\right]^{\ell}.\end{aligned}$$

where in the last inequality, we use that there are at most ℓ non-zero t_i's. As Z_p^{ℓ} and X^{ℓ} are in L^1 by assumption, we have our first result.

The previous result implies that $\sum_{i=0}^{p-1} Z_i \in \mathrm{L}^{\ell}$, as $M_p \leq \sum_{i=0}^{p-1} Z_i$ we get our second result. □

In this Sub-section, we use the multinomial coefficient defined as follows.
For all $n, m \in \mathbb{N}^*$ and natural numbers $(t_i)_{i\in[\![1,m]\!]}$ such that $\sum_{i=1}^{m} t_i = n$, we have:

$$\begin{pmatrix} n \\ t_1 \cdots t_m \end{pmatrix} = \frac{n!}{t_1! \cdots t_m!}. \tag{9.10}$$

We now give an expression of $\mathbb{E}\left[M_{n+p}^{\ell}\middle|\mathscr{F}_n\right]$ that will be used troughout this Section. Recall the equality in distribution (7.10). Using the same notations, we get for all $p, n \in \mathbb{N}$:

$$\begin{aligned}\mathbb{E}\left[M_{n+p}^{\ell} \mid \mathscr{F}_n\right] &= \sum_{j=0}^{\ell}\binom{\ell}{j}\mathbb{E}\left[M_n^{\ell-j}\left(\sum_{i=1}^{Z_n}\tilde{M}_{i,p}\right)^{j} \,\middle|\, \mathscr{F}_n\right] \\ &= \sum_{j=0}^{\ell}\binom{\ell}{j}M_n^{\ell-j}\sum_{t_1+\cdots+t_{Z_n}=j}\begin{pmatrix} j \\ t_1 \cdots t_{Z_n}\end{pmatrix}\prod_{i=1}^{Z_n}\mathbb{E}[M_p^{t_i}]. \end{aligned} \tag{9.11}$$

9.1.3.1 The sub-critical case $\mu < 1$

9.1.3.1.1 A new martingale

Before stating the main result of this Sub-section, let us state some integrability lemmas and introduce some notations.

Lemma 9.2. *Let* $\mathbf{p}$ *be a sub-critical offspring distribution that admits a moment of order* $\ell \in \mathbb{N}^*$*. Let* $\tilde{Z}_\infty$ *be the total population size of the associated Galton-Watson tree i.e.*

$$\tilde{Z}_\infty = \mathrm{Card}(\tau) = \sum_{n=0}^{+\infty} Z_n.$$

Then $\tilde{Z}_\infty$ *has a moment of order* ℓ*.*

Proof. Let us denote by g (resp. G_∞) the generating function of $\mathbf{p}$ (resp. $\tilde{Z}_\infty$). Then we have, with $(\tilde{Z}_{i,\infty})_{i\geq 1}$ i.i.d. copies of $\tilde{Z}_\infty$, for every $s \in [0,1]$,

$$G_\infty(s) = \mathbb{E}\left[s^{\tilde{Z}_\infty}\right] = \mathbb{E}\left[s^{1+\sum\limits_{i=1}^{X}\tilde{Z}_{i,\infty}}\right] = s\mathbb{E}\left[\mathbb{E}\left[s^{\sum\limits_{i=1}^{X}\tilde{Z}_{i,\infty}}\middle|\mathscr{F}_1\right]\right]$$
$$= s\mathbb{E}\left[\mathbb{E}\left[s^{\tilde{Z}_\infty}\right]^X\right] = sg(G_\infty(s)).$$

Since X has a moment of order 1 and g is $\mathcal{C}^1([0,1))$, we have $\lim\limits_{s\to 1} g'(s) = \mu$ and g is $\mathcal{C}^1([0,1])$. G_∞ is $\mathcal{C}^1([0,1))$. Differentiating this relation, we obtain, for every $s \in [0,1)$,

$$G'_\infty(s) = \frac{g(G_\infty(s))}{1 - sg'(G_\infty(s))}. \tag{9.12}$$

Then, letting $s \to 1$, we have

$$G'_\infty(s) \underset{s\to 1}{\longrightarrow} \frac{1}{1-\mu} < +\infty.$$

Therefore, $\tilde{Z}_\infty$ admits a moment of order 1, $\mathbb{E}[\tilde{Z}_\infty] = \dfrac{1}{1-\mu}$ and G_∞ is $\mathcal{C}^1([0,1])$.

Let $k \in \mathbb{N}^*$, $k < \ell$. Since X has a moment of order ℓ then g is $\mathcal{C}^{k+1}([0,1])$. Moreover, $\tilde{Z}_\infty$ has a moment of order k and G_∞ is $\mathcal{C}^k([0,1])$ and $\mathcal{C}^{k+1}([0,1))$.
We derive (9.12) k-times, then we can prove that $G_\infty^{(k+1)}$ has a finite limit when s goes to 1. Thanks to Liebniz formula and Faà di Bruno's formula, we obtain :

$$G_\infty^{(k+1)}(s) = \sum_{j=0}^{k} \begin{pmatrix} k \\ j \end{pmatrix} f^{(j)}(s) h^{(k-j)}(s),$$

with, $f^{(0)}(s) = f(s) = g(G_\infty(s))$, $h(s) = \dfrac{1}{1 - sg'(G_\infty(s))}$, for all $1 \leq i \leq k$, $\beta_i = \sum\limits_{j=1}^{i} m_j$ and:

$$f^{(i)}(s) = \sum_{\sum\limits_{j=1}^{i} jm_j = i} \begin{pmatrix} i \\ m_1 \cdots m_i \end{pmatrix} g^{(\beta_i)}(G_\infty(s)) \prod_{j=1}^{i} \left(\frac{G_\infty^{(j)}(s)}{j!}\right)^{m_j},$$

$$h^{(i)}(s) = \sum_{\sum\limits_{j=1}^{i} jm_j = i} \frac{\begin{pmatrix} i \\ m_1 \cdots m_i \end{pmatrix} (-1)^{\beta_i}\beta_i!}{(1 - sg'(G_\infty(s)))^{\beta_i+1}} \prod_{j=1}^{i} \left(\frac{-\left(s(g'\circ G_\infty)^{(j)}(s) + j(g'\circ G_\infty)^{(j-1)}(s)\right)}{j!}\right)^{m_j},$$

$$(g'\circ G_\infty)^{(i)}(s) = \sum_{\sum\limits_{j=1}^{i} jm_j = i} \begin{pmatrix} k \\ m_i \cdots m_i \end{pmatrix} g^{(\beta_i+1)}(G_\infty(s)) \prod_{j=1}^{i} \left(\frac{G_\infty^{(j)}(s)}{j!}\right)^{m_j}.$$

By induction case, $G_\infty^{(j)}(s)$ admits a limit when s goes to 1, for all $j < k+1$. Plus, for all $i \leq k+1$, $g^{(i)}(s)$ also admits a finite limit when s goes to 1. We always have $\beta_i \leq k$ for

all $0 \leq i \leq k$. In the case $j = 0$, all the derivatives are composed by functions $g^{(i)}(.)$ with $0 \leq i \leq k+1$ and $G_\infty^{(u)}(.)$ with $0 \leq u < k+1$. In the case $j = k$, the expression is composed by functions $g^{(i)}(.), G_\infty^{(i)}(.)$ with $0 \leq i < k+1$. And in the other terms, the derivatives only depends of functions $g^i(.)$ with $i \leq k$ and $G_\infty^{(j)}(.)$ with $j < k$.
Thereby all the terms admit a finite limit when s goes to 1.

Since $G_\infty^{(k+1)}(s)$ has a finite limit when s goes to 1, G_∞ is $\mathcal{C}^{k+1}([0,1])$ and $\tilde{Z}_\infty$ has a moment of order $k+1$. By induction case we can conclude, that $\tilde{Z}_\infty$ has a moment of order ℓ . □

Let us denote by M_∞ the total number of marks on the tree. As $M_\infty \leq \tilde{Z}_\infty$ a.s., M_∞ also admits a moment of order ℓ. We set for every $i \leq \ell$

$$\xi_i = \mathbb{E}[M_\infty^i]. \tag{9.13}$$

The constants ξ_i can be explicitely computed using the following recursion equation:

Proposition 9.3. *Let* $\mathbf{p}$ *be an offspring distribution such that* $\mu < 1$*, with a moment of order* $\ell \in \mathbb{N}^*$ *and let* $\mathbf{q}$ *be a mark function. Then we have*

$$(1-\mu)\xi_\ell = \sum_{j=0}^{\ell-1} \binom{\ell}{j} \mathbb{E}\left[M_1 \sum_{t_1+\cdots+t_{Z_1}=j} \binom{j}{t_1 \cdots t_{Z_1}} \prod_{i=1}^{Z_1} \xi_{t_i}\right] + \mathbb{E}\left[\sum_{\substack{t_1+\ldots+t_{Z_1}=\ell \\ t_i<\ell,\forall i}} \binom{\ell}{t_1 \cdots t_{Z_1}} \prod_{i=1}^{Z_1} \xi_{t_i}\right].$$

Remark 9.2. *For* $\ell = 1$*, the expression of* ξ_1 *is the same in Theorem 9.2 .*

Proof. First notice that, as M_1 is equal to 0 or 1, we have $M_1^j = M_1$ for every $j \geq 1$.

Moreover, for all $1 \leq j \leq k$, $\left(\mathbb{E}[M_p^j]\right)_{p\geq 0}$ is a non-decreasing sequence bounded above by $\mathbb{E}[\tilde{Z}_\infty^k]$, giving the existence and the finiteness of ξ_j according to Lemma 9.2.
Using (9.11), we can write:

$$\begin{aligned}
\mathbb{E}[M_{p+1}^k] - \mu\mathbb{E}[M_p^k] &= \sum_{j=0}^{k-1} \binom{j}{k} \mathbb{E}\left[M_1 \sum_{t_1+\ldots+t_{Z_1}} \binom{j}{t_1 \ldots t_{Z_1}} \prod_{\ell=1}^{Z_1} \mathbb{E}[M_p^{t_\ell}]\right] \\
&+ \mathbb{E}\left[\sum_{\substack{t_1+\ldots+t_{Z_1}=k \\ t_i<k,\forall i}} \binom{j}{t_1 \ldots t_{Z_1}} \prod_{\ell=1}^{Z_1} \mathbb{E}[M_p^{t_\ell}]\right] \\
&:= \sum_{j=0}^{k-1} \binom{k}{j} \mathbb{E}[A_{j,p}] + \mathbb{E}[B_p].
\end{aligned}$$

As $A_{j,p}$ and B_p are non-negative and non-decreasing sequences, we obtain ξ_k taking the limit when p goes to infinity and using the Beppo Levi's Theorem. □

We can now state the main result of this Section. Let us recall, for $\ell \in \mathbb{N}$, the definition of the function f_ℓ defined for all $s, z \in \mathbb{N}$ by:

$$f_\ell(s,z) := \frac{1}{\xi_\ell} \sum_{i=0}^{\ell} \binom{\ell}{i} s^{\ell-i} \sum_{t_1+\ldots+t_z=i} \binom{i}{t_1 \ldots t_z} \prod_{j=1}^{z} \xi_{t_j} \tag{9.14}$$

with the convention $f_\ell(s,0) = \frac{s^\ell}{\xi_\ell}$. Notice that this definition is also valid for $\ell = 0$ and we get $f_0(s,z) = 1$.

Theorem 9.4. *Let* $\mathbf{p}$ *be an offspring distribution satisfying* (6.1)*, that admits a moment of order* $\ell \in \mathbb{N}^*$ *and such that* $\mu < 1$*, and let* $\mathbf{q}$ *be a mark function satisfying* (6.2)*. For every* $n \in \mathbb{N}$ *and every* $\Lambda_n \in \mathscr{F}_n$*, we have*

$$\lim_{p\to+\infty} \frac{\mathbb{E}\left[\mathbb{1}_{\Lambda_n} M_{n+p}^\ell\right]}{\mathbb{E}\left[M_{n+p}^\ell\right]} = \mathbb{E}[\mathbb{1}_{\Lambda_n} f_\ell(M_n, Z_n)]. \tag{9.15}$$

Proof. Let $p, n \in \mathbb{N}$. Using (9.11), we get:

$$\mathbb{E}\left[\mathbb{1}_{\Lambda_n} M_{n+p}^\ell\right] = \mathbb{E}\left[\mathbb{1}_{\Lambda_n} \sum_{j=0}^{\ell} \binom{\ell}{j} M_n^{\ell-j} \sum_{t_1+\ldots+t_{Z_n}=j} \binom{j}{t_1 \cdots t_{Z_n}} \prod_{i=1}^{Z_n} \mathbb{E}\left[M_p^{t_i}\right]\right].$$

Using again Beppo Levi's theorem, we obtain

$$\lim_{p\to+\infty} \mathbb{E}\left[\mathbb{1}_{\Lambda_n} M_{n+p}^\ell\right] = \xi_\ell \mathbb{E}[\mathbb{1}_{\Lambda_n} f_\ell(M_n, Z_n)]$$

and we conclude using

$$\lim_{p\to+\infty} \mathbb{E}[M_{n+p}^\ell] = \xi_\ell.$$

□

Remark 9.3. *For* $\ell = 1$*, the obtained martingale is well*

$$B_{n,1} = Z_n + \frac{1}{\xi_1} M_n.$$

9.1.3.1.2 Change of probability and the new random tree

We still consider a subcritical offspring distribution $\mathbf{p}$ satisfying (6.1) and that admits a moment of order $\ell \in \mathbb{N}^*$, and a mark function $\mathbf{q}$ satisfying (6.2). We define a new probability measure $\mathbb{Q}_\ell$ on $(\mathbb{T}^*, \mathscr{F})$ by

$$\forall n \in \mathbb{N},\ \forall \Lambda_n \in \mathscr{F}_n,\ \mathbb{Q}_\ell(\Lambda_n) = \mathbb{E}[\mathbb{1}_{\Lambda_n} f_\ell(Z_n, M_n)]. \tag{9.16}$$

To describe this probability measure $\mathbb{Q}_\ell$, let us define a $\mathbb{T}^*$-valued random tree $\tau_\ell(\mathbf{p}, \mathbf{q})$ defined on some probability space $(\Omega, \mathscr{A}, P)$.

Definition 9.1. *The tree* $\tau_\ell(\mathbf{p}, \mathbf{q})$ *is a multitype random marked tree that is defined recursively as follows:*

— *The types of the nodes run from* 0 *to* ℓ*;*

— *The type of the root is* ℓ*;*

— *Let us suppose that the tree* $r_n^*(\tau_\ell(\mathbf{p}, \mathbf{q}))$ *is constructed and that the types of the nodes at generation* n *are known for some* $n \geq 0$*. Then all individuals at generation* n *reproduce and are marked independently of each other conditionally given* $r_n^*(\tau_\ell(\mathbf{p}, \mathbf{q}))$ *according to its type and its mass (recall the definition of the mass of a node introduced in Sub-section 9.1.1):*

— *an individual u of type 0, has $k \in \mathbb{N}$ children and a mark $\eta \in \{0,1\}$ with probability:*

$$\mathbf{p}_0(k,\eta) := \begin{cases} \mathbf{p}(k)(1-\mathbf{q}(k)) & \text{if } \eta = 0, \\ \mathbf{p}(k)\mathbf{q}(k) & \text{if } \eta = 1, \end{cases}$$

— *an individual u of type $i \in [\![1,\ell]\!]$ and with mass m_u, has $k \in \mathbb{N}$ children and a mark $\eta \in \{0,1\}$ with probability:*

$$\mathbf{p}_i(k,\eta) := \frac{\mathbf{p}_0(k,\eta) f_i(m_u+\eta,k)}{f_i(m_u,1)};$$

— *We then choose the types of the nodes at generation $n+1$. At each generation, the sum of the types must be equal to ℓ. Moreover, conditionally on $z_{n+1}(\tau_\ell(\mathbf{p},\mathbf{q})) = z_{n+1}$, the probability that the nodes at generation $n+1$ have respective types $(t_u)_{u\in z_{n+1}}$ with $\sum_{u\in z_{n+1}} t_u = \ell$, is*

$$\gamma_{\ell,(t_u)_{u\in z_{n+1}}} := \binom{\ell}{t_u, u \in z_{n+1}} \frac{\prod_{u\in z_{n+1}} f_{t_u}(m_u,1)\xi_{t_u}}{\xi_\ell f_\ell(M_{n+1}, Z_{n+1})}, \tag{9.17}$$

with the notation

$$\binom{\ell}{t_u, u \in z_{n+1}} := \frac{\ell!}{\prod_{u\in z_{n+1}} t_u!}.$$

for the usual multinomial coefficient.

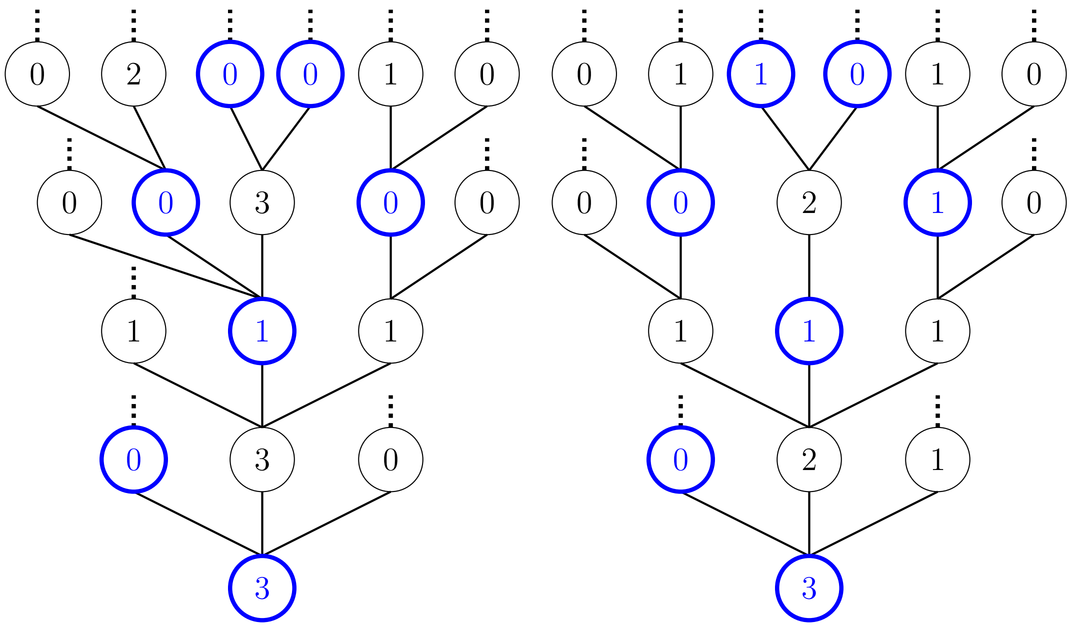


Figure 9.2 – Two examples of marked multi-type trees (marked in blue, unmarked in black), in the sub-critical case for $\ell = 3$.

Remark 9.4. *The mass m_u of a node u (and hence its offspring distribution) depends on all extinct nodes in the past and not only on the ancestors of the node. Moreover, the choice of the types of the nodes at some generation depends on the size of this generation. Consequently, the tree τ_ℓ does not satisfy the branching property.*

Remark 9.5. *If all the nodes below generation n are not marked, all the masses m_u for $u \in z_n$ are null and hence, for every node $u \in z_n$ with type $i \geq 1$, we have*

$$\mathbf{p}_i(0,0) = 0.$$

Therefore, the tree $\tau_\ell(\mathbf{p}, \mathbf{q})$ cannot die out without any marked node.

Lemma 9.3. *Formula* (9.17) *indeed defines a probability distribution on $[\![0,\ell]\!]^{Z_{n+1}}$.*

Proof. This result is equivalent to show that for all $n \in \mathbb{N}$, we have

$$f_\ell(M_n, Z_n) = \frac{1}{\xi_\ell} \sum_{\sum\limits_{u \in z_n} t_u = \ell} \binom{\ell}{t_u, u \in z_n} \prod_{u \in z_n} f_{t_u}(m_u, 1)\xi_{t_u}. \tag{9.18}$$

To obtain (9.18), we need to obtain an alternate version for the a.s. limit of $\mathbb{E}[M_{n+p}^\ell | \mathscr{F}_n]$, as we already know that this limit is $\xi_\ell f_\ell(M_n, Z_n)$ a.s. For this purpose, according to (9.3) and the branching property, conditionally on $\mathscr{F}_n$, we have:

$$M_{n+p} \overset{(d)}{=} \sum_{u \in z_n} m_u + M_{u,p},$$

where the processes $(M_{u,\cdot}, u \in \mathcal{U})$ are i.i.d. copies of M, independent of $\mathscr{F}_n$. As a result:

$$\begin{aligned}
\mathbb{E}[M_{n+p}^\ell | \mathscr{F}_n] &= \sum_{\sum\limits_{u \in z_n} t_u = \ell} \binom{\ell}{t_u, u \in z_n} \prod_{v \in z_n} \mathbb{E}\left[(m_v + M_{v,p})^{t_v} | \mathscr{F}_n\right] \\
&= \sum_{\sum\limits_{u \in z_n} t_u = \ell} \binom{\ell}{t_u, u \in z_n} \prod_{v \in z_n} \sum_{j=0}^{t_v} \binom{t_v}{j} m_v^{t_v - j} \mathbb{E}\left[M_p^j\right].
\end{aligned}$$

Using Beppo Levi's theorem:

$$\lim_{p \to +\infty} \mathbb{E}[M_{p+n}^\ell | \mathscr{F}_n] = \sum_{\sum\limits_{u \in z_n} t_u = \ell} \binom{\ell}{t_u, u \in z_n} \prod_{v \in z_n} \sum_{j=0}^{t_v} \binom{t_v}{j} m_v^{t_v - j} \xi_j,$$

and we easily conclude as, thanks to (9.14),

$$f_{t_v}(m_v, 1)\xi_{t_v} = \sum_{j=0}^{t_v} \binom{t_v}{j} m_v^{t_v - j} \xi_j.$$

□

In order to describe the new probability $\mathbb{Q}_\ell$, we need an alternative expression of $f_\ell(M_{n+1}, Z_{n+1})$:

Proposition 9.4. *For all $n \in \mathbb{N}$, we have*

$$f_\ell(M_{n+1}, Z_{n+1}) = \frac{1}{\xi_\ell} \sum_{\sum_{u\in z_n} t_u = \ell} \binom{\ell}{t_u, u \in z_n} \prod_{u \in z_n} f_{t_u}(m_u + \eta_u, k_u)\xi_{t_u}, \tag{9.19}$$

with k_u the number of children of the individual u and η_u its mark.

Proof. The proof is similar to the proof of Lemma 9.3: we write

$$M_{n+p} = \sum_{u \in z_n} m_u + \eta_u + \sum_{i=1}^{k_u} M_{i,p-1}$$

where the processes $(M_{i,.}, i \geq 0)$ are i.i.d. copies of M, independent of $\mathscr{F}_{n+1}$. Thus, $\mathbb{E}[M_{n+p}^\ell | \mathscr{F}_{n+1}]$ is equal to:

$$\begin{aligned}
&\sum_{\sum_{u\in z_n} t_u = \ell} \binom{\ell}{t_u, u \in z_n} \prod_{v \in z_n} \sum_{j=0}^{t_v} \binom{t_v}{j} (m_v + \eta_v)^{t_v - j} \mathbb{E}\left[\left(\sum_{i=1}^{k_u} M_{i,p-1} \right)^j \middle| \mathscr{F}_{n+1} \right] \\
&\quad = \sum_{\sum_{u\in z_n} t_u = \ell} \binom{\ell}{t_u, u \in z_n} \prod_{v \in z_n} \sum_{j=0}^{t_v} \binom{t_v}{j} (m_v + \eta_v)^{t_v - j} \sum_{r_1 + \cdots + r_{k_u} = j} \binom{j}{r_1 \dots r_{k_u}} \prod_{i=1}^{k_u} \mathbb{E}[M_{i,p-1}^{r_i}].
\end{aligned}$$

Taking the limit when p goes to infinity, thanks to Beppo Levi's theorem, we obtain (9.19). □

Theorem 9.5. *The probability $\mathbb{Q}_\ell$ defined by* (9.16) *is the distribution of the tree $\tau_\ell(\mathbf{p}, \mathbf{q})$ of Definition 9.1.*

Proof. According to (9.1), to prove the theorem, it suffices to prove that for all $n \in \mathbb{N}$, and all $\mathbf{t}^* \in \mathbb{T}^*$:

$$\mathbb{Q}_\ell(r_n^*(\tau^*) = r_n^*(\mathbf{t}^*)) = P(r_n^*(\tau_\ell) = r_n^*(\mathbf{t}^*)) \tag{9.20}$$

where τ^* is the canonical random variable on $\mathbb{T}^*$ and where we set τ_ℓ instead of $\tau_\ell(\mathbf{p}, \mathbf{q})$. First, note that:

$$\begin{aligned}
\mathbb{Q}_\ell(r_n^*(\tau^*) = r_n^*(\mathbf{t}^*)) &= \mathbb{E}\left[\mathbb{1}_{r_n^*(\tau^*) = r_n^*(\mathbf{t}^*)} f_\ell(M_n(\tau^*), Z_n(\tau^*)) \right] \\
&= f_\ell(M_n(\mathbf{t}^*), Z_n(\mathbf{t}^*)) \mathbb{P}(r_n^*(\tau^*) = r_n^*(\mathbf{t}^*)).
\end{aligned} \tag{9.21}$$

We prove (9.20) by induction on n.

For $n = 0$, (9.20) is true as, for every tree $\mathbf{t}^*$ (and hence also for any random tree), $r_0^*(\mathbf{t}^*)$ is the tree reduced to the root with mark 0.

Assume that (9.20) is true for $n \in \mathbb{N}$. Using the recursive definition of τ_ℓ, we have:

$$\begin{aligned}
P(r_{n+1}^*(\tau_\ell) = r_{n+1}^*(\mathbf{t}^*)) &= P(r_n^*(\tau_\ell) = r_n^*(\mathbf{t}^*)) \prod_{u \in z_n(\mathbf{t}^*)} \mathbf{p}_0(k_u(\mathbf{t}), \eta_u(\mathbf{t})) \\
&\quad \times \sum_{\sum_{u \in z_n(\mathbf{t}^*)} t_u = \ell} \gamma_{\ell,(t_u)_{u \in z_n(\mathbf{t}^*)}} \prod_{u \in z_n(\mathbf{t}^*)} \frac{f_{t_u}(m_u(\mathbf{t}^*) + \eta_u(\mathbf{t}), k_u(\mathbf{t}))}{f_{t_u}(m_u(\mathbf{t}^*), 1)}.
\end{aligned}$$

Then, the induction assumption as well as the definition of the probabilities γ give

$$\begin{aligned}
P(r_{n+1}^*(\tau_\ell) = r_{n+1}^*(\mathbf{t}^*)) &= \mathbb{Q}_\ell(r_n^*(\tau^*) = r_n^*(\mathbf{t}^*)) \prod_{u \in z_n(\mathbf{t}^*)} \mathbf{p}_0\left(k_u(\mathbf{t}), \eta_u(\mathbf{t})\right) \\
&\times \sum_{\sum_{u\in z_n(\mathbf{t}^*)} t_u = \ell} \binom{\ell}{t_u, u \in z_n(\mathbf{t}^*)} \frac{\sum_{u \in z_n(\mathbf{t}^*)} f_{t_u}(m_u(\mathbf{t}^*), 1)\xi_{t_u}}{\xi_\ell f_\ell\left(M_n(\mathbf{t}^*), Z_n(\mathbf{t}^*)\right)} \\
&\times \prod_{u \in z_n(\mathbf{t}^*)} \frac{f_{t_u}(m_u(\mathbf{t}^*) + \eta_u(\mathbf{t}), k_u(\mathbf{t}^*))}{f_{t_u}(m_u(\mathbf{t}^*), 1)}.
\end{aligned}$$

Using Formula (9.21) yields

$$\begin{aligned}
P(r_{n+1}^*(\tau_\ell) = r_{n+1}^*(\mathbf{t}^*)) = \frac{1}{\xi_\ell}\mathbb{P}(r_n^*(\tau^*) = r_n^*(\mathbf{t}^*)) &\prod_{u \in z_n(\mathbf{t}^*)} \mathbf{p}_0\left(k_u(\mathbf{t}), \eta_u(\mathbf{t})\right) \\
\times \sum_{\sum_{u\in z_n(\mathbf{t}^*)} t_u = \ell} &\binom{\ell}{t_u, u \in z_n(\mathbf{t}^*)} \prod_{u \in z_n(\mathbf{t}^*)} f_{t_u}(m_u(\mathbf{t}^*) + \eta_u(\mathbf{t}), k_u(\mathbf{t}))\xi_{t_u}.
\end{aligned}$$

Finally, Equation (7.7) and Proposition 9.4 and then Equation (9.21) give

$$\begin{aligned}
P(r_{n+1}^*(\tau_\ell) = r_{n+1}^*(\mathbf{t}^*)) &= \mathbb{P}(r_{n+1}^*(\tau^*) = r_{n+1}^*(\mathbf{t}^*)) f_\ell(M_{n+1}(\mathbf{t}^*), Z_{n+1}(\mathbf{t}^*)) \\
&= \mathbb{Q}_\ell(r_{n+1}^*(\tau^*) = r_{n+1}^*(\mathbf{t}^*)),
\end{aligned}$$

which gives the result for $n+1$. □

Remark 9.6. *Note that under $\mathbb{Q}_\ell$, the extinction probability of the population is equal to 1. Indeed, using the definition of $\mathbb{Q}_\ell$, the Lebesgue's dominated convergence theorem and the fact that $Z_\infty = 0$ $\mathbb{P}$-a.s.:*

$$\begin{aligned}
\lim_{n\to+\infty} \mathbb{Q}_\ell(Z_n = 0) &= \lim_{n\to+\infty} \mathbb{E}[\mathbb{1}_{Z_n=0} f_\ell(M_n, 0)] \\
&= \frac{1}{\xi_\ell} \lim_{n\to+\infty} \mathbb{E}[\mathbb{1}_{Z_n=0} M_n^\ell] = \frac{\mathbb{E}[M_\infty^\ell]}{\xi_\ell} = 1.
\end{aligned}$$

Moreover under $\mathbb{Q}_\ell$, one can notice that for all $n \in \mathbb{N}^*$, Z_n does not necessary admit a first moment. Just assume that Z_1 (and so Z_n) does not admit a moment of order $\ell + 1$ under $\mathbb{P}$, as:

$$\begin{aligned}
\xi_\ell \mathbb{E}^{\mathbb{Q}_\ell}[Z_n] &= \xi_\ell \mathbb{E}[Z_n f(M_n, Z_n)] \\
&\geq \mathbb{E}\left[Z_n \sum_{\substack{t_1+\ldots+t_{Z_n}=\ell \\ t_i \leq 1, \forall i}} \binom{\ell}{t_1 \cdots t_{Z_n}} \prod_{i=1}^{Z_n} \xi_{t_i} \right] = \ell! \xi_1^\ell \mathbb{E}\left[Z_n \binom{Z_n}{Z_n - \ell} \right].
\end{aligned}$$

Note that $\mathbb{E}^{\mathbb{Q}_\ell}$ denotes the expectation with respect to the probability distribution $\mathbb{Q}_\ell$.

9.1.3.2 The super-critical case $\mu > 1$

Let us begin with giving an equivalent of $\mathbb{E}[M_p^\ell]$ as $p \to +\infty$.

Proposition 9.5. *Let $\ell \in \mathbb{N}^*$. Let $\mathbf{p}$ be an offspring distribution which admits a moment of order $\ell \in \mathbb{N}^*$ and such that $\mu > 1$, and let $\mathbf{q}$ be a mark function satisfying (6.2). Then, there exists a strictly positive constant ω_ℓ such that:*

$$\mathbb{E}[M_p^\ell] \underset{p\to\infty}{\sim} \mu^{\ell p}\omega_\ell. \tag{9.22}$$

Proof. We prove this proposition by induction on ℓ.

We have $\omega_0 = 1$. According to (9.5), (9.22) is true for $\ell = 1$, with $\omega_1 := \frac{\mathbb{E}[M_1]}{\mu-1}$. Now, assume that (9.22) is true for all $1 \le i < \ell$, for some $\ell > 1$, in other words:

$$\frac{\mathbb{E}\left[M_p^i\right]}{\mu^{pi}} = \omega_i(1 + o(1)). \tag{9.23}$$

In order to obtain the equivalent for ℓ, we write:

$$\mathbb{E}[M_p^\ell] = \mu^p \sum_{m=0}^{p-1} \left(\frac{\mathbb{E}[M_{m+1}^\ell]}{\mu^{m+1}} - \frac{\mathbb{E}[M_m^\ell]}{\mu^m} \right) =: \mu^p \sum_{m=0}^{p-1} \frac{\Delta_m}{\mu^{m+1}},$$

and we study the telescopic series with general term $\dfrac{\Delta_m}{\mu^{m+1}}$. To begin with, taking the expectation in (9.11) with $n = 1$ and $p = m$, (and using again that $M_1^j = M_1$ for every $j \ge 1$), we obtain by isolating the case $j = \ell$:

$$\begin{aligned}
\frac{\Delta_m}{\mu^{m\ell}} &= \sum_{j=0}^{\ell-1} \binom{\ell}{j} \mathbb{E}\left[\frac{M_1}{\mu^{m(\ell-j)}} \sum_{t_1+\dots+t_{Z_1}=j} \binom{j}{t_1 \cdots t_{Z_1}} \prod_{i=1}^{Z_1} \frac{\mathbb{E}[M_m^{t_i}]}{\mu^{mt_i}} \right] \\
&\quad + \mathbb{E}\left[\sum_{\substack{t_1+\dots+t_{Z_1}=\ell \\ t_i<\ell, \forall i}} \binom{\ell}{t_1 \cdots t_{Z_1}} \prod_{i=1}^{Z_1} \frac{\mathbb{E}[M_m^{t_i}]}{\mu^{mt_i}} \right] \\
&:= \sum_{j=0}^{\ell-1} \binom{m}{j} \mathbb{E}[A_{j,m}] + \mathbb{E}[B_m]
\end{aligned} \tag{9.24}$$

with $B_m = 0$ if $Z_1 \le 1$.

Note that in B_m, each term of the sum has at most ℓ strictly positive t_i. Thus, setting $\omega_0 = 1$ and $\varpi_\ell := \max_{0\le j<\ell} \omega_j$ and using (9.23) for m large enough:

$$0 \le B_m \le \left(\frac{3}{2}\right)^\ell \varpi_\ell^\ell \sum_{\substack{t_1+\dots+t_{Z_1}=\ell \\ t_i<\ell, \forall i}} \binom{\ell}{t_1 \cdots t_{Z_1}} \le \left(\frac{3}{2}\varpi_\ell Z_1\right)^\ell \in L^1. \tag{9.25}$$

Using (9.23) and (9.25), Lebesgue's dominated convergence theorem ensures that:

$$\lim_{m\to+\infty} \mathbb{E}[B_m] = \mathbb{E}\left[\lim_{m\to+\infty} B_m\right] = \mathbb{E}\left[\sum_{\substack{t_1+\dots+t_{Z_1}=\ell \\ t_i<\ell, \forall i}} \binom{\ell}{t_1 \cdots t_{Z_1}} \prod_{\ell=1}^{Z_1} \omega_{t_\ell} \right] =: \tilde{c}_\ell.$$

A similar reasoning gives that:

$$\lim_{m\to+\infty}\sum_{j=1}^{\ell-1}\binom{\ell}{j}\mathbb{E}[A_{j,m}]=\lim_{m\to+\infty}\sum_{j=1}^{\ell-1}\binom{\ell}{j}\frac{1}{\mu^{m(\ell-j)}}\mathbb{E}\left[M_1\sum_{t_1+\ldots+t_{Z_1}=j}\binom{j}{t_1\cdots t_{Z_1}}\prod_{i=1}^{Z_1}\omega_{t_i}\right]=0,$$

implying that $\Delta_m\sim\mu^{m\ell}\tilde{c}_\ell$ when m goes to infinity. As a result:

$$\mathbb{E}[M_p^\ell]\;=\;\mu^p\sum_{m=0}^{p-1}\frac{\Delta_m}{\mu^{m+1}}\;\underset{p\to\infty}{\sim}\;\mu^{p-1}\sum_{m=0}^{p-1}\mu^{m(\ell-1)}\tilde{c}_\ell\;\underset{p\to\infty}{\sim}\;\mu^{\ell p}\tilde{c}_\ell(\mu^\ell\;-\;\mu)^{-1}\;=:\;\mu^{\ell p}\omega_\ell.$$

□

We deduce the limit of Theorem 6.2 in the super-critical case.

Theorem 9.6. *Let* $\mathbf{p}$ *be an offspring distribution which admits a moment of order* $\ell\in\mathbb{N}^*$ *such that* $\mu>1$ *and let* $\mathbf{q}$ *be a mark function satisfying* (6.2). *For every* $n\in\mathbb{N}$ *and every* $\Lambda_n\in\mathscr{F}_n$, *we have:*

$$\lim_{p\to+\infty}\frac{\mathbb{E}\left[\mathbb{1}_{\Lambda_n}M_{n+p}^\ell\right]}{\mathbb{E}\left[M_{n+p}^\ell\right]}=\mathbb{E}\left[\mathbb{1}_{\Lambda_n}\frac{P_\ell(Z_n)}{\mu^{\ell n}}\right],\tag{9.26}$$

where P_ℓ *is a polynomial function of degree* ℓ.

Proof. Let $p,n\in\mathbb{N}$ and note that (9.11) is equivalent to:

$$\frac{\mathbb{E}\left[M_{n+p}^\ell\middle|\mathscr{F}_n\right]}{\mu^{\ell(n+p)}}=\sum_{j=0}^{\ell}\binom{\ell}{j}\frac{M_n^{\ell-j}}{\mu^{(\ell-j)p+n\ell}}\sum_{t_1+\ldots+t_{Z_n}=j}\binom{j}{t_1\cdots t_{Z_n}}\prod_{i=1}^{Z_n}\frac{\mathbb{E}\left[M_p^{t_i}\right]}{\mu^{pt_i}}=\sum_{j=0}^{\ell}\binom{\ell}{j}D_{j,p}.$$

With the same notation and reasoning as those in the proof of Proposition 9.5, for all $j\in[\![0,\ell]\!]$:

$$D_{j,p}\le M_n^{\ell-j}\left(\frac{3}{2}\varpi_\ell Z_n\right)^j\le C\tilde{Z}_n^\ell,\tag{9.27}$$

where C is a strictly positive constant and $M_n\le\tilde{Z}_n:=\sum_{i=0}^{n}Z_i$, the total population up to generation n, belonging to L^ℓ according to Lemma 9.1.
Thanks to Proposition 9.5, almost surely:

$$\begin{aligned}\lim_{p\to+\infty}\sum_{j=0}^{\ell}\binom{\ell}{j}D_{j,p}&=\lim_{p\to+\infty}\sum_{j=0}^{\ell}\binom{\ell}{j}\frac{M_n^{\ell-j}}{\mu^{(\ell-j)p+n\ell}}\sum_{t_1+\ldots+t_{Z_n}=j}\binom{j}{t_1\cdots t_{Z_n}}\prod_{i=1}^{Z_n}\omega_{t_i}\\&=\frac{1}{\mu^{n\ell}}\sum_{t_1+\ldots+t_{Z_n}=\ell}\binom{j}{t_1\cdots t_{Z_n}}\prod_{i=1}^{Z_n}\omega_{t_i}.\end{aligned}\tag{9.28}$$

Using (9.27), (9.28) and Proposition 9.5, Lebesgue's dominated convergence theorem gives:

$$\lim_{p\to+\infty}\frac{\mathbb{E}\left[\mathbb{1}_{\Lambda_n}M_{n+p}^{\ell}\right]}{\mathbb{E}\left[M_{n+p}^{\ell}\right]}=\mathbb{E}\left[\mathbb{1}_{\Lambda_n}\frac{1}{\mu^{n\ell}\omega_\ell}\sum_{t_1+\ldots+t_{Z_n}=k}\binom{j}{t_1\cdots t_{Z_n}}\prod_{i=1}^{Z_n}\omega_{t_i}\right].$$

To prove that the sum in the right hand side of the above equation is indeed a polynomial function in Z_n, let us introduce for $1\le i\le\ell$, the set

$$S_{q,\ell}=\left\{\mathbf{n}=(n_1,\ldots,n_q)\in(\mathbb{N}^*)^q,\sum_{m=1}^{q}n_m=\ell\right\} \tag{9.29}$$

and for $\mathbf{n}\in S_{q,\ell}$, $z\in\mathbb{N}^*$,

$$A_{\mathbf{n},z}=\{(t_1,\ldots,t_z)\in\mathbb{N}^z,\ \{t_1,\ldots,t_z\}=\{n_1,\ldots,n_q,0\}\}.$$

Then, we have

$$\begin{aligned}\sum_{t_1+\ldots+t_{Z_n}=\ell}\binom{\ell}{t_1\cdots t_{Z_n}}\prod_{i=1}^{Z_n}\omega_{t_i}&=\sum_{q=1}^{\ell}\sum_{\mathbf{n}\in S_{q,\ell}}\sum_{(t_1,\ldots,t_{Z_n})\in A_{\mathbf{n},Z_n}}\binom{\ell}{t_1\cdots t_{Z_n}}\prod_{i=1}^{Z_n}\omega_{t_i}\\&=\sum_{q=1}^{\ell}\sum_{\mathbf{n}\in S_{q,\ell}}\binom{\ell}{n_1\cdots n_q}\left(\prod_{i=1}^{q}\omega_{n_i}\right)\mathrm{Card}(A_{\mathbf{n},Z_n})\end{aligned}$$

and $\mathrm{Card}(A_{\mathbf{n},Z_n})$ is clearly a polynomial of degree q in Z_n as it can be written $\binom{Z_n}{q}b_{\mathbf{n}}$ where $b_{\mathbf{n}}$ is a constant depending only of $\mathbf{n}$. □

Remark 9.7. *The martingale $\frac{P_\ell(Z_n)}{\mu^{\ell n}}$ already appears in [5]. By uniqueness property of Proposition 4.3 of [5], this is indeed the same martingale.*

Remark 9.8. *The (unmarked) random tree whose distribution is given by the change of probability using the martingale $P_\ell(Z_n)/\mu^{\ell n}$ is already described in Definition 4.4 of [5]. As this martingale does not depends on the marks, it is not difficult to see that the random marked tree associated (via the Girsanov transformation) with this martingale has the same mark function $\mathbf{q}$ that is:*

- *The unmarked tree is those of Definition 4.4 of [5]; it is a multi-type Galton-Watson tree, described as follows:*
  - *The types of the nodes run from 0 to ℓ.*
  - *The root is of type ℓ.*
  - *The sum of the the types of the offspring of one node is equal of the type of this node.*
- *Conditionally on this tree, all the nodes are marked independently of each others and a node u with k_u offspring has mark one with probability $\mathbf{q}(k_u)$.*

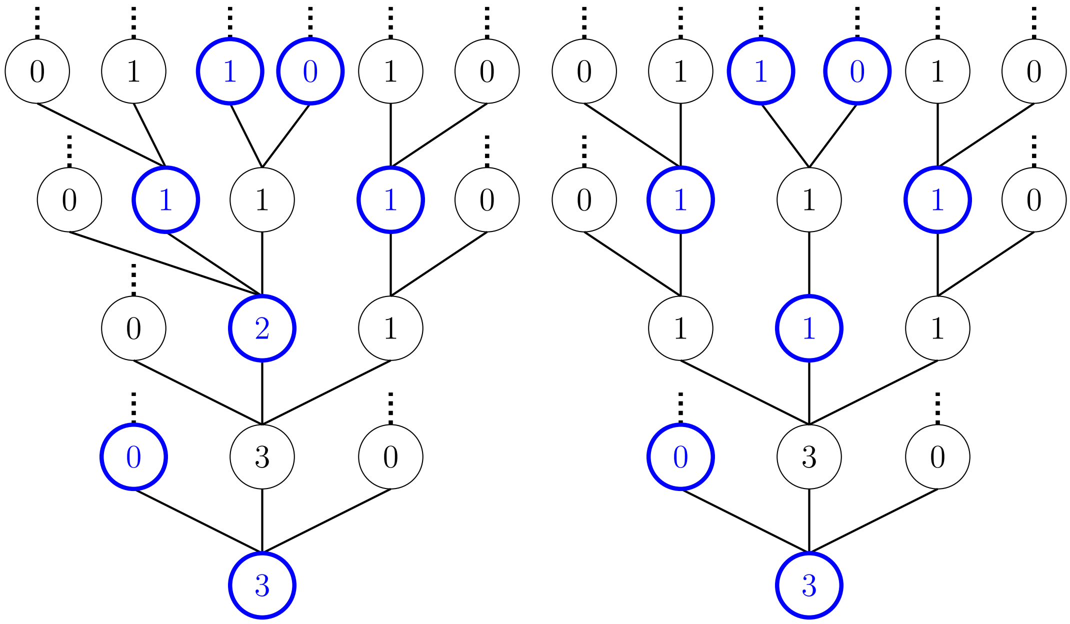


Figure 9.3 – Two examples of marked multi-type trees (marked in blue, unmarked in black), in the super-critical case for $\ell = 3$.

9.1.3.3 The critical case $\mu = 1$

As for the super-critical case, we determine the asymptotic behavior of $\mathbb{E}[M_p^\ell]$ when p goes to infinity in the critical case.

Proposition 9.6. *Let* $\mathbf{p}$ *be an offspring distribution satisfying* (6.1)*, that admits a moment of order* $\ell \in \mathbb{N}^*$ *and satisfies* $\mu = 1$*. Let* $\mathbf{q}$ *be a mark function satisfying* (6.2)*. Then, there exists a positive constant* $\tilde{\omega}_\ell$ *such that:*

$$\mathbb{E}[M_p^\ell] \underset{p\to\infty}{\sim} p^{2\ell-1}\tilde{\omega}_\ell. \tag{9.30}$$

Proof. The proof is similar to the proof of Proposition 9.5 . We have $\tilde{\omega}_0 = 1$. According to (9.5), (9.30) is true for $\ell = 1$ with $\tilde{\omega}_1 := \mathbb{E}[M_1]$. Now, assume that there exists ℓ such that (9.30) is true for all $1 \le i < \ell$, in other words:

$$\mathbb{E}\left[M_p^i\right] = p^{2i-1}\tilde{\omega}_i(1 + o(1)). \tag{9.31}$$

In order to obtain the equivalent for ℓ, we write:

$$\mathbb{E}\left[M_p^\ell\right] = \sum_{i=0}^{p-1}\mathbb{E}[M_{i+1}^\ell] - \mathbb{E}[M_i^\ell] = \sum_{i=0}^{p-1}\Delta_i,$$

and we study the telescopic series with general term Δ_i. According to Proposition 9.5 , more precisely (9.24) we obtain:

$$\frac{\Delta_m}{m^{2(\ell-1)}} = \sum_{j=0}^{\ell-1}\frac{\mathbb{E}\left[A_{j,m}\right]}{m^{2(\ell-1)}} + \frac{\mathbb{E}\left[B_m\right]}{m^{2(\ell-1)}},$$

with Δ_m, $A_{j,m}$ and B_m defined in the proof of Proposition 9.5 with $\mu = 1$ in our case.
Let $z > 1$. We set $\tilde{\omega}_0 = 1$ and $\tilde{\varpi}_\ell := \max_{0\le j<\ell} \tilde{\omega}_j$ and, for $t = (t_1, \dots, t_z)$ such that $\forall i,\ 0 \le t_i < \ell$ and $\sum_{i=1}^{z} t_i = \ell$, we set $N(t) := \sum_{i=1}^{z} \mathbb{1}_{t_i>0}$ the number of positive t_i. Thus, noting that $N(t) \ge 2$ for every term of the sum in B_m, and using (9.31) for m large enough:

$$\begin{aligned} 0 \le \frac{B_m}{m^{2(\ell-1)}} &= \sum_{\substack{t_1+\ldots+t_{Z_1}=\ell \\ t_i<\ell, \forall i}} \begin{pmatrix} \ell \\ t_1 \cdots t_{Z_1} \end{pmatrix} \frac{m^{2\ell-N(t)}}{m^{2(\ell-1)}} \prod_{i=1}^{Z_1} \tilde{\omega}_{t_i}(1+o(1)) \\ &\le \left(\frac{3}{2}\tilde{\varpi}_\ell\right)^{\ell} \sum_{t_1+\ldots+t_{Z_1}=\ell} \begin{pmatrix} \ell \\ t_1 \cdots t_{Z_1} \end{pmatrix} \le \left(\frac{3}{2}\tilde{\varpi}_\ell Z_1\right)^{\ell} \in \mathrm{L}^1. \end{aligned} \tag{9.32}$$

Note that the only terms in B_m that do not tend to 0 a.s. is when $N(t) = 2$. Then using (9.31) and (9.32), Lebesgue's dominated convergence theorem ensures that:

$$\lim_{m\to+\infty} \frac{\mathbb{E}[B_m]}{m^{2(\ell-1)}} = \mathbb{E}\left[\lim_{m\to+\infty} \frac{B_m}{m^{2(\ell-1)}}\right] = \mathbb{E}\left[\frac{Z_1(Z_1-1)}{2}\right] \sum_{\substack{t_1+t_2=\ell \\ t_i<\ell, \forall i}} \begin{pmatrix} \ell \\ t_1\ t_2 \end{pmatrix} \tilde{\omega}_{t_1}\tilde{\omega}_{t_2} =: \tilde{d}_\ell. \tag{9.33}$$

A similar reasoning gives that $\lim_{m\to+\infty} \frac{\mathbb{E}[A_{j,m}]}{m^{2(\ell-1)}} = 0$ and we can conclude with (9.33) that $\Delta_m \sim m^{2(\ell-1)}\tilde{d}_\ell$ when m goes to infinity. As a result:

$$\mathbb{E}\left[M_p^\ell\right] = \sum_{i=0}^{p-1} \Delta_i \underset{p\to+\infty}{\sim} \tilde{d}_\ell \sum_{i=0}^{p-1} i^{2(\ell-1)} \underset{p\to+\infty}{\sim} \frac{p^{2\ell-1}}{2\ell-1}\tilde{d}_\ell =: p^{2\ell-1}\tilde{\omega}_\ell.$$

For the last equivalent we use the following property

$$\sum_{i=0}^{p-1} i^m = \frac{1}{m+1} \sum_{i=0}^{m} \begin{pmatrix} m+1 \\ i \end{pmatrix} \mathscr{B}_i (p-1)^{m+1-i} \underset{p\to+\infty}{\sim} \frac{p^{m+1}}{m+1}, \tag{9.34}$$

with $2(\ell-1)$ instead of m and $\mathscr{B}_i$ are Bernoulli numbers, see [25]. □

We deduce Theorem 6.2 for the case $\mu = 1$:

Theorem 9.7. *Let* $\mathbf{p}$ *be an offspring distribution satisfying* (6.1) *that admits a moment of order* $\ell \in \mathbb{N}^*$ *and such that* $\mu = 1$ *and let* $\mathbf{q}$ *be a mark function satisfying* (6.2). *For every* $n \in \mathbb{N}$ *and for every* $\Lambda_n \in \mathscr{F}_n$, *we have:*

$$\lim_{p\to+\infty} \frac{\mathbb{E}\left[\mathbb{1}_{\Lambda_n} M_{n+p}^\ell\right]}{\mathbb{E}\left[M_{n+p}^\ell\right]} = \mathbb{E}[\mathbb{1}_{\Lambda_n} Z_n]. \tag{9.35}$$

Proof. Let $p, n \in \mathbb{N}$, here we write:

$$\begin{aligned} \frac{\mathbb{E}\left[M_p^\ell \middle| \mathscr{F}_n\right]}{p^{2\ell-1}} &= \sum_{j=0}^{\ell} \binom{\ell}{j} M_n^{\ell-j} \sum_{t_1+\ldots+t_{Z_n}=j} \begin{pmatrix} j \\ t_1 \cdots t_{Z_n} \end{pmatrix} \frac{1}{p^{2(\ell-j)-1+N(t)}} \frac{\prod_{i=1}^{Z_n} \mathbb{E}\left[M_{p-n}^{t_i}\right]}{p^{2j-N(t)}} \\ &=: \sum_{j=0}^{\ell} \binom{\ell}{j} M_n^{\ell-j} \sum_{t_1+\ldots+t_{Z_n}=j} \begin{pmatrix} j \\ t_1 \cdots t_{Z_n} \end{pmatrix} W_{j,N(t),p}, \end{aligned}$$

and (9.27) remains true replacing $\mu^{(\ell-j)p+nj}$ by $p^{2(\ell-j)-1+N(t)}$ giving us an upper bound independent of p.
Note that $W_{j,N(t),p}$ tends to 0 a.s. except for $j=\ell$ and $N(t)=1$ implying that, using (9.30):

$$\lim_{p\to+\infty}\frac{\mathbb{E}\left[M^{\ell}_{n+p}\middle|\mathscr{F}_n\right]}{p^{2\ell-1}}=\lim_{p\to+\infty}\frac{Z_n\mathbb{E}\left[M^{\ell}_{p}\right]}{p^{2\ell-1}}=Z_n\tilde{\omega}_\ell.$$

To conclude, as in Theorem 9.6, we use (9.30) again and the Lebesgue's dominated convergence to obtain (9.35). □

Remark 9.9. *As for the super-critical case, the martingale is a known one and the associated random marked tree is a size-biased tree (i.e. Kesten's tree, see [24]) associated with the offspring distribution* $\mathbf{p}$ *and with the mark function* $\mathbf{q}$ *being unchanged.*

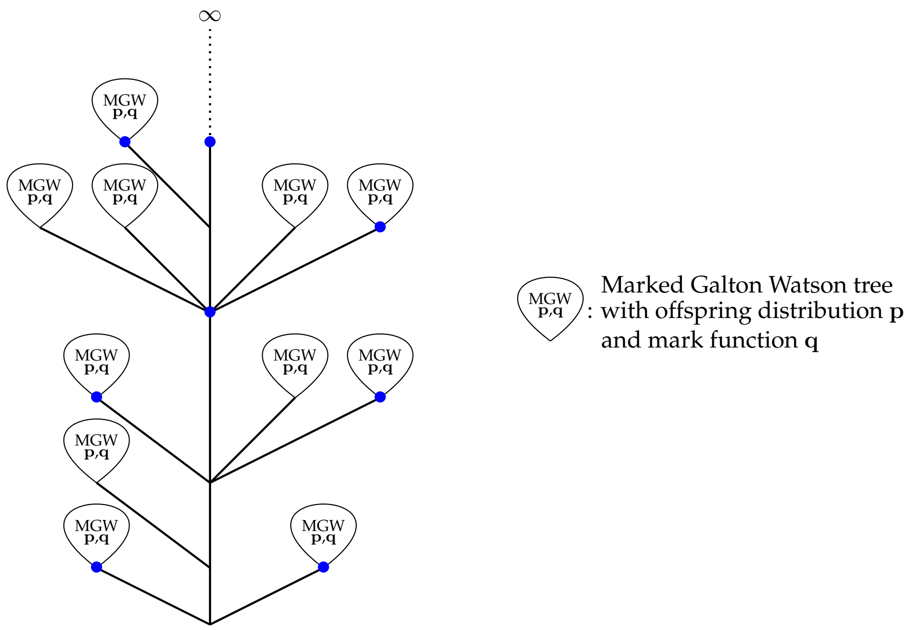


Figure 9.4 – Marked Kesten'tree

9.1.4 Penalization by $M_p^\alpha Z_p^\beta$

This part of the chapter doesn't appear in [3]. In fact, I wanted to find the martingales if we do a mixture of the considered functions in [5] and [3]. Since the results are already known, I decided to publish it only here and not in an article.

Thereby, in this Sub-section, we consider an offspring distribution $\mathbf{p}$ satisfying (6.1) and that admits a moment of order $\alpha+\beta$ with $\alpha,\beta\in\mathbb{N}^*$, and a mark function $\mathbf{q}$ satisfying (6.2). We denote by μ the mean of $\mathbf{p}$. We study the penalization by $M^\alpha_{n+p}Z^\beta_{n+p}$, i.e. we look for the limit for all $n\in\mathbb{N}$ and $\Lambda_n\in\mathscr{F}_n$ of

$$\lim_{p\to+\infty}\frac{\mathbb{E}[\mathbb{1}_{\Lambda_n}M^\alpha_{n+p}Z^\beta_{n+p}]}{\mathbb{E}[M^\alpha_{n+p}Z^\beta_{n+p}]}=\lim_{p\to+\infty}\frac{\mathbb{E}[\mathbb{1}_{\Lambda_n}\mathbb{E}[M^\alpha_{n+p}Z^\beta_{n+p}|\mathscr{F}_n]]}{\mathbb{E}[M^\alpha_{n+p}Z^\beta_{n+p}]}. \tag{9.36}$$

Remark that (6.2) implies that $\mathbb{E}[M_{n+p}^{\alpha}Z_{n+p}^{\beta}] \neq 0$ for $p \geq 1$. Thanks to Lemma 9.1 and the fact that $M_{n+p}^{\alpha}Z_{n+p}^{\beta} \leq \left(\sum_{i=0}^{p+n-1} Z_i\right)^{\alpha} Z_{n+p}^{\beta}$, $\mathbb{E}[M_{n+p}^{\alpha}Z_{n+p}^{\beta}]$ is finite.

We now give an expression of $\mathbb{E}\left[M_{n+p}^{\alpha}Z_{n+p}^{\beta}\middle|\mathscr{F}_n\right]$ that will be used throughout this Subsection. Recall the equality in distribution (7.5) and (7.10). Using the same notations, we get for all $p, n \in \mathbb{N}$:

$$\begin{aligned}\mathbb{E}\left[M_{n+p}^{\alpha}Z_{n+p}^{\beta} \mid \mathscr{F}_n\right] &= \sum_{j=0}^{\alpha}\binom{\alpha}{j}\mathbb{E}\left[M_n^{\alpha-j}\left(\sum_{i=1}^{Z_n}\tilde{M}_{i,p}\right)^j\left(\sum_{j=1}^{Z_n}\tilde{Z}_{j,p}\right)^{\beta} \,\middle|\, \mathscr{F}_n\right]\\ &= \sum_{j=0}^{\ell}\binom{\alpha}{j}M_n^{\alpha-j}\sum_{\substack{t_1+\cdots+t_{Z_n}=j\\ \tilde{t}_1+\cdots+\tilde{t}_{Z_n}=\beta}}\binom{j}{t_1\cdots t_{Z_n}}\binom{\beta}{\tilde{t}_1\cdots\tilde{t}_{Z_n}}\prod_{i=1}^{Z_n}\mathbb{E}[M_p^{t_i}Z_p^{\tilde{t}_i}]. \end{aligned} \tag{9.37}$$

We recall the expression of $\mathbb{E}[M_p]$ (9.5) and we give the one of $\mathbb{E}[Z_p^2]$:

$$\mathbb{E}[M_p] = \begin{cases}\mathbb{E}[\mathbf{q}(X)]\dfrac{1-\mu^p}{1-\mu}, & \text{if } \mu \neq 1,\\ p\mathbb{E}[\mathbf{q}(X)], & \text{otherwise.}\end{cases},$$

$$\mathbb{E}[Z_p^2] = \begin{cases}\mathrm{Var}[X]\mu^{p-1}\dfrac{1-\mu^p}{1-\mu} + \mu^{2p}, & \text{if } \mu \neq 1,\\ p\mathrm{Var}[X]+1, & \text{otherwise.}\end{cases}.$$

Moreover we have the following expression for $\mathbb{E}[M_pZ_p]$.

Proposition 9.7. *For all $p \in \mathbb{N}^*$*

$$\mathbb{E}[M_pZ_p] = \begin{cases} p\mu^{p-1}\mathbb{E}[X\mathbf{q}(X)] + \mu^p\mathbb{E}[\mathbf{q}(X)]\left(\dfrac{\mu-\mu^p}{1-\mu}\right. & \text{if } \mu \neq 1,\\ \left.+(p-1)\left(\dfrac{\mu^{-1}}{1-\mu}\mathrm{Var}[X]-1\right) - \dfrac{1-\mu^{p-1}}{(1-\mu)^2}\mathrm{Var}[X]\right), & \\ p\mathbb{E}[X\mathbf{q}(X)] + \dfrac{p(p-1)}{2}\mathrm{Var}[X]\mathbb{E}[\mathbf{q}(X)], & \text{otherwise,}\end{cases} \tag{9.38}$$

with $\mathbb{E}[M_1Z_1] = \mathbb{E}[X\mathbf{q}(X)]$.

Proof. We prove only the case $\mu = 1$, it is similar for the other case.
Let $p \in \mathbb{N}^*$. The expression is satisfied for $p = 1$.

$$\begin{aligned}\mathbb{E}[M_{p+1}Z_{p+1}] &= \mathbb{E}[M_pZ_{p+1}] + \mathbb{E}\left[\sum_{i=1}^{Z_p}\tilde{M}_{i,1}X_{i,1}^{(0)}\right] + \mathbb{E}\left[\sum_{i=1}^{Z_p}\sum_{j=1,j\neq i}^{Z_p}\tilde{M}_{i,1}X_{j,1}^{(0)}\right]\\ &= \mu\mathbb{E}[M_pZ_p] + \mu^p\mathbb{E}[X\mathbf{q}(X)] + \left(\mathbb{E}[Z_p^2]-\mu^p\right)\mu\mathbb{E}[\mathbf{q}(X)].\end{aligned}$$

We have by induction:

$$\begin{aligned}\mathbb{E}[M_{p+1}Z_{p+1}] &= \mathbb{E}[M_pZ_p] + \mathbb{E}[X\mathbf{q}(X)] + p\mathrm{Var}[X]\mathbb{E}[\mathbf{q}(X)]\\ &= p\mathbb{E}[X\mathbf{q}(X)] + \frac{p(p-1)}{2}\mathrm{Var}[X]\mathbb{E}[\mathbf{q}(X)] + \mathbb{E}[X\mathbf{q}(X)] + p\mathrm{Var}[X]\mathbb{E}[\mathbf{q}(X)]\\ &= (p+1)\mathbb{E}[X\mathbf{q}(X)] + \frac{p(p+1)}{2}\mathrm{Var}[X]\mathbb{E}[\mathbf{q}(X)].\end{aligned}$$

That concludes the proof. □

9.1.4.1 The sub-critical case : $\mu < 1$

Let us begin with giving an equivalent of $\mathbb{E}[M_p^\alpha Z_p^\beta]$ as p tends to infinity, with $\alpha, \beta \in \mathbb{N}^*$.

Proposition 9.8. *Let $\alpha, \beta \in \mathbb{N}^*$. Let $\mathbf{p}$ be an offspring distribution satisfying* (6.1) *which admits a moment of order $\alpha + \beta$ and such that $\mu < 1$, and let $\mathbf{q}$ be a mark function satisfying* (6.2). *Then, there exists a strictly positive constant $c_{\alpha,\beta}$ such that:*

$$\mathbb{E}[M_p^\alpha Z_p^\beta] \underset{p\to\infty}{\sim} \mu^p p^\alpha c_{\alpha,\beta}. \tag{9.39}$$

Proof. We prove this proposition by a double induction on α and β. We have $c_{0,0} = 1$. According to (9.38), Proposition 9.8 is true for $\alpha = \beta = 1$, with

$$c_{1,1} := \frac{\mathbb{E}[X\mathbf{q}(X)]}{\mu} + \mathbb{E}[\mathbf{q}(X)] \left(\frac{\mu^{-1}}{1-\mu} \mathrm{Var}[X] - 1 \right).$$

Now, assume that Proposition 9.8 is true for all $0 \le i \le \alpha$, $1 \le j \le \beta$, $(i,j) \neq (\alpha, \beta)$ for some $\alpha, \beta > 1$, in other words:

$$\frac{\mathbb{E}\left[M_p^i Z_p^j\right]}{p^i \mu^p} = c_{i,j}(1 + o(1)). \tag{9.40}$$

In order to obtain the equivalent for (α, β), we write:

$$\mathbb{E}[M_p^\alpha Z_p^\beta] = \mu^p \sum_{m=0}^{p-1} \left(\frac{\mathbb{E}[M_{m+1}^\alpha Z_{m+1}^\beta]}{\mu^{m+1}} - \frac{\mathbb{E}[M_m^\alpha Z_m^\beta]}{\mu^m} \right) =: \mu^p \sum_{m=0}^{p-1} \frac{\Delta_m}{\mu^{m+1}}, \tag{9.41}$$

and we study the telescopic series with general term $\dfrac{\Delta_m}{\mu^{m+1}}$. To begin with, taking the expectation in (9.37) with $n = 1$ and $p = m$, (and using again that $M_1^j = M_1$ for every $j \ge 1$), we obtain by isolating the case $j = \alpha$ and only one $\tilde{t}_i = \beta$:

$$\begin{aligned}
\frac{\Delta_m}{m^{\alpha-1}\mu^m} &= \sum_{j=0}^{\alpha-1} \binom{\alpha}{j} \mathbb{E}\left[M_1 \sum_{\substack{t_1+\cdots+t_{Z_1}=j \\ \tilde{t}_1+\cdots+\tilde{t}_{Z_1}=\beta}} \frac{m^{N(t,\tilde{t})}\mu^{mN(\tilde{t})}}{m^{\alpha-1}\mu^m} \binom{j}{t_1 \cdots t_{Z_1}} \binom{\beta}{\tilde{t}_1 \cdots \tilde{t}_{Z_1}} \prod_{i=1}^{Z_1} \frac{\mathbb{E}[M_m^{t_i} Z_m^{\tilde{t}_i}]}{m^{t_i \mathbb{1}_{t_i>0,\tilde{t}_i>0}} \mu^{m\mathbb{1}_{\tilde{t}_i>0}}} \right] \\
&\quad + \mathbb{E}\left[\sum_{\substack{t_1+\cdots+t_{Z_1}=\alpha \\ \tilde{t}_1+\cdots+\tilde{t}_{Z_1}=\beta \\ \forall i,\ (t_i,\tilde{t}_i)\neq(\alpha,\beta)}} \frac{m^{N(t,\tilde{t})}\mu^{mN(\tilde{t})}}{m^{\alpha-1}\mu^m} \binom{\alpha}{t_1 \cdots t_{Z_1}} \binom{\beta}{\tilde{t}_1 \cdots \tilde{t}_{Z_1}} \prod_{i=1}^{Z_1} \frac{\mathbb{E}[M_m^{t_i} Z_m^{\tilde{t}_i}]}{m^{t_i \mathbb{1}_{t_i>0,\tilde{t}_i>0}} \mu^{m\mathbb{1}_{\tilde{t}_i>0}}} \right] \\
&:= \sum_{j=0}^{\ell-1} \binom{m}{j} \mathbb{E}[A_{j,m}] + \mathbb{E}[B_m]
\end{aligned} \tag{9.42}$$

with $B_m = 0$ if $Z_1 \leq 1$.
Let $z > 1$. According to (9.13) $c_{t_i,0} = \xi_{t_i}$ and $\tilde{c}_{\alpha,\beta} := \max_{0\leq i<\alpha, 0\leq j<\beta} c_{i,j}$ and, for $t = (t_1, \ldots, t_z)$ and $\tilde{t} = (\tilde{t}_1, \ldots, \tilde{t}_z)$ such that $\forall i,\ 0 \leq t_i \leq j < \alpha$, $\forall i,\ 0 \leq \tilde{t}_i < \beta$ and $\sum_{i=1}^{z} t_i = j$, $\sum_{i=1}^{z} \tilde{t}_i = \beta$, we set $N(\tilde{t}) := \sum_{i=1}^{z} \mathbb{1}_{\tilde{t}_i>0}$ the number of positive $\tilde{t}_i$, and $N(t,\tilde{t}) := \sum_{i=1}^{z} t_i \mathbb{1}_{t_i>0,\tilde{t}_i>0}$ the sum of the t_i when we have a positive couple $(t_i, \tilde{t}_i)$. Note that $N(\tilde{t}) \geq \sum_{i=1}^{z} \mathbb{1}_{t_i>0,\tilde{t}_i>0} \geq 0$ and $1 \leq N(\tilde{t})$. Plus $N(t,\tilde{t}) = j$ if for all $t_i > 0$, $\tilde{t}_i$ is also positive.

In $A_{\alpha-1,m}$, using (9.40) for m large enough:

$$0 \leq A_{\alpha-1,m} \leq \left(\frac{3}{2}\tilde{c}_{\alpha,\beta}\right)^{\alpha+\beta} \sum\sum_{\substack{t_1+\cdots+t_{Z_1}=\alpha-1 \\ \tilde{t}_1+\cdots+\tilde{t}_{Z_1}=\beta}} \binom{\alpha-1}{t_1 \cdots t_{Z_1}} \binom{\beta}{\tilde{t}_1 \cdots \tilde{t}_{Z_1}} \leq \left(\frac{3}{2}\tilde{c}_{\alpha,\beta} Z_1\right)^{\alpha+\beta} \in L^1. \quad (9.43)$$

Note that the only terms in $A_{\alpha-1,m}$ that do not tend to 0 a.s. is when $N(\tilde{t}) = 1$ and $N(t,\tilde{t}) = \alpha - 1$. Using (9.40) and (9.43), Lebesgue's dominated convergence theorem ensures that:

$$\lim_{m\to+\infty} \mathbb{E}[A_{\alpha-1,m}] = \mathbb{E}\left[\lim_{m\to+\infty} A_{\alpha-1,m}\right] = \mathbb{E}\left[M_1 Z_1\right] c_{\alpha-1,\beta}.$$

A similar reasoning gives that $\lim_{m\to+\infty} \mathbb{E}[A_{j,m}] = 0$, for all $1 \leq j < \alpha - 1$ and

$$\lim_{m\to+\infty} \mathbb{E}[B_m] = \mathbb{E}\left[\lim_{m\to+\infty} B_m\right] = \mathbb{E}\left[Z_1(Z_1-1)\right] \alpha c_{\alpha-1,\beta} c_{1,0},$$

implying that $\Delta_m \sim m^{\alpha-1}\mu^m \tilde{r}_{\alpha,\beta}$, with $\tilde{r}_{\alpha,\beta} := \alpha c_{\alpha-1,\beta} c_{1,0} \mathbb{E}\left[Z_1(Z_1-1)\right] + \mathbb{E}\left[M_1 Z_1\right] c_{\alpha-1,\beta}$ when m goes to infinity. As a result:

$$\mathbb{E}[M_p^\alpha Z_p^\beta] = \mu^p \sum_{m=0}^{p-1} \frac{\Delta_m}{\mu^{m+1}} \underset{p\to\infty}{\sim} \mu^{p-1} \sum_{m=0}^{p-1} m^{\alpha-1} \tilde{r}_{\alpha,\beta} \underset{p\to\infty}{\sim} \mu^p p^\alpha c_{\alpha,\beta},$$

using the property (9.34).

□

We deduce the limit (9.36) in the sub-critical case.

Theorem 9.8. *Let* $\mathbf{p}$ *be an offspring distribution satisfying* (6.1) *which admits a moment of order* $\alpha + \beta$ *with*
$\alpha, \beta \in \mathbb{N}^*$ *such that* $\mu < 1$ *and let* $\mathbf{q}$ *be a mark function satisfying* (6.2). *For every* $n \in \mathbb{N}$ *and every* $\Lambda_n \in \mathscr{F}_n$, *we have:*

$$\lim_{p\to+\infty} \frac{\mathbb{E}\left[\mathbb{1}_{\Lambda_n} M_{n+p}^\alpha Z_{n+p}^\beta\right]}{\mathbb{E}\left[M_{n+p}^\alpha Z_{n+p}^\beta\right]} = \mathbb{E}\left[\mathbb{1}_{\Lambda_n} \frac{Z_n}{\mu^n}\right]. \quad (9.44)$$

Proof. Let $p, n \in \mathbb{N}$ and note that (9.37) is equivalent to:

$$\begin{aligned}\frac{\mathbb{E}\left[M_{n+p}^{\alpha} Z_{n+p}^{\beta} \mid \mathscr{F}_n\right]}{p^{\alpha}\mu^{n+p}} &= \sum_{j=0}^{\ell} \binom{\alpha}{j} M_n^{\alpha-j} \\ &\times \sum_{\substack{t_1+\cdots+t_{Z_n}=j \\ \tilde{t}_1+\cdots+\tilde{t}_{Z_n}=\beta}}\sum \frac{p^{N(t,\tilde{t})}\mu^{pN(\tilde{t})}}{p^{\alpha}\mu^{n+p}} \binom{j}{t_1 \cdots t_{Z_n}} \binom{\beta}{\tilde{t}_1 \cdots \tilde{t}_{Z_n}} \prod_{i=1}^{Z_n} \frac{\mathbb{E}[M_p^{t_i} Z_p^{\tilde{t}_i}]}{p^{t_i \mathbb{1}_{t_i>0,\tilde{t}_i>0}}\mu^{p\mathbb{1}_{\tilde{t}_i>0}}} \\ &= \sum_{j=0}^{\alpha} \binom{\ell}{\alpha} D_{j,p}.\end{aligned}$$

With the same notation and reasoning as those in the proof of Proposition 9.8 for all $j \in [\![0, \alpha]\!]$:

$$D_{j,p} \leq M_n^{\alpha-j} \left(\frac{3}{2}\tilde{c}_{\alpha,\beta} Z_n\right)^{j+\beta} \leq C\tilde{Z}_n^{\alpha+\beta}, \tag{9.45}$$

where C is a strictly positive constant and $M_n \leq \tilde{Z}_n := \sum_{i=0}^{n} Z_i$, the total population up to generation n, belonging to $L^{\alpha+\beta}$ according to Lemma 9.1.
Thanks to Proposition 9.8, almost surely:

$$\begin{aligned}\lim_{p\to+\infty} \sum_{j=0}^{\ell} \binom{\ell}{j} D_{j,p} &= \lim_{p\to+\infty} \sum_{j=0}^{\ell} \binom{\alpha}{j} M_n^{\alpha-j} \sum_{\substack{t_1+\cdots+t_{Z_n}=j \\ \tilde{t}_1+\cdots+\tilde{t}_{Z_n}=\beta}}\sum \frac{p^{N(t,\tilde{t})}\mu^{pN(\tilde{t})}}{p^{\alpha}\mu^{n+p}} \binom{j}{t_1 \cdots t_{Z_n}} \binom{\beta}{\tilde{t}_1 \cdots \tilde{t}_{Z_n}} \prod_{i=1}^{Z_n} c_{t_i,\tilde{t}_i} \\ &= \frac{Z_n}{\mu^n} c_{\alpha,\beta}.\end{aligned} \tag{9.46}$$

Using (9.45), (9.46) and Proposition 9.8, Lebesgue's dominated convergence theorem gives:

$$\lim_{p\to+\infty} \frac{\mathbb{E}\left[\mathbb{1}_{\Lambda_n} M_{n+p}^{\alpha} Z_{n+p}^{\beta}\right]}{\mathbb{E}\left[M_{n+p}^{\alpha} Z_{n+p}^{\beta}\right]} = \mathbb{E}\left[\mathbb{1}_{\Lambda_n} \frac{Z_n}{\mu^n}\right].$$

□

Remark 9.10. *As in Remark 9.9, we obtain a marked Kesten tree in the sub-critical.*

9.1.4.2 The critical case : $\mu = 1$

As for the sub-critical case, we determine the asymptotic behavior of $\mathbb{E}[M_p^{\alpha} Z_p^{\beta}]$ when p tends to infinity, with $\alpha, \beta \in \mathbb{N}^*$, in the critical case.

Proposition 9.9. *Let $\alpha, \beta \in \mathbb{N}^*$. Let $\mathbf{p}$ be an offspring distribution satisfying* (6.1) *which admits a moment of order $\alpha + \beta$ and such that $\mu = 1$, and let $\mathbf{q}$ be a mark function satisfying* (6.2). *Then, there exists a strictly positive constant $c_{\alpha,\beta}$ such that:*

$$\mathbb{E}[M_p^{\alpha} Z_p^{\beta}] \underset{p\to\infty}{\sim} p^{2\alpha+\beta-1} c_{\alpha,\beta}. \tag{9.47}$$

Proof. We prove this proposition by a double induction on α and β. We have $c_{0,0} = 1$. According to (9.38), (Proposition 9.9) is true for $\alpha = \beta = 1$, with $c_{1,1} := \dfrac{\mathrm{Var}[X]\mathbb{E}[\mathbf{q}(X)]}{2}$. Now, assume that (Proposition 9.9) is true for all $0 \le i \le \alpha$, $0 \le j \le \beta$, $(i,j) \notin \{(\alpha,\beta),(0,0)\}$ for some $\alpha, \beta > 1$, in other words:

$$\frac{\mathbb{E}\left[M_p^i Z_p^j\right]}{p^{2i+j-1}} = c_{i,j}(1+o(1)). \tag{9.48}$$

In order to obtain the equivalent for (α,β), we use again (9.41):

$$\mathbb{E}[M_p^\alpha Z_p^\beta] = \sum_{m=0}^{p-1} \Delta_m,$$

and we study the telescopic series with general term Δ_m. To begin with, taking the expectation in (9.37) with $n = 1$ and $p = m$, (and using again that $M_1^j = M_1$ for every $j \ge 1$), we obtain by isolating the case $j = \alpha$ and only one $\tilde{t}_i = \beta$:

$$\begin{aligned}
\frac{\Delta_m}{m^{2(\alpha-1)+\beta}} &= \sum_{j=0}^{\alpha-1}\binom{\alpha}{j} \qquad\qquad (9.49)\\
&\times \mathbb{E}\left[M_1 \sum_{\substack{t_1+\cdots+t_{Z_1}=j\\ \tilde{t}_1+\cdots+\tilde{t}_{Z_1}=\beta}} \frac{m^{2j+\beta-N(t,\tilde{t})}}{m^{2(\alpha-1)+\beta}}\binom{j}{t_1\cdots t_{Z_1}}\binom{\beta}{\tilde{t}_1\cdots\tilde{t}_{Z_1}}\prod_{i=1}^{Z_1}\frac{\mathbb{E}[M_m^{t_i}Z_m^{\tilde{t}_i}]}{m^{(2t_i+\tilde{t}_i-1)(\mathbb{1}_{t_i>0}+\mathbb{1}_{\tilde{t}_i>0}-\mathbb{1}_{t_i>0}\mathbb{1}_{\tilde{t}_i>0})}}\right]\\
&+ \mathbb{E}\left[\sum_{\substack{t_1+\cdots+t_{Z_1}=\alpha\\ \tilde{t}_1+\cdots+\tilde{t}_{Z_1}=\beta\\ \forall i\ (t_i,\tilde{t}_i)\neq(\alpha,\beta)}} \frac{m^{2\alpha+\beta-N(t,\tilde{t})}}{m^{2(\alpha-1)+\beta}}\binom{\alpha}{t_1\cdots t_{Z_1}}\binom{\beta}{\tilde{t}_1\cdots\tilde{t}_{Z_1}}\prod_{i=1}^{Z_1}\frac{\mathbb{E}[M_m^{t_i}Z_m^{\tilde{t}_i}]}{m^{(2t_i+\tilde{t}_i-1)(\mathbb{1}_{t_i>0}+\mathbb{1}_{\tilde{t}_i>0}-\mathbb{1}_{t_i>0}\mathbb{1}_{\tilde{t}_i>0})}}\right]\\
&:= \sum_{j=0}^{\ell-1}\binom{m}{j}\mathbb{E}[A_{j,m}] + \mathbb{E}[B_m] \qquad\qquad (9.50)
\end{aligned}$$

with $B_m = 0$ if $Z_1 \le 1$.
Let $z > 1$. We set $\tilde{c}_{\alpha,\beta} := \max_{0\le i<\alpha, 0\le j<\beta} c_{i,j}$ and, for $t = (t_1,\ldots,t_z)$ and $\tilde{t} = (\tilde{t}_1,\ldots,\tilde{t}_z)$ such that $\forall i,\ 0 \le t_i \le j < \alpha, \forall i,\ 0 \le \tilde{t}_i < \beta$ and $\sum_{i=1}^{z} t_i = j$, $\sum_{i=1}^{z}\tilde{t}_i = \beta$,
we set $N(t,\tilde{t}) := \sum_{i=1}^{z}\mathbb{1}_{t_i>0} + \mathbb{1}_{\tilde{t}_i>0} - \mathbb{1}_{t_i>0}\mathbb{1}_{\tilde{t}_i>0}$ the number of couple $(t_i,\tilde{t}_i) \neq (0,0)$.

Thus, noting that $N(t) \ge 2$ for every term of the sum in B_m, and using (9.48) for m large enough:

$$0 \le B_m \le \left(\frac{3}{2}\tilde{c}_{\alpha,\beta}\right)^{\alpha+\beta} \sum_{\substack{t_1+\cdots+t_{Z_1}=\alpha\\ \tilde{t}_1+\cdots+\tilde{t}_{Z_1}=\beta\\ \forall i\ (t_i,\tilde{t}_i)\neq(\alpha,\beta)}} \binom{\alpha}{t_1\cdots t_{Z_1}}\binom{\beta}{\tilde{t}_1\cdots\tilde{t}_{Z_1}} \le \left(\frac{3}{2}\tilde{c}_{\alpha,\beta}Z_1\right)^{\alpha+\beta} \in L^1. \tag{9.51}$$

Note that the only terms in B_m that do not tend to 0 a.s. is when $N(t,\tilde{t}) = 2$. Using (9.48) and (9.51), Lebesgue's dominated convergence theorem ensures that:

$$\begin{aligned}\lim_{m\to+\infty} \mathbb{E}[B_m] &= \mathbb{E}\left[\lim_{m\to+\infty} B_m\right] \\ &= \mathbb{E}\left[\sum_{\substack{t_1+t_2=\alpha \\ \tilde{t}_1+\tilde{t}_2=\beta \\ \forall i\ (t_i,\tilde{t}_i)\neq(\alpha,\beta)}}\sum \binom{\alpha}{t_1 t_2}\binom{\beta}{\tilde{t}_1\tilde{t}_2} c_{t_1,\tilde{t}_1} c_{t_2,\tilde{t}_2}\right] =: \tilde{r}_{\alpha,\beta}.\end{aligned}$$

A similar reasoning gives that $\lim_{m\to+\infty} \mathbb{E}[A_{j,m}] = 0$, for all $1 \le j < \alpha$ and implying that $\Delta_m \sim m^{2(\alpha-1)+\beta} r_{\alpha,\beta}$, when m goes to infinity. As a result:

$$\mathbb{E}[M_p^\alpha Z_p^\beta] = \sum_{m=0}^{p-1} \Delta_m \underset{p\to\infty}{\sim} \sum_{m=0}^{p-1} m^{2(\alpha-1)+\beta} r_{\alpha,\beta} \underset{p\to\infty}{\sim} p^{2\alpha+\beta-1} c_{\alpha,\beta},$$

using the property (9.34).

□

We deduce the limit (9.36) in the critical case.

Theorem 9.9. *Let* $\mathbf{p}$ *be an offspring distribution satisfying* (6.1) *which admits a moment of order* $\alpha+\beta$ *with*
$\alpha, \beta \in \mathbb{N}^*$ *such that* $\mu = 1$ *and let* $\mathbf{q}$ *be a mark function satisfying* (6.2). *For every* $n \in \mathbb{N}$ *and every* $\Lambda_n \in \mathscr{F}_n$*, we have:*

$$\lim_{p\to+\infty} \frac{\mathbb{E}\left[\mathbb{1}_{\Lambda_n} M_{n+p}^\alpha Z_{n+p}^\beta\right]}{\mathbb{E}\left[M_{n+p}^\alpha Z_{n+p}^\beta\right]} = \mathbb{E}\left[\mathbb{1}_{\Lambda_n} Z_n\right]. \tag{9.52}$$

Proof. Let $p, n \in \mathbb{N}$ and note that (9.37) is equivalent to:

$$\begin{aligned}\frac{\mathbb{E}\left[M_{n+p}^\alpha Z_{n+p}^\beta \mid \mathscr{F}_n\right]}{p^{2\alpha+\beta-1}} &= \sum_{j=0}^{\ell}\binom{\alpha}{j} M_n^{\alpha-j} \sum_{\substack{t_1+\cdots+t_{Z_n}=j \\ \tilde{t}_1+\cdots+\tilde{t}_{Z_n}=\beta}}\sum \frac{p^{2j+\beta-N(t,\tilde{t})}}{p^{2\alpha+\beta-1}}\binom{j}{t_1\cdots t_{Z_n}}\binom{\beta}{\tilde{t}_1\cdots\tilde{t}_{Z_n}} \\ &\times \prod_{i=1}^{Z_n} \frac{\mathbb{E}[M_p^{t_i} Z_p^{\tilde{t}_i}]}{p^{(2t_i+\tilde{t}_i-1)(\mathbb{1}_{t_i>0}+\mathbb{1}_{\tilde{t}_i>0}-\mathbb{1}_{t_i>0}\mathbb{1}_{\tilde{t}_i>0})}} \\ &= \sum_{j=0}^{\alpha}\binom{\ell}{\alpha} D_{j,p}.\end{aligned}$$

With the same notation and reasoning as those in the proof of Proposition 9.9 for all
$j \in [\![0,\alpha]\!]$:

$$D_{j,p} \le M_n^{\alpha-j}\left(\frac{3}{2}\tilde{c}_{\alpha,\beta} Z_n\right)^{j+\beta} \le C\tilde{Z}_n^{\alpha+\beta}, \tag{9.53}$$

where C is a strictly positive constant and $M_n \leq \tilde{Z}_n := \sum_{i=0}^{n} Z_i$, the total population up to generation n, belonging to $L^{\alpha+\beta}$ according to Lemma 9.1.
Thanks to Proposition 9.9, almost surely:

$$\begin{aligned}\lim_{p\to+\infty}\sum_{j=0}^{\ell}\binom{\ell}{j}D_{j,p} &= \lim_{p\to+\infty}\sum_{j=0}^{\ell}\binom{\alpha}{j}M_n^{\alpha-j}\sum_{\substack{t_1+\cdots+t_{Z_n}=j\\ \tilde{t}_1+\cdots+\tilde{t}_{Z_n}=\beta}}\sum\frac{p^{2j+\beta-N(t,\tilde{t})}}{p^{2\alpha+\beta-1}}\binom{j}{t_1\cdots t_{Z_n}}\binom{\beta}{\tilde{t}_1\cdots\tilde{t}_{Z_n}}\prod_{i=1}^{Z_n}c_{t_i,\tilde{t}_i}\\ &= Z_n c_{\alpha,\beta}.\end{aligned} \tag{9.54}$$

Using (9.53), (9.54) and Proposition 9.9, Lebesgue's dominated convergence theorem gives:

$$\lim_{p\to+\infty}\frac{\mathbb{E}\left[\mathbb{1}_{\Lambda_n}M_{n+p}^{\alpha}Z_{n+p}^{\beta}\right]}{\mathbb{E}\left[M_{n+p}^{\alpha}Z_{n+p}^{\beta}\right]} = \mathbb{E}\left[\mathbb{1}_{\Lambda_n}Z_n\right].$$

□

Remark 9.11. *The Remark 9.9, is the same in this case.*

9.1.4.3 The super-critical case : $\mu > 1$

As for the sub-critical and critical cases, we determine the asymptotic behavior of $\mathbb{E}[M_p^{\alpha}Z_p^{\beta}]$ when tends to infinity, with $\alpha, \beta \in \mathbb{N}^*$, in the super-critical case.

Proposition 9.10. *Let $\alpha, \beta \in \mathbb{N}^*$. Let $\mathbf{p}$ be an offspring distribution satisfying* (6.1) *which admits a moment of order $\alpha+\beta$ and such that $\mu > 1$, and let $\mathbf{q}$ be a mark function satisfying* (6.2). *Then, there exists a strictly positive constant $c_{\alpha,\beta}$ such that:*

$$\mathbb{E}[M_p^{\alpha}Z_p^{\beta}] \underset{p\to\infty}{\sim} \mu^{p(\alpha+\beta)}c_{\alpha,\beta}. \tag{9.55}$$

Proof. We prove this proposition by a double induction on α and β. We have $c_{0,0} = 1$. According to (9.38), Proposition 9.10 is true for $\alpha = \beta = 1$, with $c_{1,1} := \frac{\mathbb{E}[\mathbf{q}(X)]}{1-\mu} + \frac{\mathrm{Var}[X]}{\mu(1-\mu)^2}$.
Now, assume that Proposition 9.10 is true for all $0 \leq i \leq \alpha$, $0 \leq j \leq \beta$, $(i,j) \neq (\alpha,\beta)$ for some $\alpha, \beta > 1$, in other words:

$$\frac{\mathbb{E}\left[M_p^i Z_p^j\right]}{\mu^{p(i+j)}} = c_{i,j}(1+o(1)). \tag{9.56}$$

In order to obtain the equivalent for (α,β), we recall (9.41):

$$\mathbb{E}[M_p^{\alpha}Z_p^{\beta}] = \mu^p\sum_{m=0}^{p-1}\left(\frac{\mathbb{E}[M_{m+1}^{\alpha}Z_{m+1}^{\beta}]}{\mu^{m+1}} - \frac{\mathbb{E}[M_m^{\alpha}Z_m^{\beta}]}{\mu^m}\right) = \mu^p\sum_{m=0}^{p-1}\frac{\Delta_m}{\mu^{m+1}},$$

and we study the telescopic series with general term $\frac{\Delta_m}{\mu^{m+1}}$. To begin with, taking the expectation in (9.37) with $n = 1$ and $p = m$, (and using again that $M_1^j = M_1$ for every $j \geq 1$), we obtain by isolating the case $j = \alpha$ and only one $\tilde{t}_i = \beta$:

$$
\begin{aligned}
\frac{\Delta_m}{\mu^{m(\alpha+\beta)}} &= \sum_{j=0}^{\alpha-1} \binom{\alpha}{j} \mathbb{E}\left[\frac{M_1}{\mu^{m(\alpha-j)}} \sum\sum_{\substack{t_1+\cdots+t_{Z_1}=j \\ \tilde{t}_1+\cdots+\tilde{t}_{Z_1}=\beta}} \binom{j}{t_1 \cdots t_{Z_1}} \binom{\beta}{\tilde{t}_1 \cdots \tilde{t}_{Z_1}} \prod_{i=1}^{Z_1} \frac{\mathbb{E}[M_m^{t_i} Z_m^{\tilde{t}_i}]}{\mu^{m(t_i+\tilde{t}_i)}} \right] \\
&\quad + \mathbb{E}\left[\sum\sum_{\substack{t_1+\cdots+t_{Z_1}=\alpha \\ \tilde{t}_1+\cdots+\tilde{t}_{Z_1}=\beta \\ \forall i,\ (t_i,\tilde{t}_i)\neq(\alpha,\beta)}} \binom{\alpha}{t_1 \cdots t_{Z_1}} \binom{\beta}{\tilde{t}_1 \cdots \tilde{t}_{Z_1}} \prod_{i=1}^{Z_1} \frac{\mathbb{E}[M_m^{t_i} Z_m^{\tilde{t}_i}]}{\mu^{m(t_i+\tilde{t}_i)}} \right] \\
&:= \sum_{j=0}^{\ell-1} \binom{m}{j} \mathbb{E}[A_{j,m}] + \mathbb{E}[B_m]
\end{aligned} \tag{9.57}
$$

with $B_m = 0$ if $Z_1 \leq 1$.

We set $\tilde{c}_{\alpha,\beta} := \max_{0\leq i<\alpha, 0\leq j<\beta} c_{i,j}$, in B_m, we have at most $\alpha+\beta$ couple $(t_i, \tilde{t}_i) \neq (0,0)$, and using (9.56) for m large enough:

$$
0 \leq B_m \leq \left(\frac{3}{2}\tilde{c}_{\alpha,\beta}\right)^{\alpha+\beta} \sum\sum_{\substack{t_1+\cdots+t_{Z_1}=\alpha-1 \\ \tilde{t}_1+\cdots+\tilde{t}_{Z_1}=\beta}} \binom{\alpha-1}{t_1 \cdots t_{Z_1}} \binom{\beta}{\tilde{t}_1 \cdots \tilde{t}_{Z_1}} \leq \left(\frac{3}{2}\tilde{c}_{\alpha,\beta} Z_1\right)^{\alpha+\beta} \in L^1. \tag{9.58}
$$

Using (9.56) and (9.58), Lebesgue's dominated convergence theorem ensures that:

$$
\begin{aligned}
\lim_{m\to+\infty} \mathbb{E}[B_m] &= \mathbb{E}\left[\lim_{m\to+\infty} B_m\right] \\
&= \mathbb{E}\left[\sum\sum_{\substack{t_1+\cdots+t_{Z_1}=\alpha \\ \tilde{t}_1+\cdots+\tilde{t}_{Z_1}=\beta \\ \forall i,\ (t_i,\tilde{t}_i)\neq(\alpha,\beta)}} \binom{\alpha}{t_1 \cdots t_{Z_1}} \binom{\beta}{\tilde{t}_1 \cdots \tilde{t}_{Z_1}} \prod_{i=1}^{Z_1} c_{t_i,\tilde{t}_i} \right] =: \tilde{r}_{\alpha,\beta}.
\end{aligned}
$$

A similar reasoning gives that $\lim_{m\to+\infty} \mathbb{E}[A_{j,m}] = 0$, for all $1 \leq j < \alpha$ and implying that $\Delta_m \sim \mu^{m(\alpha+\beta)} \tilde{r}_{\alpha,\beta}$ when m goes to infinity. As a result:

$$
\mathbb{E}[M_p^\alpha Z_p^\beta] = \mu^p \sum_{m=0}^{p-1} \frac{\Delta_m}{\mu^{m+1}} \underset{p\to\infty}{\sim} \mu^{p-1} \sum_{m=0}^{p-1} \mu^{m(\alpha+\beta)-1} \tilde{r}_{\alpha,\beta} \underset{p\to\infty}{\sim} \mu^{p(\alpha+\beta)} c_{\alpha,\beta}.
$$

□

We deduce the limit (9.36) in the super-critical case.

Theorem 9.10. *Let* $\mathbf{p}$ *be an offspring distribution satisfying* (6.1) *which admits a moment of order* $\alpha+\beta$ *with* $\alpha, \beta \in \mathbb{N}^*$ *such that* $\mu > 1$ *and let* $\mathbf{q}$ *be a mark function satisfying* (6.2). *For every* $n \in \mathbb{N}$ *and every* $\Lambda_n \in \mathscr{F}_n$*, we have:*

$$\lim_{p\to+\infty} \frac{\mathbb{E}\left[\mathbb{1}_{\Lambda_n} M_{n+p}^{\alpha} Z_{n+p}^{\beta}\right]}{\mathbb{E}\left[M_{n+p}^{\alpha} Z_{n+p}^{\beta}\right]} = \mathbb{E}\left[\mathbb{1}_{\Lambda_n} \frac{P_{\alpha+\beta}(Z_n)}{\mu^{n(\alpha+\beta)}}\right], \tag{9.59}$$

where $P_{\alpha+\beta}$ *is a polynomial function of degree* $\alpha+\beta$.

Proof. Let $p, n \in \mathbb{N}$ and note that (9.37) is equivalent to:

$$\begin{aligned}
\frac{\mathbb{E}\left[M_{n+p}^{\alpha} Z_{n+p}^{\beta} \mid \mathscr{F}_n\right]}{\mu^{(n+p)(\alpha+\beta)}} &= \sum_{j=0}^{\ell} \binom{\alpha}{j} M_n^{\alpha-j} \sum_{\substack{t_1+\cdots+t_{Z_n}=j \\ \tilde{t}_1+\cdots+\tilde{t}_{Z_n}=\beta}}\sum \frac{\mu^{p(j+\beta)}}{\mu^{(n+p)(\alpha+\beta)}} \binom{j}{t_1 \cdots t_{Z_n}} \binom{\beta}{\tilde{t}_1 \cdots \tilde{t}_{Z_n}} \prod_{i=1}^{Z_n} \frac{\mathbb{E}[M_p^{t_i} Z_p^{\tilde{t}_i}]}{\mu^{p(t_i+\tilde{t})}} \\
&= \sum_{j=0}^{\alpha} \binom{\ell}{\alpha} D_{j,p}.
\end{aligned}$$

With the same notation and reasoning as those in the proof of Proposition 9.10 for all $j \in [\![0, \alpha]\!]$:

$$D_{j,p} \leq M_n^{\alpha-j} \left(\frac{3}{2} \tilde{c}_{\alpha,\beta} Z_n\right)^{j+\beta} \leq C \tilde{Z}_n^{\alpha+\beta}, \tag{9.60}$$

where C is a strictly positive constant and $M_n \leq \tilde{Z}_n := \sum_{i=0}^{n} Z_i$, the total population up to generation n, belonging to $L^{\alpha+\beta}$ according to Lemma 9.1.
Thanks to Proposition 9.10, almost surely:

$$\begin{aligned}
\lim_{p\to+\infty} \sum_{j=0}^{\ell} \binom{\ell}{j} D_{j,p} &= \lim_{p\to+\infty} \sum_{j=0}^{\ell} \binom{\alpha}{j} M_n^{\alpha-j} \sum_{\substack{t_1+\cdots+t_{Z_n}=j \\ \tilde{t}_1+\cdots+\tilde{t}_{Z_n}=\beta}}\sum \frac{\mu^{p(j+\beta)}}{\mu^{(n+p)(\alpha+\beta)}} \binom{j}{t_1 \cdots t_{Z_n}} \binom{\beta}{\tilde{t}_1 \cdots \tilde{t}_{Z_n}} \prod_{i=1}^{Z_n} c_{t_i,\tilde{t}_i} \\
&= \sum_{\substack{t_1+\cdots+t_{Z_1}=\alpha \\ \tilde{t}_1+\cdots+\tilde{t}_{Z_1}=\beta}}\sum \frac{1}{\mu^{n(\alpha+\beta)}} \binom{\alpha}{t_1 \cdots t_{Z_n}} \binom{\beta}{\tilde{t}_1 \cdots \tilde{t}_{Z_n}} \prod_{i=1}^{Z_n} c_{t_i,\tilde{t}_i}. \qquad (9.61)
\end{aligned}$$

Using (9.60), (9.61) and Proposition 9.8, Lebesgue's dominated convergence theorem gives:

$$\begin{aligned}
\lim_{p\to+\infty} \frac{\mathbb{E}\left[\mathbb{1}_{\Lambda_n} M_{n+p}^{\alpha} Z_{n+p}^{\beta}\right]}{\mathbb{E}\left[M_{n+p}^{\alpha} Z_{n+p}^{\beta}\right]} &= \mathbb{E}\left[\mathbb{1}_{\Lambda_n} \sum_{\substack{t_1+\cdots+t_{Z_n}=\alpha \\ \tilde{t}_1+\cdots+\tilde{t}_{Z_n}=\beta}}\sum \frac{1}{c_{\alpha,\beta}\mu^{n(\alpha+\beta)}} \binom{\alpha}{t_1 \cdots t_{Z_n}} \binom{\beta}{\tilde{t}_1 \cdots \tilde{t}_{Z_n}} \prod_{i=1}^{Z_n} c_{t_i,\tilde{t}_i}\right] \\
&= \mathbb{E}\left[\mathbb{1}_{\Lambda_n} \frac{P_{\alpha+\beta}(Z_n)}{\mu^{n(\alpha+\beta)}}\right].
\end{aligned}$$

□

Remark 9.12. *We can do the same remarks of Remark 9.7 and Remark 9.8, but with* $\ell = \alpha+\beta$.

9.2 Exponential penalization

We now consider a penalization of the form $s^{M_n}t^{Z_n}$ with $s,t \in [0,1]$. Contrary to the previous study, this function gives weight to trees with a small number of marks.

The case $s=1$ has already been studied in [5].

To begin with, let us set for $p \geq 1$, $s,t \in [0,1]$, $\varphi_p(s,t) = \mathbb{E}[s^{M_p}t^{Z_p}]$. For $s \in (0,1)$ fixed, we consider the function f_s defined for $t \in [0,1]$ by

$$f_s(t) = \varphi_1(s,t) = \sum_{k=0}^{+\infty} \mathbf{p}(k)t^k(s\mathbf{q}(k) + 1 - \mathbf{q}(k)).$$

Proposition 9.11. *We have:*

$$\varphi_p(s,t) = f_s^p(t), \tag{9.62}$$

with $f^p = f \circ \ldots \circ f$ is the p-th composition of f.

Proof. We prove this result by induction on p. For $p=1$ we have, $\varphi_1(s,t) = f_s(t) = f_s^1(t)$. Thus the assertion is checked.

Let $p \in \mathbb{N}^*$ such that $\varphi_p(s,t) = f_s^p(t)$, we have:

$$\begin{aligned}
\varphi_{p+1}(s,t) &= \mathbb{E}\left[s^{M_{p+1}}t^{Z_{p+1}}\right] = \mathbb{E}\left[s^{M_p + \sum\limits_{i=1}^{Z_p}\tilde{M}_{i,1}} t^{\sum\limits_{i=1}^{Z_p} X_{i,p}}\right] = \mathbb{E}\left[\mathbb{E}\left[s^{M_p} s^{\sum\limits_{i=1}^{Z_p}\tilde{M}_{i,1}} t^{\sum\limits_{i=1}^{Z_p} X_{i,p}} \,\middle|\, \mathscr{F}_p\right]\right] \\
&= \mathbb{E}\left[s^{M_p}\mathbb{E}\left[s^{M_1}t^{Z_1}\right]^{Z_p}\right] = \varphi_p(s,\varphi_1(s,t)) = \varphi_p(s,f_s(t)) \\
&= f_s^p(f_s(t)) = f_s^{p+1}(t).
\end{aligned}$$

That concludes the proof. □

As f_s is $\mathscr{C}^\infty$ on $[0,1)$, increasing, convex and satisfies $0 \leq f_s \leq 1$, $f_s(1) < 1$, the function f_s admits a unique fixed point denoted $\kappa(s)$ on $[0,1]$. Indeed:

Proposition 9.12. *f_s has an unique fixed point on $[0,1]$, denoted by $\kappa(s)$.*

Proof. We suppose the distribution of X satisfies (6.1). In the other case, 0 is a fixed point. Since f_s is increasing we have for all $t \in [0,1]$:

$$0 < f_s(0) \leq f_s(t) \leq f_s(1) < 1.$$

Let $h(t) := f_s(t) - t$, since f_s is $\mathcal{C}^\infty([0,1])$, it is the same for h. Moreover, $h(0) = f_s(0) - 0 > 0$ and $h(1) = f_s(1) - 1 < 0$, according to the intermediate value theorem, it exists $x \in]0,1[$, such that $h(x) = 0$, thus $f_s(x) = x$.

We suppose it exists two fixed points denoted by x_1 and x_2, such that $x_1 < x_2$, the case $x_2 < x_1$ is similar. Since f_s is convex we can apply the "inégalité des pentes" to $x_1 < x_2 < 1$, and since $x_2 < f_s(1) < 1$, we obtain:

$$1 \leq \frac{f_s(1) - x_1}{1 - x_1} \leq \frac{f_s(1) - x_2}{1 - x_2} < 1.$$

Thereby $x_1 = x_2$ and f_s has an unique fixed point. □

Using the Proposition 9.11, we obtain,

Lemma 9.4. *Le* $s \in (0,1)$ *we have:*

$$\forall t \in [0,1], \lim_{p\to+\infty} \mathbb{E}[s^{M_p} t^{Z_p}] = \lim_{p\to+\infty} f_s^p(t) = \kappa(s).$$

Proof. f_s is defined as follow:

$$f_s(t) \begin{cases} > t & \text{if } t \in [\,0, \kappa(s)\,[, \\ = t & \text{if } t = \kappa(s), \\ < t & \text{if } t \in]\,\kappa(s), 1\,]. \end{cases}$$

We define the sequence $(u_n)_{n\in\mathbb{N}}$, such that $u_0 \in [0,1]$ and $u_{n+1} = f_s(u_n)$, then $u_n = \varphi_n(s, u_0)$.
If $u_0 = \kappa(s)$, u_n is stationary and the limit is $\kappa(s)$.
If $u_0 < \kappa(s)$, then $u_n \in [\,0, \kappa(s)\,[$ for all $n \in \mathbb{N}$.
Indeed, it is true for u_0, if we suppose it il also true for u_n, we have by increasing of f_s

$$u_{n+1} = f(u_n) \in [f_s(u_0), f_s(\kappa(s))[\subset [\,0, \kappa(s)\,[.$$

By induction, it is true for u_{n+1}. Thereby, by definition of f_s, we have $u_{n+1} = f_s(u_n) > u_n$. (u_n) is increasing and increased, so it converges to $\ell \in [\,0, \kappa(s)\,]$. When n goes to infinity in $u_{n+1} = f_s(u_n)$, we have $f_s(\ell) = \ell$, thereby, by uniqueness of the fixed point, we have $\ell = \kappa(s)$.
If $u_0 > \kappa(s)$, then $u_n \in]\,\kappa(s), 1\,]$ for all $n \in \mathbb{N}$. the proof is similar of the case $u_0 < \kappa(s)$ Thereby, by definition of f_s, we have $u_{n+1} = f_s(u_n) < u_n$. (u_n) is decreasing and reduced, so it converges to $\ell \in [\,0, \kappa(s)\,]$. We have $\ell = \kappa(s)$, in the same way of the preceding case.
That concludes the proof. □

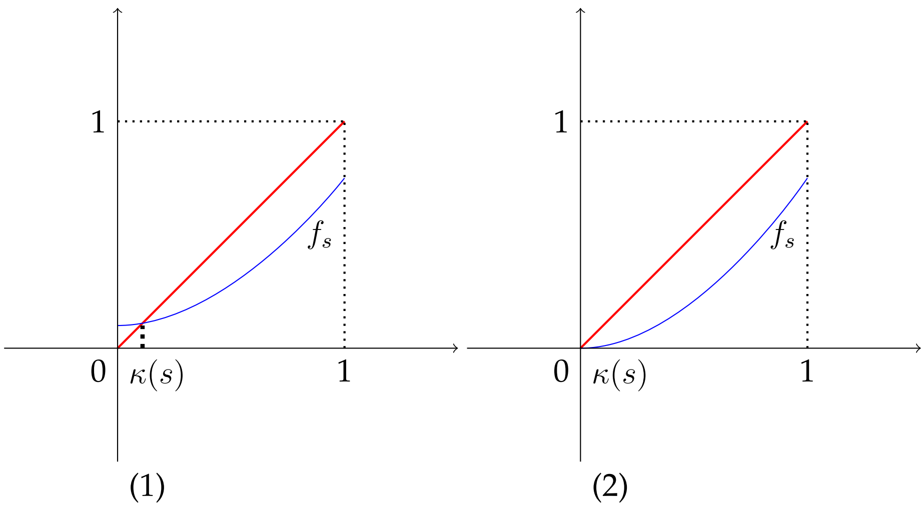


Figure 9.5 – Examples of the function f_s and its fixed point $\kappa(s)$. Case $\mathbf{p}(0) > 0$ (1) and case $\mathbf{p}(0) = 0$ (2).

Note that $\kappa(s) = 0$ if and only if $\mathbf{p}(0) = 0$. Therefore, we must study separately the two cases $\mathbf{p}(0) > 0$ and $\mathbf{p}(0) = 0$.

9.2.1 Case $\mathbf{p}(0) > 0$

Theorem 9.11. *1. Let $\mathbf{p}$ be an offspring distribution satisfying* (6.1) *and such that $\mathbf{p}(0) > 0$, and let $\mathbf{q}$ be a mark function satisfying* (6.2)*. For every $t \in [0,1]$ and $s \in (0,1)$, for every $n \in \mathbb{N}$, $\Lambda_n \in \mathscr{F}_n$, we have:*

$$\frac{\mathbb{E}\left[\mathbb{1}_{\Lambda_n} s^{M_{n+p}} t^{Z_{n+p}}\right]}{\mathbb{E}\left[s^{M_{n+p}} t^{Z_{n+p}}\right]} \underset{p\to+\infty}{\longrightarrow} \mathbb{E}[\mathbb{1}_{\Lambda_n} s^{M_n} \kappa(s)^{Z_n-1}]. \tag{9.63}$$

2. The probability measure $\mathbb{Q}_{s,1}$ defined on $(\mathbb{T}^*, \mathscr{F})$ by

$$\forall n \in \mathbb{N}, \Lambda_n \in \mathscr{F}_n, \quad \mathbb{Q}_{s,1}(\Lambda_n) = \mathbb{E}\left[\mathbb{1}_{\Lambda_n} s^{M_n} \kappa(s)^{Z_n-1}\right],$$

is the distribution of a MGW with reproduction-marking law $\mathbf{p}_{s,0}$ defined by :

$$\forall k \in \mathbb{N},\ \forall \eta \in \{0,1\}, \quad \mathbf{p}_{s,0}(k,\eta) = \mathbf{p}_0(k,\eta) s^{\eta} \kappa(s)^{k-1} \tag{9.64}$$

where $\mathbf{p}_0(k,\eta)$ is defined by (7.9)*.*

Proof. 1. Thanks to the branching property and (7.10), for $p, n \in \mathbb{N}$, we have:

$$\mathbb{E}\left[s^{M_{n+p}} t^{Z_{n+p}} | \mathscr{F}_n\right] = s^{M_n} \mathbb{E}\left[s^{M_p} t^{Z_p}\right]^{Z_n} = s^{M_n} f_s^p(t)^{Z_n}. \tag{9.65}$$

As $s^{M_n} f_s^p(t)^{Z_n} \leq 1$, by dominated convergence theorem and according to Lemma 9.4 , we obtain:

$$\lim_{p\to+\infty} \frac{\mathbb{E}\left[\mathbb{1}_{\Lambda_n} s^{M_{n+p}} t^{Z_{n+p}}\right]}{\mathbb{E}\left[s^{M_{n+p}} t^{Z_{n+p}}\right]} = \lim_{p\to+\infty} \frac{\mathbb{E}\left[\mathbb{1}_{\Lambda_n} s^{M_n} f_s^p(t)^{Z_n}\right]}{f_s^{n+p}(t)} \qquad = \mathbb{E}\left[\mathbb{1}_{\Lambda_n} s^{M_n} \kappa(s)^{Z_n-1}\right].$$

2. Let $\tau_{s,1}$ be a MGW with the reproduction-marking law $\mathbf{p}_{s,0}$ defined on some probability space $(\Omega, \mathscr{A}, P)$. According to (9.1), to obtain our result, it suffices to prove that:

$$\forall \mathbf{t}^* \in \mathbb{T}^*, \forall n \in \mathbb{N}, \quad \mathbb{Q}_{s,1}(r_n^*(\tau^*) = r_n^*(\mathbf{t}^*)) = P(r_n^*(\tau_{s,1}) = r_n^*(\mathbf{t}^*)).$$

Let $\mathbf{t}^* = (\mathbf{t}, (\eta_u)_{u\in\mathbf{t}}) \in \mathbb{T}^*$. As $\sum_{u\in r_{n-1}(\mathbf{t})} (k_u(\mathbf{t}) - 1) = Z_n(\mathbf{t}^*) - 1$, we obtain our result since:

$$\begin{aligned}
P(r_n^*(\tau_{s,1}) = r_n^*(\mathbf{t}^*)) &= \prod_{u\in r_{n-1}(\mathbf{t})} \mathbf{p}_{s,0}(k_u(\mathbf{t}), \eta_u(\mathbf{t})) \\
&= \prod_{u\in r_{n-1}(\mathbf{t})} \mathbf{p}_0(k_u(\mathbf{t}), \eta_u(\mathbf{t})) \kappa(s)^{k_u(\mathbf{t})-1} s^{\eta_u(\mathbf{t})} \\
&= s^{M_n(\mathbf{t}^*)} \kappa(s)^{\sum_{u\in r_{n-1}(\mathbf{t})} k_u(\mathbf{t})-1} \prod_{u\in r_{n-1}(\mathbf{t})} \mathbf{p}_0(k_u(\mathbf{t}), \eta_u(\mathbf{t})) \\
&= s^{M_n(\mathbf{t}^*)} \kappa(s)^{Z_n(\mathbf{t}^*)-1} \mathbb{P}(r_n^*(\tau^*) = r_n^*(\mathbf{t}^*)) \\
&= \mathbb{Q}_{s,1}(r_n^*(\tau^*) = r_n^*(\mathbf{t}^*)).
\end{aligned}$$

$\square$

Remark 9.13. *Note that , we already obtained this reproduction-marking law. Indeed, when we do the conditioning with a change of measure in Section 8.2. It is sufficient to consider $s = c_\theta$ and $\theta = \kappa(s)$ in* (6.3).

Remark 9.14. *By definition of $\mathbf{p}_{s,0}$* (9.64)*, the greater the number of children, the lower the probability. Then contrary to Section 9.1 which privilege trees with a large number of children and a large number of marks. Here, we privilege trees with a less number of marks and a less number of individuals.*

As in [5], the previous penalization can be generalized by looking at the penalization $H_\ell(Z_p)s^{M_p}t^{Z_p}$ where H_ℓ is the ℓ-th Hilbert polynomial defined for $\ell \geq 1$ by

$$H_\ell(x) = \frac{1}{\ell!}\prod_{j=0}^{\ell-1}(x-j).$$

Recall the definition of the set $S_{i,k}$ of (9.29), for $k \leq i \in \mathbb{N}$:

$$S_{i,k} := \left\{(n_1,\dots,n_i) \in (\mathbb{N}^*)^i, \sum_{j=1}^{i} n_j = k\right\}.$$

Using Faà di Bruno's formula, we have for any functions f, h of class $\mathscr{C}^k$:

$$\begin{aligned}
\sum_{i=1}^{k}\frac{k!}{i!}f^{(i)}(h(s))\sum_{\substack{(n_1,\dots,n_i)\\ \in S_{i,k}}}\prod_{j=1}^{i}\frac{h^{(n_j)}(s)}{n_j!} &= \sum_{i=1}^{k}\frac{k!}{i!}f^{(i)}(h(s))\sum_{\substack{m_1+\dots+m_k=i,\\ \sum_{j=1}^{k} jm_j=k,\\ m_j\geq 0\ \forall j}}\binom{i}{m_1\cdots m_k}\prod_{j=1}^{k}\left(\frac{h^{(j)}(s)}{j!}\right)^{m_j}\\
&= \sum_{i=1}^{k}\sum_{\substack{m_1+\dots+m_k=i\\ \sum_{j=1}^{k} jm_j=k,\ m_j\geq 0\ \forall j}}\frac{k!}{m_1!\dots m_k!}f^{(i)}(h(s))\prod_{j=1}^{k}\left(\frac{h^{(j)}(s)}{j!}\right)^{m_j}\\
&= \sum_{\sum_{j=1}^{k} jm_j=k}\binom{k}{m_1\cdots m_k}f^{(m_1+\dots+m_k)}(h(s))\prod_{j=1}^{k}\left(\frac{h^{(j)}(s)}{j!}\right)^{m_j}\\
&= (f\circ h)^{(k)}(s).
\end{aligned}\tag{9.66}$$

We first need asymptotics for the derivatives of the p-th composition of f_s as p tends to infinity.

Lemma 9.5. *Let $\mathbf{p}$ be an offspring distribution satisfying* (6.1)*, and that admits a moment or order $\ell \in \mathbb{N}^*$, and let $\mathbf{q}$ be a mark function satisfying* (6.2)*. Then, for all $s \in (0,1)$, there exists a positive function C_ℓ such that for all $t \in [0,1]$:*

$$(f_s^p)^{(\ell)}(t) \underset{p\to+\infty}{\sim} f_s'(\kappa(s))^p C_\ell(t). \tag{9.67}$$

Proof. Note that $f_s'(\kappa(s)) \in (0,1)$ since f_s is convex, increasing on $[0,1]$, $f_s(0) > 0$ and by (6.2) $f_s(1) < 1$.

To prove (9.67) for $\ell = 1$, we use a similar proof to that in [8] p.38. We study the function

$$Q_p(t) := f_s'(\kappa(s))^{-p}\left(f_s^p(t) - \kappa(s)\right). \tag{9.68}$$

This function can be derived and we have:

$$Q_p'(t) = \frac{(f_s^p)'(t)}{f_s'(\kappa(s))^p} = \prod_{j=0}^{p-1} \frac{f_s'(f_s^j(t))}{f_s'(\kappa(s))} = \prod_{j=0}^{p-1}\left(1 + \left(\frac{f_s'(f_s^j(t))}{f_s'(\kappa(s))} - 1\right)\right).$$

We fix $t \in [0,1]$ and show first that $Q_p'(t)$ admits a positive limit $Q'(t)$ when p goes to infinity. The positivity is clear as $f_s^j(t) \in [0,1)$ implying that $f_s'(f_s^j(t)) > 0$.
The existence of this limit is equivalent to the convergence of the series with general term $a_j := |f_s'(\kappa(s)) - f_s'(f_s^j(t))|$.
Let $\varepsilon > 0$ be such that $0 < \kappa(s) + \varepsilon < 1$ and let $\gamma := f_s'(\kappa(s)+\varepsilon) < 1$. According to Lemma 9.4, there exists j_0 such that for all $j \geq 0$, $f_s^{j+j_0}(t) < \kappa(s) + \varepsilon$. Let $j \in \mathbb{N}$. Since f_s'' is increasing and $\max(\kappa(s), f_s^{j+j_0}(t)) < \kappa(s) + \varepsilon$, thanks to the mean value inequality, we have:

$$a_{j+j_0} = |f_s'(\kappa(s)) - f_s'(f_s^{j+j_0}(t))| \leq f_s''(\kappa(s)+\varepsilon)|\kappa(s) - f_s^{j+j_0}(t)|.$$

With the same arguments applied to f_s' and using the fact that $f_s(\kappa(s)) = \kappa(s)$:

$$|\kappa(s) - f_s^{j+j_0}(t)| \leq \gamma |f_s^{j+j_0-1}(\kappa(s)) - f_s^{j+j_0-1}(t)|.$$

By iteration, we obtain: $a_{j+j_0} \leq f_s''(\kappa(s)+\varepsilon)\gamma^j$, since $0 < \gamma < 1$, we have $\displaystyle\sum_{j=0}^{\infty} a_j < +\infty$.

Therefore, $\frac{(f_s^p)'(t)}{f_s'(\kappa(s))^p} \underset{p\to+\infty}{\longrightarrow} Q'(t)$, implying (9.67) for $\ell = 1$ with $C_1(t) = Q'(t)$.

The proof is similar of the induction step of Lemma 3.2 in [5]. We assume that (9.67) is true for all $j \leq \ell - 1$. We use Faà di Bruno's Formula (9.66), and we obtain:

$$\frac{(f_s^{p+1})^{(\ell)}(t)}{(f_s^{p+1})'(t)} - \frac{(f_s^p)^{(\ell)}(t)}{(f_s^p)'(t)} = \frac{1}{(f_s^{p+1})'(t)} \sum_{i=2}^{\ell} \frac{\ell!}{i!} f_s^{(i)}(f_s^p(t)) \sum_{(n_1,\ldots,n_i)\in S_{i,\ell}} \prod_{j=1}^{i} \frac{(f_s^p)^{(n_j)}(t)}{n_j!} \tag{9.69}$$

Since for every $2 \leq i \leq \ell$ and every $(n_1,\ldots,n_i) \in S_{i,\ell}$, $n_j < \ell$ for all $j \leq i$, we can use the induction hypothesis:

$$\sum_{(n_1,\ldots,n_i)\in S_{i,\ell}} \prod_{j=1}^{i} \frac{(f_s^p)^{(n_j)}(t)}{n_j!} \underset{p\to+\infty}{\sim} \sum_{(n_1,\ldots,n_i)\in S_{i,\ell}} \prod_{i=1}^{i} \frac{f_s'(\kappa(s))^p C_{n_j}(t)}{n_j!} =: f_s'(\kappa(s))^{pi} K_i(t), \tag{9.70}$$

with K_i a positive function.

Lemma 9.4 and the continuity of $f_s^{(j)}$ imply that

$$\lim_{p\to+\infty} f_s^{(j)}(f_s^p(t)) = f_s^{(j)}(\kappa(s))$$

for all $j \geq 1$.

Moreover, since $f_s'(\kappa(s)) < 1$, Equations (9.69) and (9.70) imply that:

$$\frac{(f_s^{p+1})^{(\ell)}(t)}{(f_s^{p+1})'(t)} - \frac{(f_s^{p})^{(\ell)}(t)}{(f_s^{p})'(t)} \underset{p\to+\infty}{\sim} \frac{\ell!}{2!} f_s''(\kappa(s)) \frac{f_s'(\kappa(s))^{2p}}{(f_s^{p+1})'(t)} K_2(t) \qquad \underset{p\to+\infty}{\sim} f_s'(\kappa(s))^p \tilde{K}(t),$$

with $\tilde{K}(t)$ a positive function.

Since $0 < f_s'(\kappa(s)) < 1$, we have on $[0,1]$:

$$0 \ < \ \tilde{C}_\ell(t) \ := \ \lim_{p\to+\infty} \frac{(f_s^p)^{(\ell)}(t)}{(f_s^p)'(t)} \ = \ \frac{(f_s)^{(\ell)}(t)}{f_s'(t)} + \sum_{p\geq 1} \left(\frac{(f_s^{p+1})^{(\ell)}(t)}{(f_s^{p+1})'(t)} - \frac{(f_s^{p})^{(\ell)}(t)}{(f_s^{p})'(t)} \right) \ < \ +\infty.$$

The previous result is equivalent to $(f_s^p)^{(\ell)}(t) \underset{p\to+\infty}{\sim} \tilde{C}_\ell(t)\,(f_s^p)'(t)$, and we conclude with the induction hypothesis and $C_\ell(t) := C_1(t)\tilde{C}_\ell(t)$. □

We can now show the results of the limit with the penalization function $H_\ell(Z_p)s^{M_p}t^{Z_p-\ell}$.

Theorem 9.12. *Let* $\mathbf{p}$ *be an offspring distribution satisfying* (6.1) *and that admits a moment of order* $\ell \in \mathbb{N}^*$*, and let* $\mathbf{q}$ *be a mark function satisfying* (6.2). *Let* $n \in \mathbb{N}$, $\Lambda_n \in \mathscr{F}_n$, $t \in [0,1]$ *and* $s \in (0,1)$ *fixed, we have*

$$\frac{\mathbb{E}\left[\mathbb{1}_{\Lambda_n} H_\ell(Z_{n+p}) s^{M_{n+p}} t^{Z_{n+p}}\right]}{\mathbb{E}\left[H_\ell(Z_{n+p}) s^{M_{n+p}} t^{Z_{n+p}}\right]} \underset{p\to+\infty}{\longrightarrow} \mathbb{E}\left[\mathbb{1}_{\Lambda_n} \frac{s^{M_n} Z_n \kappa(s)^{Z_n-1}}{f_s'(\kappa(s))^n}\right].$$

Proof. The proof is inspired by those of Theorem 3.3 in [5].

Let $s \in (0,1)$ be fixed. Let $p, n \in \mathbb{N}$, according to (9.65) and Faà di Bruno's formula (9.66), we have for every $t \in [0,1]$:

$$\begin{aligned}
\mathbb{E}[H_\ell(Z_p)s^{M_{n+p}}t^{Z_{n+p}-\ell}|\mathscr{F}_n] &= \frac{s^{M_n}}{\ell!}\frac{\partial^\ell}{\partial t^\ell}\left(f_s^p(t)^{Z_n}\right) \\
&= s^{M_n}\sum_{i=1}^{\ell} H_i(Z_n) f_s^p(t)^{Z_n-i} \sum_{(n_1,\dots,n_i)\in S_{i,\ell}} \prod_{j=1}^{i} \frac{(f_s^p)^{(n_j)}(t)}{n_j!} \\
&= s^{M_n}\sum_{i=1}^{\ell} H_i(Z_n) f_s^p(t)^{Z_n-i} R_i(t). \qquad (9.71)
\end{aligned}$$

Applying (9.67), as $0 < f_s'(\kappa(s)) < 1$, we obtain for all $i \geq 1$ and $t \in [0,1]$, as tends to infinity,

$$\frac{R_i(t)}{f'(\kappa(s))^p} = c_i(t) f_s'(\kappa(s))^{p(i-1)} + o\left(f_s'(\kappa(s))^{p(i-1)}\right) \qquad (9.72)$$

for some strictly positive function c_i. Thereby, using Lemma 9.4 in (9.71), we get a.s.:

$$\lim_{p\to+\infty} \frac{\mathbb{E}\left[H_\ell(Z_{p+n})s^{M_{p+n}}t^{Z_{p+n}-\ell}|\mathscr{F}_n\right]}{f'(\kappa(s))^p} = \frac{c_1(t)}{\ell!} s^{M_n} Z_n \kappa(s)^{Z_n-1}.$$

Note that (9.72) implies the existence of a positive function C, such that, for all $t \in [0,1]$:

$$\max_{1\leq i\leq \ell} \frac{R_i(t)}{f'(\kappa(s))^p} \leq C(t),$$

and since $s < 1$, $f_s^p(t) < 1$ for all $t \in [0,1]$:

$$0 \le \frac{\mathbb{E}\left[H_\ell(Z_{p+n}) s^{M_{p+n}} t^{Z_{p+n}-\ell} | \mathscr{F}_n\right]}{f'(\kappa(s))^p} \le C(t) \sum_{i=1}^{\ell} H_i(Z_n).$$

As $H_i(Z_n)$ is integrable, we have by dominated convergence:

$$\lim_{p\to+\infty} \frac{\mathbb{E}\left[\mathbb{1}_{\Lambda_n} H_\ell(Z_{p+n}) s^{M_{p+n}} t^{Z_{p+n}-\ell}\right]}{f'(\kappa(s))^p} = \mathbb{E}\left[\mathbb{1}_{\Lambda_n} \frac{c_1(t)}{\ell!} s^{M_n} Z_n \kappa(s)^{Z_n-1}\right]. \tag{9.73}$$

Note that, since in the case $i = 1$, there is only one term in $R_i(t)$, according to Lemma 9.5, we have $c_1(t) = C_\ell(t)$.

Furthermore, according to Lemma 9.5,

$$\lim_{p\to+\infty} \frac{\mathbb{E}\left[H_\ell(Z_{p+n}) s^{M_{p+n}} t^{Z_{p+n}-\ell}\right]}{f'(\kappa(s))^p} = \lim_{p\to+\infty} \frac{(f_s^{p+n})^\ell(t)}{\ell! f'(\kappa(s))^p} = f'(\kappa(s))^n \frac{c_1(t)}{\ell!}.$$

We obtain our result taking taking the ratio of these two quantities. □

In order to describe the new probability $\mathbb{Q}_{s,2}$ defined on $(\mathbb{T}^*, \mathscr{F})$ by:

$$\forall \Lambda_n \in \mathscr{F}_n,\ \mathbb{Q}_{s,2}(\Lambda_n) = \mathbb{E}\left[\mathbb{1}_{\Lambda_n} \frac{s^{M_n} Z_n \kappa(s)^{Z_n-1}}{f_s'(\kappa(s))^n}\right], \tag{9.74}$$

we need to introduce the following marked random tree:

Definition 9.2. *A special random marked tree is a multi-type random marked tree $\tau_{s,2}$ described as follows:*

- *the nodes are normal or special;*
- *the root is special;*
- *the reproduction-marking law of a normal node is $\mathbf{p}_{s,0}$ (see* (9.64)*);*
- *the reproduction-marking law of a special node is*

$$\mathbf{p}_{s,1}(k,\eta) = \mathbf{p}_{s,0}(k,\eta) \frac{k}{f_s'(\kappa(s))};$$

- *at each generation, conditionally on the number of individuals at that generation, there is a unique special node chosen uniformly among all the individuals of this generation.*

Remark 9.15. *Contrary to the Kesten tree, here, the special node can be choose uniformly among all the individuals. So there isn't a special lineage.*

Using the same arguments as in the proof of Theorem 9.5, we have the following result:

Theorem 9.13. *The probability $\mathbb{Q}_{s,2}$ defined by* (9.74) *is the distribution of a special random marked tree defined in Definition 9.2.*

9.2.2 Case $\mathbf{p}(0) = 0$

Henceforth, we consider the case where $\mathbf{p}(0) = 0$ implying $\kappa(s) = 0$. We denote by $r := \min\{j > 0;\ \mathbf{p}(j) > 0\}$ and $\alpha_r(s) := \mathbf{p}(r)(s\mathbf{q}(r) + 1 - \mathbf{q}(r))$. We begin to give an equivalent of Lemma 9.5 in this case. It permits to prove the result about the penalization with $H_k(Z_p)s^{M_p}t^{Z_p-k}$ for all $k \in \mathbb{N}$.

Lemma 9.6. *Assume $r \geq 2$. For every $\ell \in \mathbb{N}$, every $s \in (0,1)$ and every $t \in (0,1]$, there exists a positive function $K_{s,\ell}(t)$, such that, as $p \to +\infty$,*

$$(f_s^p)^{(\ell)}(t) = K_{s,\ell}(t) r^{p\ell} e^{r^p b(t)}(1 + o(1)), \tag{9.75}$$

where

$$b(t) := \ln(t) + \sum_{j=0}^{+\infty} r^{-(j+1)} \ln\left(\frac{f_s^{j+1}(t)}{f_s^j(t)^r}\right).$$

Proof. The case $\ell = 0$ is inspired by the proof of Lemma 10 in [17]. First, note that for all $t \in (0,1]$,

$$f_s(t) = \mathbb{E}\left[s^{M_1}t^{Z_1}\right] \leq \mathbb{E}\left[s^{M_1}\right] =: \gamma(s) \in (0,1).$$

f_s being non decreasing, we have:

$$f_s^2(t) \leq f_s(\gamma(s)) = \sum_{j \geq r} \gamma(s)^j \alpha_j(s) \leq \gamma(s)^r,$$

and with an obvious induction reasoning $f_s^p(t) \leq \gamma(s)^{r^{p-1}}$ for all $p \in \mathbb{N}^*$.

Moreover, as for $0 < t \leq 1$, $f_s^p(t) \neq 0$, we can write:

$$\frac{f_s^{p+1}(t)}{\alpha_r(s) f_s^p(t)^r} = 1 + \sum_{j \geq 1} \frac{\alpha_{j+r}(s)}{\alpha_r(s)} f_s^p(t)^j, \tag{9.76}$$

implying:

$$0 \leq \frac{f_s^{p+1}(t)}{\alpha_r(s) f_s^p(t)^r} - 1 \leq f_s^p(t) \sum_{j \geq 1} \frac{\alpha_{j+r}(s)}{\alpha_r(s)} \leq \gamma(s)^{r^{p-1}} \sum_{j \geq 1} \frac{\alpha_{j+r}(s)}{\alpha_r(s)} =: \gamma(s)^{r^{p-1}} C_r. \tag{9.77}$$

As $\ln(1+u) \leq u$ for every nonnegative u, we have:

$$\ln\left(\frac{f_s^{p+1}(t)}{\alpha_r(s) f_s^p(t)}\right) \leq \frac{f_s^{p+1}(t)}{\alpha_r(s) f_s^p(t)^r} - 1 \leq \gamma(s)^{r^{p-1}} C_r,$$

and therefore, $b : (0,1] \to \mathbb{R}$ is well defined since:

$$-\frac{\ln(\alpha_r(s))}{r-1} + \sum_{p \geq 0} \frac{1}{r^{p+1}} \ln\left(\frac{f_s^{p+1}(t)}{f_s^p(t)^r}\right) = \sum_{p \geq 0} \frac{1}{r^{p+1}} \ln\left(\frac{f_s^{p+1}(t)}{\alpha_r(s) f_s^p(t)^r}\right) < \infty.$$

Using the telescopic property of the series:

$$b(t) - \sum_{j \geq p} \frac{1}{r^{j+1}} \ln\left(\frac{f_s^{j+1}(t)}{f_s^j(t)^r}\right) = \ln t + \sum_{j=0}^{p-1} \frac{1}{r^{j+1}} \ln\left(\frac{f_s^{j+1}(t)}{f_s^j(t)^r}\right) = \frac{1}{r^p} \ln(f_s^p(t)).$$

and

$$\sum_{j\geq p}\frac{1}{r^{j+1}}\ln\left(\frac{f_s^{j+1}(t)}{f_s^j(t)^r}\right)=\ln(\alpha_r(s))\frac{r^{-p}}{r-1}+\sum_{j\geq p}\frac{1}{r^{j+1}}\ln\left(\frac{f_s^{j+1}(t)}{\alpha_r(s)f_s^j(t)^r}\right)$$
$$=\ln(\alpha_r(s))\frac{r^{-p}}{r-1}+O\left(\frac{1}{r^p}\gamma(s)^{r^{p-1}}\right).$$

As a result:

$$\ln(f_s^p(t))=r^pb(t)-\frac{\ln(\alpha_r(s))}{r-1}+O\left(\gamma(s)^{r^{p-1}}\right),$$

giving:

$$f_s^p(t)=\alpha_r(s)^{\frac{-1}{r-1}}e^{r^pb(t)}(1+o(1))=:K_{s,0}(t)e^{r^pb(t)}(1+o(1)). \tag{9.78}$$

The result for $\ell=1$ is very similar to the previous one. As $Z_1\in L^1$, for all $s\in(0,1)$, $f_s\in\mathscr{C}^1((0,1])$, and for $t\in(0,1]$, we have:

$$\left(f_s^{p+1}\right)'(t)=\left(f_s^p\right)'(t)f_s'(f_s^p(t))=\left(f_s^p\right)'(t)\sum_{j\geq0}(r+j)\alpha_{j+r}f_s^p(t)^{r+j-1}.$$

This implies with (9.78):

$$0\leq\frac{(f_s^{p+1})'(t)}{(f_s^p)'(t)r\alpha_r(s)f_s^p(t)^{r-1}}-1=\sum_{j\geq1}\frac{(r+j)\alpha_{j+r}(s)}{r\alpha_r(s)}f_s^p(t)^j$$
$$\leq f_s^p(t)\sum_{j\geq1}\frac{(r+j)\alpha_{j+r}(s)}{r\alpha_r(s)}\leq\widetilde{K}_s(t)e^{r^pb(t)},$$

where $\tilde{K}_s$ is a positive function. Using again that, for all $u>0$, $\ln(1+u)<u$, we have:

$$\sum_{p\geq0}\ln\left(\frac{(f_s^{p+1})'(t)}{(f_s^p)'(t)r\alpha_r(s)f_s^p(t)^{r-1}}\right)<\infty.$$

Thanks to Equation (9.78), we have:

$$\ln\left(\frac{(f_s^{p+1})'(t)}{(f_s^p)'(t)r\alpha_r(s)f_s^p(t)^{r-1}}\right)\underset{p\to+\infty}{\sim}\ln\left(\frac{(f_s^{p+1})'(t)}{(f_s^p)'(t)re^{r^p(r-1)b(t)}}\right),$$

thereby, it gives:

$$\tilde{b}(t):=\sum_{p\geq0}\ln\left(\frac{(f_s^{p+1})'(t)}{(f_s^p)'(t)re^{r^p(r-1)b(t)}}\right)=\sum_{p\geq0}\ln\left(\frac{(f_s^{p+1})'(t)r^{-p-1}e^{-r^{p+1}b(t)}}{(f_s^p)'(t)r^{-p}e^{-r^pb(t)}}\right)<\infty.$$

$\tilde{b}(t)$ being telescopic, we have:

$$\ln\left(\frac{\left(f_s^p\right)'(t)}{r^pe^{(r^p-1)b(t)}}\right)=\tilde{b}(t)-\sum_{j\geq p}\ln\left(\frac{(f_s^{j+1})'(t)}{(f_s^j)'(t)re^{r^j(r-1)b(t)}}\right)=\tilde{b}(t)+o(1), \tag{9.79}$$

and as a result:

$$\left(f_s^p\right)'(t) = r^p e^{(r^p-1)b(t)} e^{\tilde{b}(t)}(1+o(1)), \tag{9.80}$$

which is the claimed result for $\ell = 1$ with $K_{s,1} := e^{\tilde{b}(t)-b(t)}$.
We conclude the proof by induction on ℓ: assume that there exists $\ell \geq 2$ such that (9.75) is true for $j < \ell$ and recall (9.69):

$$\frac{(f_s^{p+1})^{(\ell)}(t)}{(f_s^{p+1})'(t)} - \frac{(f_s^{p})^{(\ell)}(t)}{(f_s^{p})'(t)} = \sum_{i=2}^{\ell} \frac{\ell!}{i!} f_s^{(i)}(f_s^p(t)) \sum_{(n_1,\dots,n_i)\in S_{i,\ell}} \frac{1}{(f_s^{p+1})'(t)} \prod_{j=1}^{i} \frac{(f_s^p)^{(n_j)}(t)}{n_j!}.$$

For all $1 \leq i \leq \ell$, according to the induction hypothesis, there exists a positive function $C_{i,s}$ such that:

$$\sum_{(n_1,\dots,n_i)\in S_{i,\ell}} \frac{1}{(f_s^{p+1})'(t)} \prod_{j=1}^{i} \frac{(f_s^p)^{(n_j)}(t)}{n_j!} \sim C_{i,s}(t) r^{p(\ell-1)} e^{r^p(i-r)b(t)}. \tag{9.81}$$

Note that when t goes to 0:

$$f_s^{(i)}(t) = \begin{cases} t^{r-i}\frac{r!}{(r-i)!}\alpha_r(s) + o(t^{r-i}), & \text{if } i < r \\ i!\alpha_i(s) + o(1), & \text{otherwise.} \end{cases}$$

Consequently, as $f_s^p(t)$ goes to 0 when p goes to infinity, using (9.81):

$$\begin{aligned} \frac{(f_s^{p+1})^{(\ell)}(t)}{(f_s^{p+1})'(t)} - \frac{(f_s^{p})^{(\ell)}(t)}{(f_s^{p})'(t)} &\sim r^{p(\ell-1)} \sum_{i=2}^{r-1} \frac{\ell!}{i!} \left(f_s^p(t)\right)^{r-i} \frac{r!}{(r-i)!} \alpha_r(s) C_{i,s}(t) e^{r^p(i-r)b(t)} \\ &\quad + r^{p(\ell-1)} \sum_{i=r}^{\ell} \ell! \alpha_i(s) C_{i,s}(t) e^{r^p(i-r)b(t)} \\ &\sim r^{p(\ell-1)} \alpha_r(s) \ell! \sum_{i=2}^{r} \binom{r}{i} K_{s,0}(t)^{r-i} C_{i,s}(t) =: \breve{K}_s(t) r^{p(\ell-1)}, \end{aligned}$$

where $\breve{K}_s$ is a positive function.
Since $r^{\ell-1} > 1$, the series diverge and the partial sums are equivalent, which gives:

$$\frac{\left(f_s^p\right)^{(\ell)}(t)}{\left(f_s^p\right)'(t)} \underset{p\to+\infty}{\sim} \sum_{j=1}^{p-1} r^{j(\ell-1)} \breve{K}_s(t), \underset{p\to+\infty}{\sim} r^{p(\ell-1)} \breve{K}_s(t),$$

and we conclude using (9.75) for $\ell = 1$. □

Now we can state the second part of Theorem 6.5 concerning the limit with the penalization function $H_\ell(Z_p) s^{M_p} t^{Z_p-\ell}$ for all $\ell \in \mathbb{N}$ when $\mathbf{p}(0) = 0$.

Theorem 9.14. *1. Let* $\mathbf{p}$ *be an offspring distribution satisfying the two last properties of* (6.1), $\mathbf{p}(0) = 0$ *and which admits a moment of order* $\ell \in \mathbb{N}^*$ *and let* $\mathbf{q}$ *be mark function* $\mathbf{q}$ *satisfying* (6.2).

Let $r = \min\{k \in \mathbb{N},\ \mathbf{p}(k) > 0\}$. Let $t \in (0,1]$ and $s \in (0,1)$ be fixed. We have for all $n \in \mathbb{N}$, and all $\Lambda_n \in \mathscr{F}_n$,

$$\frac{\mathbb{E}\left[\mathbb{1}_{\Lambda_n} H_\ell(Z_{n+p}) s^{M_{n+p}} t^{Z_{n+p}}\right]}{\mathbb{E}\left[H_\ell(Z_{n+p}) s^{M_{n+p}} t^{Z_{n+p}}\right]} \underset{p\to+\infty}{\longrightarrow} \mathbb{E}\left[\mathbb{1}_{\Lambda_n} B_{s,n,0,r}\right], \tag{9.82}$$

with

$$B_{s,n,0,r} = \begin{cases} s^{M_n}\alpha_1(s)^{-n}\mathbb{1}_{Z_n=1} & \text{if } r = 1, \\ s^{M_n}\alpha_r(s)^{\frac{-(r^n-1)}{r-1}}\mathbb{1}_{Z_n=r^n} & \text{if } r > 1. \end{cases}$$

2. *The probability measure $\mathbb{Q}_{s,0,r}$ defined on $(\mathbb{T}^*, \mathscr{F})$ by*

$$\forall n \in \mathbb{N}, \Lambda_n \in \mathscr{F}_n, \quad \mathbb{Q}_{s,0,r}(\Lambda_n) = \mathbb{E}\left[\mathbb{1}_{\Lambda_n} B_{s,n,0,r}\right],$$

is the distribution of a regular r-ary tree whose each node is marked, independently of each others, with probability

$$\frac{s\mathbf{q}(r)}{s\mathbf{q}(r) + 1 - \mathbf{q}(r)}.$$

Proof. We only prove part (1). The proof of (2) follows the same lines as in the proof of Theorem 9.11, (2).

We begin with the case $\ell = 0$ and $r = 1$. Using (9.65), we obtain:

$$0 \le \frac{\mathbb{E}\left[s^{M_{n+p}} t^{Z_{n+p}} | \mathscr{F}_n\right]}{\mathbb{E}\left[s^{M_{n+p}} t^{Z_{n+p}}\right]} = \frac{s^{M_n} f_s^p(t)^{Z_n}}{f_s^{n+p}(t)} \le \frac{s^{M_n} f_s^p(t)^{Z_n - 1}}{(f_s^n)'(0)} \le \frac{1}{(f_s^n)'(0)}, \tag{9.83}$$

since the mean value inequality, applied to f_s^n on $[0,1]$, gives:

$$f_s^{n+p}(t) = f_s^n(f_s^p(t)) - f_s^n(0) \ge (f_s^n)'(0) f_s^p(t).$$

As $0 < (f_s^n)'(0) = \mathbb{E}\left[s^{M_n}\mathbb{1}_{Z_n=1}\right] < 1$, we can use (9.83) to apply Lebesgue's dominated convergence theorem and we just study the a.s. limit to obtain our results.

For $r = 1$, according to Lemma 10 in [8] and taking the notations in the proof of Lemma 9.5, we can define:

$$Q(t) = \int_{\kappa(s)}^t Q'(x)\mathrm{d}x,\ t \in [0,1],$$

such that $\lim_{p\to+\infty} Q_p(t) = Q(t)$ and Q is a positive function.

Since $\kappa(s) = 0$, and thanks to the expression (9.68) we have

$$f_s^p(t) \underset{p\to+\infty}{\sim} f_s'(0)^p Q(t), \tag{9.84}$$

with $f_s'(0) = \mathbb{E}[s^{M_1}\mathbb{1}_{Z_1=1}]$. Thereby:

$$\frac{\mathbb{E}\left[s^{M_{n+p}} t^{Z_{n+p}} | \mathscr{F}_n\right]}{\mathbb{E}\left[s^{M_{n+p}} t^{Z_{n+p}}\right]} \underset{p\to+\infty}{\sim} s^{M_n} f_s'(0)^{p(Z_n-1)-n} Q(t)^{Z_n-1} \qquad \underset{p\to\infty}{\longrightarrow} s^{M_n} f_s'(0)^{-n}\mathbb{1}_{Z_n=1},$$

since $s < 1$, $f_s'(0) < 1$, giving us our result.

Henceforth, we consider $\ell = 0$ and $r > 1$. According to (9.65) and Lemma 9.6, we get:

$$\frac{\mathbb{E}\left[s^{M_{n+p}}t^{Z_{n+p}}\,|\mathscr{F}_n\right]}{\mathbb{E}\left[s^{M_{n+p}}t^{Z_{n+p}}\right]} = \frac{s^{M_n} f_s^p(t)^{Z_n}}{f_s^{n+p}(t)} \underset{p\to+\infty}{\sim} s^{M_n}\alpha_r^{\frac{-1}{r-1}(Z_n-1)} e^{r^{n+p}b(t)(Z_n r^{-n}-1)} \underset{p\to\infty}{\longrightarrow} s^{M_n}\alpha_r^{\frac{-(r^n-1)}{r-1}}\mathbb{1}_{Z_n=r^n}$$

since $b(t) < 0$. To conclude this case, Lemma 9.6 ensures that for p large enough:

$$\frac{1}{2} \le \frac{f_s^p(t)}{\alpha_r(s)^{\frac{-1}{r-1}}e^{r^p b(t)}} \le \frac{3}{2}$$

implying that for p large enough, as $Z_n \ge r^n$:

$$\frac{\mathbb{E}\left[s^{M_{n+p}}t^{Z_{n+p}}\,|\mathscr{F}_n\right]}{\mathbb{E}\left[s^{M_{n+p}}t^{Z_{n+p}}\right]} = \frac{s^{M_n} f_s^p(t)^{Z_n}}{f_s^{n+p}(t)} \le \frac{\left(f_s^p(t)\right)^{r^n}}{f_s^{n+p}(t)} \le 2\left(\frac{3}{2}\right)^{r^n}\alpha_r(s)^{\frac{-(r^n-1)}{r-1}},$$

thus we can apply dominated convergence.

Now, we suppose $\ell \ge 1$ and $r = 1$. According to Lemma 9.5, (9.71) and (9.84) we have:

$$\begin{aligned}
\frac{\mathbb{E}\left[H_\ell(Z_{n+p})s^{M_{n+p}}t^{Z_{n+p}-\ell}|\mathscr{F}_n\right]}{f_s'(0)^p} &= \frac{s^{M_n}}{f_s'(0)^p}\sum_{i=1}^{\ell} H_i(Z_n) f_s^p(t)^{Z_n-i}\sum_{(n_1,\dots,n_i)\in S_{i,\ell}}\prod_{j=1}^{i}\frac{\left(f_s^p\right)^{(n_j)}(t)}{n_j!}\\
&\sim \frac{s^{M_n}}{f_s'(0)^p}\sum_{i=1}^{\ell} H_i(Z_n)\left(f_s'(0)^p Q(t)\right)^{Z_n-i}\sum_{(n_1,\dots,n_i)\in S_{i,\ell}}\prod_{j=1}^{i}\frac{f_s'(0)^p C_{n_j}(t)}{n_j!}\\
&= s^{M_n} f_s'(0)^{p(Z_n-1)}\sum_{i=1}^{\ell} H_i(Z_n) Q(t)^{Z_n-i}\sum_{(n_1,\dots,n_i)\in S_{i,\ell}}\prod_{j=1}^{i}\frac{C_{n_j}(t)}{n_j!}\\
&\underset{p\to\infty}{\longrightarrow} s^{M_n}\frac{C_\ell(t)}{\ell!}\mathbb{1}_{Z_n=1}.
\end{aligned}$$

The rest of the proof is the same as the one of Theorem 9.12 as the following upper bound remains true in our case:

$$0 \le \frac{\mathbb{E}\left[H_\ell(Z_{n+p})s^{M_{n+p}}t^{Z_{n+p}-\ell}|\mathscr{F}_n\right]}{f_s'(0)^p} \le C\sum_{i=1}^{\ell} H_i(Z_n).$$

Finally, we consider the case $\ell \ge 1$ and $r > 1$. According to Lemma 9.6, we have for every $1 \le i \le \ell$ and every $(n_1,\dots,n_i) \in S_{i,\ell}$, when $p \to +\infty$:

$$\sum_{(n_1,\dots,n_i)\in S_{i,\ell}}\prod_{j=1}^{i}\frac{\left(f_s^p\right)^{(n_j)}(t)}{n_j!} \sim r^{p\ell}e^{r^p i b(t)}\tilde{K}_i(t),$$

with $\tilde{K}_i(t)$ a positive function, since all the terms in the sum are positive and of the same order. According to (9.71) and (9.78), when p goes to infinity:

$$\begin{aligned}
\frac{\mathbb{E}\left[H_\ell(Z_{n+p})s^{M_{n+p}}t^{Z_{n+p}-\ell}|\mathscr{F}_n\right]}{r^{p\ell}e^{r^{m+p}b(t)}} &\sim s^{M_n}\sum_{i=1}^{\ell} H_i(Z_n)\left(e^{r^p b(t)}K_0(t)\right)^{Z_n-i} e^{r^p(i-r^n)b(t)}\tilde{K}_i(t)\\
&\sim s^{M_n}e^{r^p(Z_n-r^n)b(t)}\sum_{i=1}^{\ell} H_i(Z_n)K_0(t)^{Z_n-i}\\
&\underset{p\to\infty}{\longrightarrow} \mathbb{1}_{Z_n=r^n}s^{M_n}\sum_{i=1}^{\ell} H_i(r^n)K_0(t)^{r^n-i} =: s^{M_n}a_n\mathbb{1}_{Z_n=r^n},
\end{aligned}$$

where a_n is a positive constant. Note that for p large enough, as $Z_n \geq r^n$:

$$\frac{\mathbb{E}\left[H_\ell(Z_{n+p})s^{M_{n+p}}t^{Z_{n+p}-\ell}|\mathscr{F}_n\right]}{r^{p\ell}e^{r^{n+p}b(t)}} \leq \frac{s^{M_n}}{r^{p\ell}e^{r^{n+p}b(t)}}\sum_{i=1}^{\ell} H_i(Z_n)f_s^p(t)^{r^n-i}\sum_{(n_1,\dots,n_i)\in S_{i,\ell}}\prod_{j=1}^{i}\frac{\left(f_s^p\right)^{(n_j)}(t)}{n_j!}$$
$$\leq s^{M_n}\left(\frac{3}{2}\right)^{r^n}\sum_{i=1}^{\ell} H_i(Z_n)K_0(t)^{r^n-i}\sum_{(n_1,\dots,n_i)\in S_{i,\ell}}\prod_{j=1}^{i}\frac{K_{n_j}}{n_j!} \in L^1$$

as $Z_n \in L^\ell$. As a result, Lebesgue's dominated convergence theorem gives:

$$\frac{\mathbb{E}\left[\mathbb{1}_{\Lambda_n}H_\ell(Z_{n+p})s^{M_{n+p}}t^{Z_{n+p}-\ell}\right]}{r^{p\ell}e^{r^{n+p}b(t)}} = \mathbb{E}\left[\mathbb{1}_{\Lambda_n}s^{M_n}a_n\mathbb{1}_{Z_n=r^n}\right].$$

The previous formula is true for $n=0$ and $\Lambda_0 = \mathbb{T}^*$, consequently:

$$\lim_{p\to+\infty}\frac{\mathbb{E}\left[\mathbb{1}_{\Lambda_n}H_\ell(Z_{n+p})s^{M_{n+p}}t^{Z_{n+p}-\ell}\right]}{\mathbb{E}\left[H_\ell(Z_{n+p})s^{M_{n+p}}t^{Z_{n+p}-\ell}\right]} = \mathbb{E}\left[\mathbb{1}_{\Lambda_n}s^{M_n}r^{-n\ell}\frac{a_n}{a_0}\mathbb{1}_{Z_n=r^n}\right] =: \mathbb{E}\left[\mathbb{1}_{\Lambda_n}s^{M_n}r^{-n\ell}b_n\mathbb{1}_{Z_n=r^n}\right].$$

Since $s^{M_n}r^{-n\ell}b_n\mathbb{1}_{Z_n=r^n}$ is a martingale with mean 1, $b_n = r^{n\ell}\alpha_r(s)^{\frac{-(r^n-1)}{r-1}}$. □

9.2.3 Case $s=0$

For the sake of completeness, we consider the case $s=0$, that is we penalize by the quantity

$$H_\ell(Z_p)\mathbb{1}_{M_p=0}t^{Z_p}.$$

As the proofs are very close to that of the previous sections, we only state the main results and give some ideas for the proofs.

Here, we introduce $\psi(t) := \mathbb{E}\left[\mathbb{1}_{M_1=0}t^{Z_1}\right]$, $\tilde{r} := \min\{j \geq 0;\ \mathbf{p}(j)(1-\mathbf{q}(j)) > 0\}$ and $\tilde{\kappa}$ the smallest positive fixed point of ψ.

We begin with the case $\tilde{r}=0$, where $0<\tilde{\kappa}<1$ and we obtain:

Theorem 9.15. *Let* $\mathbf{p}$ *be an offspring distribution satisfying* (6.1) *that admits a moment of order* $\ell \in \mathbb{N}^*$ *and let* $\mathbf{q}$ *be a mark function satisfying* (6.2) *such that* $\tilde{r}=0$*, and let* $t \in (0,1]$*. We have for all* $n \in \mathbb{N}$*,* $\Lambda_n \in \mathscr{F}_n$*:*

$$\frac{\mathbb{E}\left[\mathbb{1}_{\Lambda_n}H_\ell(Z_{n+p})\mathbb{1}_{M_{n+p}=0}t^{Z_{n+p}}\right]}{\mathbb{E}\left[H_\ell(Z_{n+p})\mathbb{1}_{M_{n+p}=0}t^{Z_{n+p}}\right]} \underset{p\to+\infty}{\longrightarrow} \begin{cases} \mathbb{E}\left[\mathbb{1}_{\Lambda_n}B_{0,n,1}\right], & \text{if } \ell=0 \\ \mathbb{E}\left[\mathbb{1}_{\Lambda_n}B_{0,n,2}\right], & \text{otherwise}, \end{cases} \tag{9.85}$$

where $B_{0,n,1} := \tilde{\kappa}^{Z_n-1}\mathbb{1}_{M_n=0}$ *and* $B_{0,n,2} := \frac{Z_n\tilde{\kappa}^{Z_n-1}}{\psi'(\tilde{\kappa})^n}\mathbb{1}_{M_n=0}$.

The proof of this result is based on that of Theorem 9.11 and Theorem 9.12, and the following asymptotic behavior:

$$\forall t \in [0,1],\ (\psi^p)^{(\ell)}(t) \underset{p\to+\infty}{\sim} \psi'(\tilde{\kappa})^p C_\ell(t). \tag{9.86}$$

where C_ℓ is a positive function (the proof is similar to that of Lemma 9.5).
Under the probability $\mathbb{Q}_{0,i}$ defined on $(\mathbb{T}^*, \mathscr{F})$ by $\mathbb{Q}_{0,i}(\Lambda_n) = \mathbb{E}\left[\mathbb{1}_{\Lambda_n}B_{0,n,i}\right]$, for:

— $i = 1$, τ^* is a MGW with the reproduction-marking law:

$$p_{0,1}(k, \eta) = \mathbf{p}_0(k, \eta)\tilde{\kappa}^{k-1}\delta_{\eta 0};$$

— $i = 2$, τ^* is a two-typed random marked tree, there are normal and special nodes. The reproduction-marking law of the normal ones is $p_{0,1}$ and the special ones' is:

$$p_{0,2}(k, \eta) = \mathbf{p}_0(k, \eta)\frac{k\tilde{\kappa}^{k-1}}{\psi'(\tilde{\kappa})}\delta_{\eta 0},$$

and at each generation, there is a unique special node chosen uniformly among all the individuals of this generation.

Henceforth, we consider the case $\tilde{r} \geq 1$. We begin to present a lemma similar to Lemma 9.6 for the case $\tilde{r} > 1$.

Lemma 9.7. *Suppose $\tilde{r} \geq 2$. For every $\ell \in \mathbb{N}$, and every $t \in (0, 1]$, there exists a positive constant $K_\ell(t)$, such that*

$$(\psi^p)^{(\ell)}(t) = K_\ell(t) r^{p\ell} e^{\tilde{r}^p b(t)}(1 + o(1)),$$

where

$$b(t) := \ln(t) + \sum_{j=0}^{+\infty} \tilde{r}^{-(j+1)} \ln\left(\frac{\psi^{j+1}(t)}{\psi^j(t)^{\tilde{r}}}\right).$$

Thereby we obtain the following theorem:

Theorem 9.16. *Let $\mathbf{p}$ be an offspring distribution satisfying the two last points of* (6.1) *that admits a moment of order $\ell \in \mathbb{N}^*$, and let $\mathbf{q}$ be a mark function satisfying* (6.2) *such that $\tilde{r} \geq 1$, and let $t \in (0, 1]$. We have for all $n \in \mathbb{N}$, $\Lambda_n \in \mathscr{F}_n$:*

$$\frac{\mathbb{E}\left[\mathbb{1}_{\Lambda_n} H_\ell(Z_{n+p})\mathbb{1}_{M_{n+p}=0} t^{Z_{n+p}}\right]}{\mathbb{E}\left[H_\ell(Z_{n+p})\mathbb{1}_{M_{n+p}=0} t^{Z_{n+p}}\right]} \underset{p\to+\infty}{\longrightarrow} \begin{cases} \mathbb{E}\left[\mathbb{1}_{\Lambda_n}\mathbf{p}_0(1, 0)^{-n}\mathbb{1}_{M_n=0, Z_n=1}\right], & \text{if } \tilde{r} = 1, \\ \mathbb{E}\left[\mathbb{1}_{\Lambda_n}\mathbf{p}_0(r, 0)^{\frac{-(\tilde{r}^n-1)}{\tilde{r}-1}}\mathbb{1}_{M_n=0, Z_n=\tilde{r}^n}\right], & \text{if } \tilde{r} > 1. \end{cases}$$

The probabilities obtained with these martingales are Dirac masses at $\tilde{r}$-ary regular trees with no marks.

Chapter 10

Conclusion

The conditioning of Galton–Watson trees began in 1986 and has since evolved, becoming increasingly general. It was observed that conditioning on the number of leaves or the total number of individuals could be reduced to conditioning on the event "the number of individuals whose number of offspring belongs to a certain set $\mathcal{A}$". This type of conditioning can be interpreted as marking individuals depending on whether their number of children belongs to $\mathcal{A}$ or not, and then conditioning on the number of such marks in the tree.

The natural next step was to introduce some randomness in determining which individuals would be of interest. In other words, knowing that an individual has k children, with $k \in \mathbb{N}$, does not necessarily determine whether that individual is marked. To address this, we introduced a function $\mathbf{q} : \mathbb{N} \to [0, 1]$ that gives the probability that an individual with k children is marked.

One of the goals of my thesis is to characterize local convergence under conditioning on the number of marks(8).

I began by investigating the distribution of a MGW conditioned to have a large number of marks. Assuming that the offspring distribution has a finite second moment, we obtain, in the critical case, local convergence towards a marked Kesten tree. This tree is itself a Galton–Watson tree conditioned on non-extinction, with an infinite lineage. Consequently, it is infinitely large.

I considered the sub-critical case, still with the hypothesis of moment of order 2. We can distinguish two cases. The generic case, there is a critical biased law, depending of the initial reproduction-marking law, such that we obtain a local convergence into a marked Kesten tree with the biased reproduction law and the biased marking function. The non-generic case, it is not possible to make a link with the critical case. According to the study, in the deterministic case, of Abraham and Delmas [6], the expected result was a local convergence into a marked condensation tree. Contrary to the Kesten tree, this one has an infinite height and it is infinitely large. Indeed, one of the node of the special lineage can have an infinity of offspring. However, to obtain this local convergence, with marked Galton-Watson trees, it is necessary to understand how we mark the "infinity node". That is why, it was necessary to add additional hypothesis on the mark function and on the offspring distribution. Assuming that the reproduction law is α-stable and the marking function admits a finite limit at infinity, we obtain a local convergence towards a marked condensation tree.

Thanks to these results, I completed the one obtained by Abraham, Bouaziz and Delmas [2]. Indeed, in the sub-critical case, conditioning to have a large number of marks, a generic

GW converges to a Kesten tree with a biased reproduction law and a non generic GW converges to a condensation tree. Contrary to the results of [2] obtained with only a moment of order 1, we need to add a moment of order 2 on the offspring distribution. Unfortunately, we did not obtain a general result in the non-generic sub-critical case. However, we conjecture that similar results can be obtained. To achieve this, we need more general results on local probabilities of randomly indexed sums.

Another way of getting a tree with an "abnormal" number of marked vertices is to make a change of measure via a Girsanov transformation with a martingale which gives more weight to trees with a large number of marks or with a very small number of marks. Thereby, we can generalize conditioning to have an abnormal number of marks, by penalizing our MGW by its number of marks (9).

The first penalization function used is the following $\phi(Z_p, M_p) = M_p^{\alpha} Z_p^{\beta}$, with $\alpha, \beta \in \mathbb{N}$, then we assume that the offspring distribution admits a moment of order $\alpha + \beta$. The case $\alpha = 0$ was already treated in [5].

I began with the case $\beta = 0$ and described the MGW's behavior under the new measure of probability, defined thanks to the obtained martingale. In the critical case, the behavior of the MGW under the new probability is the same of a marked Kesten tree. In the super-critical case, the behavior is described by a multi-type tree, see [5], marked with the same mark function $\mathbf{q}$. Contrary to the two preceding cases, the sub-critical case wasn't already know. The difficulty was to interpret the obtained martingale to describe the trees under the new measure of probability. Thereby, we need to introduce the notion of mass (9.1.1), the obtained tree is a multi-type tree, where all type have an associated reproduction-marking law. In other words, the individual obtains its mark and its number of children at the same time. Since the mass is in reproduction-marking law, we lose the independence between the generation and the tree obtained is not a MGW.

After that, I was interested by the case $\alpha, \beta \in \mathbb{N}^*$. In the sub-critical and critical cases, we obtain the behavior of a marked Kesten tree and in the super critical case, we obtain the same multi-type marked tree describe in [5], with the same mark function $\mathbf{q}$.

This penalization function privileges trees with a large number of marks. We decided, to consider a penalization which makes weight on tree with a less number of marks. For that, I consider the function $\phi(Z_p, M_p) = H_\ell(Z_p) s^{M_p} t^{Z_p}$, with $s, t \in (0,1)$ and H_ℓ, the ℓ-th Hilbert polynomial, assuming the offspring distribution admits a moment of order ℓ. We distinguish different cases, depending of the fact that we have a probability positive or not for to have 0 children. If the probability to have any children is strictly positive, then we can obtained two types of trees. If $\ell = 0$, we obtain a MGW with a marking-reproduction law defined thanks to the initials marking et reproduction law. Otherwise, $\ell > 0$, we obtain a multitype tree with two types. The marking-reproduction law of the normal nodes is the same of the preceding case, the one of the special nodes is biaised. Contrary to the Kesten tree, there isn't special lineage and at each generation the special node is chosen uniformly among all the individual of the generation. In the case where the probability to have any children is null, we obtain a r-ary marked tree, with r the smallest integer such that the probability to have r children is positive. We also considered the case where $s = 0$, we obtained similar trees with the condition we had any marks.

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Sonia Boulal

Conditionnement et pénalisation d'arbres de Galton-Watson marqués

Résumé :

Le processus de Galton-Watson est un modèle très simple d'évolution de population où chaque individu se reproduit aléatoirement et indépendamment de tous les autres selon une même loi fixée. Il est connu que dans le cas où le nombre moyen d'enfants par individu est inférieur ou égale à 1 (sous-critique ou critique), la population s'éteint en temps fini presque sûrement. Certains auteurs ont décrit cette population (ou plutôt l'arbre généalogique associé) lorsqu'elle est anormalement grande et ont mis en évidence l'évènement exceptionnel permettant d'obtenir ce comportement particulier. D'un point de vue mathématique, cela consiste à conditionner la loi de l'arbre T par un évènement de la forme $\{A(T) = n\}$ où A est une fonctionnelle de l'arbre T (par exemple, $A(T)$ est la dernière génération survivante, ou bien le nombre total d'individus dans la population,...) et à regarder la limite (en loi) lorsque n tend vers l'infini.

Dans un récent travail, Abraham, Bouaziz et Delmas ont considéré d'autres types d'évènements faisant intervenir une source d'aléa supplémentaire. Dans leur article, chaque individu est marqué aléatoirement avec une probabilité dépendant de son nombre d'enfants. Ils ont conditionné par l'évènement "Le nombre d'individus marqués est égal à n". En utilisant des techniques similaires aux cas classiques, ils ont obtenu la limite en loi de l'arbre conditionné dans le cas critique. Une partie de ma thèse poursuit ce travail pour obtenir des résultats dans le cas sous-critique. Ainsi, j'ai pu décrire la limite en loi de l'arbre marqué dans les cas critique et sous-critique, sous certaines conditions de régularité concernant la loi de reproduction et la loi de marquage.

Une deuxième partie de ma thèse consiste à considérer une autre façon d'obtenir des arbres avec un nombre anormal de sommets marqués. Une technique possible est de faire un changement de mesure via une transformation de Girsanov avec une martingale. Afin de trouver une telle martingale, nous avons utilisé une méthode générale appelée *pénalisation* qui a été introduite par Roynette, Vallois et Yor en 2006 pour le mouvement Brownien. J'ai adapté cette technique dans le cadre des arbres discrets. En effet, j'ai biaisé la loi de l'arbre par une fonction du nombre de marques, qui soit les favorise, soit les défavorise, et j'ai décrit l'arbre obtenu sous cette nouvelle loi. J'ai traité les trois cas possibles, sous-critique, critique et sur-critique (le nombre moyen d'enfants par individu est strictement supérieur à 1). Dans les deux derniers cas, j'obtiens des arbres limites déjà connus, tandis que dans le premier cas, je décris un nouvel arbre, que l'on nommera "arbre à poids". Ce dernier est un arbre multi-type marqué, caractérisé par une notion de masse se transmettant de génération en génération. Il faut noter que, cet arbre n'est pas un arbre de Galton-Watson marqué.

Mots clés : Arbre de Galton-Watson marqué · Pénalisation · Transformation de Girsanov · Comportement asymptotique · Arbres aléatoires · Conditionnement

Conditioning and penalization of marked Galton-Watson trees

Abstract :

The Galton–Watson process is a very simple model of population evolution where each individual reproduces randomly and independently of all others according to the same given distribution. It is well known that in the case where the average number of offspring per individual is less than 1 (sub-critical or critical case), the population becomes extinct in finite time. Many authors have focused on the description of this population (more precisely, the genealogical tree associated with it) under the condition that it reaches an abnormally large size. From a mathematical point of view, this consists in conditioning the distribution of the tree T by some event of the form $\{A(T) = n\}$ where A is a functional of the tree T (for instance, $A(T)$ is the height of the last surviving generation, or the total number of individuals in the population,...) and to study the limit (in distribution) of the tree when n tends to infinity.

In a recent paper of Abraham, Bouaziz and Delmas, the authors explored alternative types of conditioning involving an additional source of randomness. In their paper, each individual is randomly marked with a probability depending on their number of children. They then conditioned the tree on the event "The number of marked individuals is equal to n". By adapting techniques from the classical framework, they obtained the limiting distribution of the conditioned tree in the critical case. Part of my PhD research extends this work to the sub-critical regime. Rather than studying the limiting distribution of the tree T, my focus is on the marked tree with its marks, allowing me to describe its asymptotic behavior in both critical and sub-critical cases. These results were obtained under certain regularity assumptions on both the offspring distribution and the marking mechanism.

The second part of my thesis investigates a different approach to generating trees with an unusual number of marked vertices. One natural method is to perform a change of measure using a martingale, through a Girsanov-type transformation. To find such a martingale, I used a general method called *penalization*, introduced by Roynette, Vallois, and Yor in 2006 for Brownian motion. I adapt this method to the discrete tree setting by biasing the distribution of the tree according to a function of the number of marks, either favoring or disfavoring them, and then analyzing the obtained trees under this new probability measure. I treated all three regimes : sub-critical, critical, and supercritical (where the average number of offspring per individual exceeds 1). In the critical and supercritical cases, I recovered known limiting objects. In contrast, the sub-critical case led me to identify a new limiting structure, which I refer to as the "weighted tree". This is a multi-type marked tree, characterized by a notion of mass that is transmitted from one generation to the next. Note that, it does not correspond to a marked Galton-Watson tree.

Keywords : Marked Galton–Watson tree · Penalization · Girsanov transformation · Asymptotic behavior · Random trees · Conditioning

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